Solutions

Find the maximum and minimum values **April 2023**

2166. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let $x_1, \dots, x_n$ be nonnegative real numbers with $x_1 + \cdots + x_n = 1$ and $n \geq 2$. Determine the minimum and maximum values of the following function

$$\frac{x_1 + x_2}{1 + x_1 x_2} + \frac{x_2 + x_3}{1 + x_2 x_3} + \cdots + \frac{x_n + x_1}{1 + x_n x_1}.$$

When do the extreme values occur?

Composite solution by Stan Dolan, Charmouth, UK, and the proposer.
Put

$$f_n(x_1, \dots, x_n) = \frac{x_1 + x_2}{1 + x_1 x_2} + \frac{x_2 + x_3}{1 + x_2 x_3} + \cdots + \frac{x_n + x_1}{1 + x_n x_1}$$

with $x_i \geq 0$ and $x_1 + \cdots + x_n = 1$. Since

$$f_n(x_1, \dots, x_n) \leq (x_1 + x_2) + (x_2 + x_3) + \cdots + (x_n + x_1) = 2,$$

the maximum value is 2, which occurs if and only if $x_1 x_2 = x_2 x_3 = \cdots = x_n x_1 = 0$.

To minimize f_n, we consider three cases. If $n = 2$, then

$$f_2(x, y) = \frac{2(x + y)}{1 + xy} = \frac{2}{1 + xy}.$$

This is minimized, then xy is maximized, i.e., only when $x = y = 1/2$ and the minimum value is thus $8/5$.

Without loss of generality, when $n = 3$, we may consider $f_3(x, y, z)$ with $0 \leq x \leq y \leq z \leq 1$. Note that this forces $0 \leq x \leq 1/3$. We claim that in this case

$$f_3(x, y, z) \geq f_3\left(x, \frac{y + z}{2}, \frac{y + z}{2}\right) = f_3\left(x, \frac{1 - x}{2}, \frac{1 - x}{2}\right). \tag{1}$$

After some computations, this is equivalent to

$$(1 - x)(y - z)^2(2 - 4x - x^2 - xyz(5 - 4x + 2x^2)) \geq 0. \tag{2}$$

Since $0 \leq x \leq 1/3$ and $0 \leq xyz \leq ((x + y + z)/3)^3 = 1/27$, we have

$$\begin{aligned}
&49 > 104\left(\frac{1}{3}\right) + 29\left(\frac{1}{3}\right)^2 \geq 104x + 29x^2 \\
&\Rightarrow 54 - 108x - 27x^2 > 5 - 4x + 2x^2 \\
&\Rightarrow 2 - 4x - x^2 > \frac{1}{27}(5 - 4x + 2x^2) \geq xyz(5 - 4x + 2x^2) \\
&\Rightarrow 2 - 4x - x^2 - xyz(5 - 4x + 2x^2) > 0.
\end{aligned}$$

Therefore (2) and equivalently (1) are true, and the equality holds if and only if $y = z$.

We have

$$f_3\left(x, \frac{y+z}{2}, \frac{y+z}{2}\right) = f_3\left(x, \frac{1-x}{2}, \frac{1-x}{2}\right)$$
$$= \frac{9}{5} + \frac{x(1-3x)^2}{5(2-x)(5-2x+x^2)},$$

so the minimum value of f_3 is $9/5$, which occurs only when $x = 0$ or $x = 1/3$. This corresponds to $(x, y, z) = (1/3, 1/3, 1/3)$ or a permutation of $(0, 1/2, 1/2)$.

We now consider the case when $n \geq 4$. Each term of $f_n(x_1, \dots, x_n)$ involves neither x_1 nor x_3 or involves just one of them (since $n \geq 4$) and is a concave function of those arguments. Therefore $f_n(x_1, \dots, x_n)$ is a concave function of x_1 if $x_1 + x_3$ and all other arguments are kept constant. Hence its minimum as x_1 varies is attained at the boundary values where either x_1 or x_3 is zero.

Without loss of generality, suppose that the minimum occurs when $x_1 = 0$. Then

$$f_n(x_1, \dots, x_n) = f_{n-1}(x_2, \dots, x_n) + x_2 + x_n - \frac{x_2 + x_n}{1 + x_2 x_n} \geq f_{n-1}(x_2, \dots, x_n)$$

with $x_2 + \cdots x_n = 1$ and equality occurring if and only if $x_2 x_n = 0$. Repeating this procedure leads to the case $n = 3$ with one of the arguments being zero. From the results for the case $n = 3$, the three arguments must be a permutation of $(1/2, 1/2, 0)$ and the minimum value is $9/5$. Hence the minimum value of f_n is also $9/5$ and this occurs only when $(x_1, \dots x_n)$ is a cyclic permutation of $(1/2, 1/2, 0, \dots, 0)$.

Also solved by Paul Bracken, Prithwijit De (India), and Harris Kwong. There was one incomplete or incorrect solution.

The limit of an exponential product **April 2023**

2167. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

Prove that

$$\lim_{n\to\infty} e^{n/2} \prod_{i=2}^{n} e^{i^2} \left(1 - \frac{1}{i^2}\right)^{i^4} = \pi \exp\left(-\frac{5}{4} + \frac{3\zeta(3)}{\pi^2}\right),$$

where $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.
Let

$$P_n := e^{n/2} \prod_{i=2}^{n} e^{i^2} \left(1 - \frac{1}{i^2}\right)^{i^4}.$$

Taking logarithms yields

$$\ln P_n = \frac{n}{2} + \sum_{i=2}^{n} i^2 + \sum_{i=2}^{n} i^4 \ln\left(1 - \frac{1}{i^2}\right).$$

Applying the power series

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } x \in [-1, 1),$$

we have

$$\begin{aligned}
\ln P_n &= \frac{n}{2} + \sum_{i=2}^{n} i^2 - \sum_{i=2}^{n}\sum_{k=1}^{\infty} \frac{i^4}{k\, i^{2k}} \\
&= \frac{n}{2} + \sum_{i=2}^{n} i^2 - \sum_{i=2}^{n}\left(i^2 + \frac{1}{2} + \sum_{k=3}^{\infty} \frac{1}{k\, i^{2(k-2)}}\right) \\
&= \frac{1}{2} - \sum_{i=2}^{n}\sum_{k=3}^{\infty} \frac{1}{k\, i^{2(k-2)}} \qquad (\text{use } k-2 \to k) \\
&= \frac{1}{2} - \sum_{i=2}^{n}\sum_{k=1}^{\infty} \frac{1}{(k+2)\, i^{2k}}.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \ln P_n = \frac{1}{2} - \sum_{i=2}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(k+2)\, i^{2k}}.$$

Exchanging the order of the summation gives

$$S := \sum_{k=1}^{\infty} \frac{1}{k+2} \sum_{i=2}^{\infty} \frac{1}{i^{2k}} = \sum_{k=1}^{\infty} \frac{1}{k+2}(\zeta(2k) - 1),$$

where $\zeta(2k) = \sum_{n=1}^{\infty} 1/n^{2k}$. Thus

$$\begin{aligned}
S &= \sum_{k=1}^{\infty} 2\int_0^1 (\zeta(2k) - 1)x^{2k+3}\, dx \\
&= \int_0^1 \left(1 - \pi x\cot(\pi x) - \frac{2x^2}{1-x^2}\right) x^3\, dx \\
&= \int_0^1 \left(3 - \pi x\cot(\pi x) - \frac{2}{1-x^2}\right) x^3\, dx,
\end{aligned}$$

where we have used $2x^2/(1-x^2) = -2 + 2/(1-x^2)$ and

$$\sum_{k=1}^{\infty} \zeta(2k)x^{2k} = \frac{1}{2}(1 - \pi x\cot(\pi x))$$

(https://proofwiki.org/wiki/Riemann_Zeta_Function_at_Even_Integers). Since $x^3/(1-x^2) = -x + 2x/(1-x^2)$, we have

$$\begin{aligned}
S &= \frac{7}{4} - \int_0^1 \left(\pi x^4\cot(\pi x) + \frac{2x}{1-x^2}\right) dx \\
&= \frac{7}{4} - \lim_{b\to 1^-} \int_0^b \left(\pi x^4\cot(\pi x) + \frac{2x}{1-x^2}\right) dx \\
&= \frac{7}{4} - \lim_{b\to 1^-} \left(b^4\ln(\sin(\pi b)) - \ln(1-b^2)\right) + 4\int_0^1 x^3\ln(\sin(\pi x))\, dx
\end{aligned}$$

$$= \frac{7}{4} - \ln(\pi/2) + 4\int_0^1 x^3 \ln(\sin(\pi x))\, dx.$$

Using

$$\ln(\sin x) = -\ln 2 - \sum_{n=1}^{\infty} \frac{\cos(2nx)}{n}$$

(https://proofwiki.org/wiki/Fourier_Series/Logarithm_of_Sine_of_x_over_0_to_Pi), and

$$\int_0^1 x^3 \cos(2n\pi x)\, dx = \frac{3}{4n^2\pi^2}$$

we find that

$$\begin{aligned} S &= \frac{7}{4} - \ln(\pi/2) - 4\ln 2\int_0^1 x^3\, dx - 4\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^3\cos(2n\pi x)\, dx \\ &= \frac{7}{4} - \ln\pi - 4\left(\frac{3}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3}\right) \\ &= \frac{7}{4} - \ln\pi - \frac{3\zeta(3)}{\pi^2}. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \ln P_n = \frac{1}{2} - S = -\frac{5}{4} + \ln\pi + \frac{3\zeta(3)}{\pi^2}.$$

Taking exponent yields the desired result.

Also solved by Paul Bracken, Robert Calcaterra, Kee-Wai Lau (China), Raymond Mortini (Luxembourg) & Rudolf Rupp (Germany), Albert Stadler (Switzerland), Seán Stewart (Saudi Arabia), and the proposer. There was one incomplete or incorrect solution.

Polynomial identities for a recursive sequence **April 2023**

2168. *Proposed by C. J. Hillar, San Francisco, CA.*

Let $\alpha, r \in \mathbb{R}$ and let a_n be the sequence

$$a_0 = 0, a_1 = \alpha \text{ and } a_n = ra_{n-1} + a_{n-2} \text{ for } n > 1.$$

Prove that for each odd integer k, there are polynomials $p_k, q_k \in \mathbb{R}[x]$ such that for all nonnegative integers n,

$$a_{kn} = p_k(a_n) \text{ for } n \text{ even and } a_{kn} = q_k(a_n) \text{ for } n \text{ odd.}$$

For example, for the Fibonacci sequence (where $\alpha = r = 1$),

$$F_{3n} = 5F_n^3 + 3F_n \text{ for } n \text{ even and } F_{3n} = 5F_n^3 - 3F_n \text{ for } n \text{ odd.}$$

Solution by Kyle Gatesman, Fairfax, VA.
If $\alpha = 0$, then $a_n = 0$ for all whole numbers n, so for all k, the polynomials $p_k(z) = q_k(z) = z$ satisfy the desired criteria. Now suppose $\alpha \neq 0$. The recurrence has the characteristic polynomial $\lambda^2 - r\lambda - 1 = 0$, yielding solutions

$$\lambda_1 = \frac{r + \sqrt{r^2+4}}{2} \quad \text{and} \quad \lambda_2 = \frac{r - \sqrt{r^2+4}}{2}.$$

The explicit form of the solution to the recurrence is then $a_n = c_1\lambda_1^n + c_2\lambda_2^n$. Taking $n = 0$ gives $c_2 = -c_1$, and taking $n = 1$ gives

$$c_1 = \frac{\alpha}{\sqrt{r^2+4}} \in \mathbb{R} - \{0\}.$$

Since $\lambda_1\lambda_2 = -1$,

$$a_n = c_1\left(\lambda_1^n - \left(-\frac{1}{\lambda_1}\right)^n\right).$$

We claim that for all odd integers $k \geq -1$, there exist polynomials P_k and Q_k with real coefficients such that

$$P_k\left(x + \frac{1}{x}\right) = x^k + \left(\frac{1}{x}\right)^k \quad \text{and} \quad Q_k\left(x - \frac{1}{x}\right) = x^k - \left(\frac{1}{x}\right)^k$$

for all x. We prove this claim by induction. The case $k = -1$ is satisfied when $P_{-1}(z) = z$ and $Q_{-1}(z) = -z$, and the case $k = 1$ is satisfied when $P_1(z) = Q_1(z) = z$. Now suppose that for some odd $k > 1$, the polynomials P_{k-2}, P_{k-4}, Q_{k-2}, and Q_{k-4} exist. For $s \in \{-1, +1\}$, we have

$$\left(x^{k-2} + s\left(\frac{1}{x}\right)^{k-2}\right)\left(x + \frac{s}{x}\right)^2 =$$
$$\left(x^k + s\left(\frac{1}{x}\right)^k\right) + 2s\left(x^{k-2} + s\left(\frac{1}{x}\right)^{k-2}\right) + \left(x^{k-4} + s\left(\frac{1}{x}\right)^{k-4}\right).$$

Therefore if we define polynomials P_k and Q_k by

$$P_k(z) = P_{k-2}(z)z^2 - 2P_{k-2}(z) - P_{k-4}(z)$$

and

$$Q_k(z) = Q_{k-2}(z)z^2 + 2Q_{k-2}(z) - Q_{k-4}(z),$$

then

$$P_k\left(x + \frac{1}{x}\right) = x^k + \left(\frac{1}{x}\right)^k \quad \text{and} \quad Q_k\left(x - \frac{1}{x}\right) = x^k - \left(\frac{1}{x}\right)^k,$$

as desired.

For every odd $k \geq 1$, let

$$p_k(z) = c_1 P_k(z/c_1) \quad \text{and} \quad q_k(z) = c_1 Q_k(z/c_1).$$

This makes sense since $c_1 \neq 0$. Note that p_k and q_k have real coefficients since P_k and Q_k do and $c_1 \in \mathbb{R}$. For odd n,

$$\begin{aligned} p_k(a_n) &= c_1 P_k\left(\lambda_1^n + \frac{1}{\lambda_1^n}\right) = c_1\left(\left(\lambda_1^n\right)^k + \left(\frac{1}{\lambda_1^n}\right)^k\right) \\ &= c_1\left(\lambda_1^{kn} - \left(-\frac{1}{\lambda_1}\right)^{kn}\right) = a_{kn}, \end{aligned}$$

and for even n,

$$\begin{aligned} q_k(a_n) &= c_1 Q_k\left(\lambda_1^n - \frac{1}{\lambda_1^n}\right) = c_1\left(\left(\lambda_1^n\right)^k - \left(\frac{1}{\lambda_1^n}\right)^k\right) \\ &= c_1\left(\lambda_1^{kn} - \left(-\frac{1}{\lambda_1}\right)^{kn}\right) = a_{kn}. \end{aligned}$$

Also solved by Robert Calcaterra, Hongwei Chen, Eagle Problem Solvers (Georgia Southern University), Dmitry Fleischman, Russell Gordon, Eugene A. Herman, Walther Janous (Austria), Shin Hin Jimmy Pa (China), Harris Kwong, Northwestern University Math Problem Solving Group, Angel Plaza (Spain), Randy K. Schwartz, Albert Stadler (Switzerland), and the proposer.

A functional equation that mirrors polynomial factorization **April 2023**

2169. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.*

Let a, b, and c be distinct positive real numbers, which are not equal to 1, and let d be one of them. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions:

(i) $f(f(f(x))) - (a+b+c)f(f(x)) + (ab+ac+bc)f(x) - abcx = 0$ for all $x \in \mathbb{R}$.

(ii) f is continuous.

(iii) There exists an $x_0 \in (0, \infty)$ such that $f(x_0) = dx_0$, and $f(f(x_0)) = d^2x_0$.

Prove that $f(x) = dx$ for all $x \in [0, \infty)$.

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI.
The notation f^n will be used to denote f composed with itself n times. For example, $f^3(x) = f(f(f(x)))$. Also, we may assume $d = c$ due to the inherit symmetry in the problem. Note that if $f(x) = f(y)$, then condition (i) implies $abcx = abcy$ and thus $x = y$. Hence f is one-to-one. Since f is continuous, f must be strictly monotone on $\mathbb{R}$. Moreover,

$$\frac{f(f(x_0)) - f(x_0)}{f(x_0) - x_0} = \frac{c^2x_0 - cx_0}{cx_0 - x_0} = c > 0.$$

Thus f is strictly increasing. Assume f has an upper bound. Then $\lim_{x\to\infty} f(x) = s$, the supremum of f, and by (i),

$$\lim_{x\to\infty} abcx = f^2(s) - (a+b+c)f(s) + (ab+ac+bc)s,$$

which is finite. This is a contradiction, so f does not have an upper bound. Similarly, f does not have a lower bound, so f is a bijection from $\mathbb{R}$ onto $\mathbb{R}$. In particular, f^{-1} exists.

We claim that for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}$,

$$f^{k+3}(x) - (a+b+c)f^{k+2}(x) + (ab+ac+bc)f^{k+1}(x) - abcf^k(x) = 0.$$

The hypotheses imply the claim is true for $k = 0$. Moreover, replacing the argument x by $f(x)$ shows we can increment k by 1 and replacing x by $f^{-1}(x)$ shows we can decrement k by 1. Therefore a bidirectional induction argument validates the claim.

We next claim that $f^k(x_0) = c^k x_0$ for all integers k. The claim is trivially true if $k = 0$ and (iii) implies the claim is true for $k = 1$ and $k = 2$. Next assume

$$f^k(x_0) = c^k x_0, \quad f^{k+1}(x_0) = c^{k+1}x_0, \quad \text{and} \quad f^{k+2}(x_0) = c^{k+2}x_0,$$

for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} f^{k+3}(x_0) &= (a+b+c)f^{k+2}(x_0) - (ab+ac+bc)f^{k+1}(x_0) + abcf^k(x_0) \\ &= (a+b+c)c^{k+2}x_0 - (ab+ac+bc)c^{k+1}x_0 + (abc)c^k x_0 \\ &= c^{k+3}x_0 \end{aligned}$$

and

$$\begin{aligned} abcf^{k-1}(x_0) &= f^{k+2}(x_0) - (a+b+c)f^{k+1}(x_0) + (ab+ac+bc)f^k(x_0) \\ &= c^{k+2}x_0 - (a+b+c)c^{k+1}x_0 + (ab+ac+bc)c^k x_0 \\ &= abc^k x_0. \end{aligned}$$

Hence $f^{k-1}(x_0) = c^{k-1}x_0$ and the claim is verified.

Finally, suppose t is an arbitrary positive real number. Denote the closed interval with endpoints at u and v (regardless of which is larger) by $|u, v|$. Choose $L \in \mathbb{Z}$ so that $t \in \left|c^L x_0, c^{L+1}x_0\right|$ and let $t_0 = t/c^L$. For $k \in \mathbb{Z}$, let

$$z_k = f^k(t_0) - cf^{k-1}(t_0).$$

Since

$$f^{k+2}(t_0) - (a+b+c)f^{k+1}(t_0) + (ab+ac+bc)f^k(t_0) - abcf^{k-1}(t_0) = 0$$

implies that

$$\left(f^{k+2}(t_0) - cf^{k+1}(t_0)\right) - (a+b)\left(f^{k+1}(t_0) - cf^k(t_0)\right) + ab\left(f^k(t_0) - cf^{k-1}(t_0)\right) = 0,$$

we have $z_{k+2} = (a+b)z_{k+1} - abz_k$. This recurrence relation has characteristic equation $r^2 - (a+b)r + ab = 0$ with roots $r = a$ and $r = b$. Therefore there exist real constants A and B such that

$$z_k = Aa^k + Bb^k \text{ and hence } \frac{z_k}{c^k} = A\left(\frac{a}{c}\right)^k + B\left(\frac{b}{c}\right)^k$$

for all $k \in \mathbb{Z}$. Note that $t_0 \in |x_0, cx_0|$. Since cf^{k-1} and f^k are increasing functions,

$$cf^{k-1}(t_0) \in \left|cf^{k-1}(x_0), cf^{k-1}(cx_0)\right| = \left|c^k x_0, c^{k+1}x_0\right|$$

and

$$f^k(t_0) \in \left| f^k(x_0), f^k(cx_0) \right| = \left| c^k x_0, c^{k+1} x_0 \right|.$$

Therefore,

$$|z_k| = \left| f^k(t_0) - cf^{k-1}(t_0) \right| \le \left| c^{k+1} x_0 - c^k x_0 \right|, \text{ so } \left| \frac{z_k}{c^k} \right| \le |cx_0 - x_0|.$$

This implies that $A(a/c)^k + B(b/c)^k$ is bounded as k ranges from $-\infty$ to ∞. Since a, b, and c are distinct positive constants, it follows that $A = B = 0$, $z_k = 0$, and hence $f^k(t_0) = cf^{k-1}(t_0)$ for every integer k. Since $f^0(t_0) = t_0$, a routine induction argument yields $f^k(t_0) = c^k t_0$ for all $k \in \mathbb{Z}$. Hence

$$f(t) = f\left(c^L t_0\right) = f\left(f^L(t_0)\right) = f^{L+1}(t_0) = c^{L+1} t_0 = ct$$

for all $t \in (0, \infty)$. In addition,

$$f(0) = \lim_{t \to 0^+} f(t) = 0$$

and the proof is complete.

Also solved by Anthony Kindness & Dylan Strohl and the proposer.

A partition of the k-element subsets of $\mathbb{Z}_p$ **April 2023**

2170. *Proposed by George Stoica, Saint John, NB, Canada.*

For a fixed prime number p, let A_k be the set of all subsets of $\{0, \ldots, p-1\}$ having k elements, $1 \le k \le p-1$. Let

$$A_{k,m} = \left\{ \{i_1, \ldots, i_k\} \in A_k \,\middle|\, \sum_{j=1}^{k} i_j \equiv m \pmod{p} \right\}$$

with $0 \le m \le p-1$.

Prove that

$$|A_{k,m}| = \frac{1}{p}\binom{p}{k}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

It is obvious that $A_{k,s} \cap A_{k,t} = \emptyset$ whenever $s \not\equiv t \pmod{p}$. Thus $\left\{A_{k,0}, \ldots, A_{k,p-1}\right\}$ form a partition of A_k. Since $|A_k| = \binom{p}{k}$, it suffices to show that $\left|A_{k,s}\right| = \left|A_{k,t}\right|$ whenever $s \not\equiv t \pmod{p}$. Consider $\{i_1, \ldots, i_k\} \in A_{k,s}$. Since $t - s \not\equiv 0 \pmod{p}$, there exists a positive integer k' such that $kk' \equiv t - s \pmod{p}$. For $j = 1, \ldots, k$, define $\ell_j \equiv i_j + k' \pmod{p}$. Clearly $|\{\ell_1 \ldots, \ell_k\}| = k$. We have

$$\sum_{j=1}^{k} \ell_j = \left(\sum_{j=1}^{k} i_j \right) + kk' \equiv s + (t - s) = t \pmod{p}.$$

We deduce that $\{\ell_1, \ldots, \ell_k\} \in A_{k,t}$. This proves that $\left|A_{k,s}\right| \le \left|A_{k,t}\right|$. In a similar manner, we also have $\left|A_{k,t}\right| \le \left|A_{k,s}\right|$. Therefore $\left|A_{k,s}\right| = \left|A_{k,t}\right|$, which completes the proof.

Also solved by Robert Calcaterra, Dimitry Fleischman, Kyle Gatesman, Shing Hin Jimmy Pa (China), Albert Stadler (Switzerland), and the proposer.

Answers

Solutions to the Quickies from page 224.

A1139. Since

$$\binom{p-1}{i} = \frac{(p-1)(p-2)\cdots(p-i)}{i!} \equiv \frac{(-1)(-2)\cdots(-i)}{i!} \equiv (-1)^i \pmod{p},$$

we have

$$\sum_{i=1}^{p-1} \frac{1}{\binom{p-1}{i}} \equiv \sum_{i=1}^{p-1}(-1)^i \equiv 0 \pmod{p}.$$

A1140. Consider the generating function

$$h(x) := \sum_{\nu=0}^{\infty} \left(\sum_{k=0}^{\nu} (-1)^k \binom{m+k}{k} \binom{m+n+1}{\nu-k} \right) x^\nu.$$

This is the product of the two series

$$f(x) := \sum_{j=0}^{\infty} \binom{m+j}{j} (-1)^j x^j = \frac{1}{(1+x)^{m+1}}$$

and

$$g(x) := \sum_{j=0}^{\infty} \binom{m+n+1}{j} x^j = (1+x)^{m+n+1}.$$

Hence

$$h(x) = f(x)g(x) = (1+x)^n.$$

Thus the nth Taylor coefficient of h is $\binom{n}{n} = 1$.

Solutions

Bitstrings which contain neither 11 **nor** 000 **as substrings** **February 2023**

2161. *Proposed by Didier Pinchon, Toulouse, France, and George Stoica, Saint John, NB, Canada.*

Let x_n denote the number of bitstrings of length n which contain neither 11 nor 000 as substrings. Find a recursive formula for x_n.

Solution by the Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.
The answer is $x_n = x_{n-2} + x_{n-3}$, with initial conditions $x_1 = 2$, $x_2 = 3$, $x_3 = 4$.

Assume $n > 3$ and divide the set of bitstrings with the desired property into two subsets: those starting with 0 and those starting with 1. Let y_n be the number in the first set and z_n the number in the second.

Bitstrings starting with 0 can be followed by a string starting with 1, or by another 0 and then a string starting with 1. Hence

$$y_n = z_{n-1} + z_{n-2}\,.$$

Bitstrings starting with 1 can only be followed by a string starting with 0, hence

$$z_n = y_{n-1}.$$

This, together with the first equation, implies $y_n = y_{n-2} + y_{n-3}$ and $z_n = z_{n-2} + z_{n-3}$. Since $x_n = y_n + z_n$, we get the recursive formula:

$$x_n = x_{n-2} + x_{n-3}.$$

The initial conditions can be obtained by enumerating the bitstrings with the desired property. For $n = 1 : 0, 1$, for $n = 2 : 00, 01, 10$, and for $n = 3 : 001, 010, 100, 101$. Hence $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$.

Also solved by Ricardo Bittencourt (Brazil), Robert Calcaterra, Kyle Calderhead, Eagle Problem Solvers (Georgia Southern University), John Ferdinands, Eugene A. Herman, Aykhan Ismayilov (Azerbaijan), Walther Janous (Austria), Kee Wai Lau (Hong Kong), Kent E. Morrison, José Heber Nieto (Venezuela), Michelle Nogin, Shing Hin Jimmy Pa (China), Angel Plaza (Spain), Rob Pratt, Edward Schmeichel, Randy Schwartz, Albert Stadler (Switzerland), Paul Stockmeyer, and the proposers. There were two incomplete or incorrect solution.

An infinite series involving skew-harmonic numbers **February 2023**

2162. *Proposed by Narendra Bhandari, Bajura, Nepal.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n(2n+1)4^n}\binom{2n}{n} = 4G + \frac{\pi^2}{12} - 2\pi \ln 2,$$

where

$$\overline{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$$

is the nth skew-harmonic number and

$$G = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2}$$

is Catalan's constant.

Solution by the proposer.
It is well known that

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n(2n+1)}\binom{2n}{n} = \arcsin x - x.$$

Dividing by x^2 and multiplying by $\ln(1+x)$ on both sides, gives us

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}\ln(1+x)}{4^n(2n+1)}\binom{2n}{n} = \frac{\arcsin x \ln(1+x)}{x^2} - \frac{\ln(1+x)}{x}. \tag{1}$$

Now

$$\int_0^1 \frac{1-x^{2n}}{1+x}\,dx = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} = \overline{H}_{2n}.$$

Integrating by parts yields

$$\int_0^1 \frac{1-x^{2n}}{1+x}\,dx = 2n\int_0^1 x^{2n-1}\ln(1+x)\,dx.$$

Therefore,

$$\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{\overline{H}_{2n}}{2n}.$$

Integrating (1) from 0 to 1 and using the above, we obtain

$$\sum_{n=1}^{\infty}\frac{\overline{\mathcal{H}}_{2n}}{2n(2n+1)4^n}\binom{2n}{n} = \int_0^1 \frac{\arcsin x\ln(1+x)}{x^2}dx - \int_0^1 \frac{\ln(1+x)}{x}dx.$$

The second integral can be evaluated as

$$\mathcal{J} = \int_0^1 \frac{\ln(1+x)}{x}dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{k-1}dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}.$$

For the first integral $\mathcal{I}$, substituting $x = \sin y$ and applying integration by parts (with $u = y\ln(1+\sin y)$ and $dv = \cos y/\sin^2 y$), we get

$$\mathcal{I} = \int_0^{\frac{\pi}{2}} \frac{y\ln(1+\sin y)\cos y}{\sin^2 y}dy = -\frac{\pi}{2}\ln(2) + \int_0^{\frac{\pi}{2}}\frac{\ln(1+\sin y)}{\sin y}dy$$
$$-\int_0^{\frac{\pi}{2}}\frac{y\cos y}{1+\sin y}dy + \int_0^{\frac{\pi}{2}}\frac{y}{\tan y}dy.$$

We have

$$A = \int_0^{\frac{\pi}{2}}\frac{\ln(1+\sin y)}{\sin y}dy \overset{\text{y=2 arctan t}}{=} \int_0^1 \frac{\ln\left(1+\frac{2t}{1+t^2}\right)}{\frac{2t}{1+t^2}}\frac{2t\,dt}{1+t^2}$$
$$= \int_0^1 \frac{2\ln(1+t)-\ln(1+t^2)}{t}dt = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$$
$$= \frac{3}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{8}$$

and

$$B = \int_0^{\frac{\pi}{2}}\frac{y\cos y}{1+\sin y}dy \overset{\text{IBP}}{=} \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\ln(1+\sin y)dy$$
$$= \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\ln(1+\cos y)dy = \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\frac{y\sin y}{1+\cos y}dy$$
$$= \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\frac{y\tan y}{\sec y+1}dy = \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\left(\frac{y}{\sin y} - \frac{y}{\tan y}\right)dy$$
$$= \frac{\pi}{2}\ln(2) - 2G + \int_0^{\frac{\pi}{2}}\frac{y}{\tan y}dy,$$

where we have used the well-known fact that $G = \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} \dfrac{y}{\sin y} dy$. Therefore,

$$\mathcal{I} = -\frac{\pi}{2}\ln(2) + A - B + \int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = -\frac{\pi}{2}\ln(2) + \frac{\pi^2}{8} + 2G - \frac{\pi}{2}\ln(2)$$
$$= 2G - \pi\ln(2) + \frac{\pi^2}{8},$$

and hence

$$\sum_{n=1}^{\infty} \frac{\overline{\mathcal{H}}_{2n}}{n(2n+1)4^n}\binom{2n}{n} = 2(\mathcal{I} - \mathcal{J}) = 4G + \frac{\pi^2}{12} - 2\pi\ln(2).$$

Also solved by Paul Bracken, Hongwei Chen, Walther Janous, and Albert Stadler (Switzerland). There was one incomplete or incorrect solution.

Sequences on which σ (respectively ϕ) is decreasing **February 2023**

2163. *Proposed by Philippe Fondanaiche, Paris, France.*

Recall that for a positive integer n, $\sigma(n)$ denotes the sum of the positive divisors of n, and $\phi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n. Show that the following hold.

(a) There are arbitrarily long sequences $n_1 < n_2 < \cdots < n_k$ such that $\sigma(n_1) > \sigma(n_2) > \cdots > \sigma(n_k)$.
(b) There are arbitrarily long sequences $n_1 < n_2 < \cdots < n_k$ such that $\phi(n_1) > \phi(n_2) > \cdots > \phi(n_k)$.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela, (part (a)) and the proposer (part (b))
(a) We shall prove, by induction on k, that for any $k > 1$ there is a sequence

$$n_1 < n_2 < \cdots < n_k \text{ such that } \sigma(n_1) > \sigma(n_2) > \cdots > \sigma(n_k).$$

For $k = 2$, put $n_1 = 4$, $n_2 = 5$. Then $n_1 < n_2$ and $\sigma(n_1) = 7 > 6 = \sigma(n_2)$. Now assume that the result is true for k. Take a prime $r > n_k$. It is well known that, given $\epsilon > 0$, there exists an integer N such that, for any $n > N$, there is a prime p such that $n < p < (1+\epsilon)n$. Take $\epsilon = 1/r$ and a prime $q > r$ such that $q > N/r$ as well. Then $rq > N$ and there exists a prime p such that $rq < p < rq(1 + 1/r)$, Put $m_i = rqn_i$ for $i = 1, \ldots, k$ and $m_{k+1} = n_k p$. Clearly

$$m_1 < m_2 < \cdots < m_k < m_{k+1} \text{ and } \sigma(m_1) > \sigma(m_2) > \cdots > \sigma(m_k).$$

Moreover

$$\sigma(m_{k+1}) = \sigma(n_k)(p+1) < \sigma(n_k)(rq+q+1) < \sigma(n_k)(r+1)(q+1) = \sigma(rqn_k) = \sigma(m_k)$$

and we are done.

(b) Let us reason again by induction. For $k = 2$, put $n_1 = 5$ and $n_2 = 6$. Then $n_1 < n_2$ and $\phi(n_1) = 4 > 2 = \phi(n_2)$. Assume that we have a sequence

$$n_1 < n_2 < \cdots < n_k \text{ such that } \phi(n_1) > \phi(n_2) > \cdots > \phi(n_k).$$

We consider an integer y whose prime factors are strictly greater than all the prime factors of the n_i, $i = 1, 2, \ldots, k$. Since ϕ is multiplicative, $\phi(yn_i) = \phi(y)\phi(n_i)$, hence

$$yn_1 < yn_2 < \cdots < yn_k \text{ and } \phi(yn_1) > \phi(yn_2) > \cdots > \phi(yn_k).$$

We claim that there is an integer y having the properties above, which satisfies the relation $4\phi(yn_1) < yn_1$. To see this, note that if $q_1, \ldots, q_r$ are the prime factors of y, then

$$\frac{\phi(y)}{y} = \prod_{i=1}^{r}\left(1 - \frac{1}{q_i}\right)$$

and this product tends to zero as r goes to infinity. In particular, we can find a y such that $\phi(y)/y < n_1/(4\phi(n_1))$ and the claim follows.

Since, for such a y, $4\phi(yn_1) < yn_1$, there is an integer m such that $\phi(yn_1) < 2^m < 2^{m+1} < yn_1$. Taking $z = 2^{m+1}$, we have $z < yn_1$ and $\phi(z) = 2^m > \phi(yn_1)$.

Also solved by Robert Calcaterra (part (b)). There was one incomplete or incorrect solution.

A locus comprising (almost) two lines **February 2023**

2164. *Proposed by Dixon J. Jones, Coralville, IA.*

In the plane of a triangle ABC, let P_1 and P_2 be fixed points such that P_1P_2 is not perpendicular to AB, BC, or CA. Find the set of points P_3 for which

$$\frac{L_1L_2}{L_2L_3} \cdot \frac{M_1M_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3} = +1 \ ,$$

where L_i, M_{i+1}, N_{i+2} are the feet of the perpendiculars from P_i to BC, CA, AB, respectively (indices taken modulo 3), and the quantities are directed distances.

Solution by the proposer.
The solution set comprises a line ℓ_1^* from which two points have been removed, along with a complete line ℓ_2.

First, let ℓ_1^* be the line through P_1 and P_2, but with those two points excluded. If P_3 lies on ℓ_1^*, then P_1L_1, P_2L_2, and P_3L_3 are parallel, distinct, and of nonzero length. Therefore,

$$\frac{L_1L_2}{L_2L_3} = \frac{P_1P_2}{P_2P_3} \ ,$$

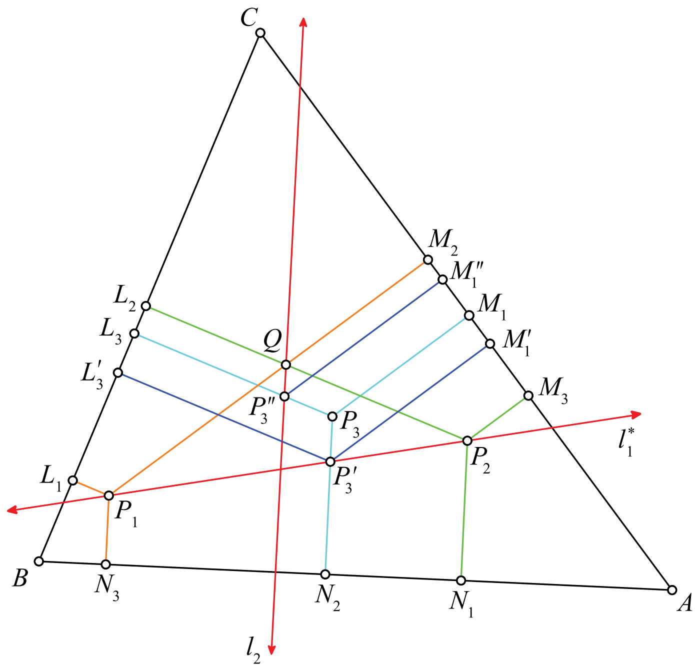
and similarly

$$\frac{M_1M_2}{M_2M_3} = \frac{P_3P_1}{P_1P_2} \ , \quad \text{and} \quad \frac{N_1N_2}{N_2N_3} = \frac{P_2P_3}{P_3P_1} \ .$$

Multiplying these equalities and canceling like terms yields the claim.

Next, let P_1M_2 meet P_2L_2 at Q. Let ℓ_2 be the perpendicular to AB through Q. We assert that if P_3 lies on ℓ_2, the claim follows. We have

$$L_1L_2 = QP_1 \sin\angle P_2QP_1 \ .$$



Similarly

$$L_2L_3 = QP_3 \sin\angle P_3QP_2\ ,$$

$$M_1M_2 = QP_3 \sin\angle P_3QP_1\ , \quad M_2M_3 = QP_2 \sin\angle P_2QP_1\ ,$$

$$N_1N_2 = QP_2 \sin\angle P_3QP_1\ , \quad N_2N_3 = QP_1 \sin\angle P_3QP_2\ .$$

Again, forming quotients and canceling produces the claim.

We now show that if the claim holds, then P_3 necessarily lies on ℓ_1^* or ℓ_2. Let P_3N_2 meet ℓ_1^* at P_3', and let the feet of the perpendiculars from P_3' to BC and CA be L_3' and M_1', respectively. Let P_3L_3 meet ℓ_2 at P_3'', and let M_1'' be the foot of the perpendicular from P_3'' to CA. Set

$$x = \frac{L_1L_2}{L_2L_3} \cdot \frac{M_1M_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3}\ . \tag{1}$$

Since P_3' lies on ℓ_1^*, the claim applies; that is,

$$\frac{L_1L_2}{L_2L_3'} \cdot \frac{M_1'M_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3} = +1\ ,$$

which can be rearranged as

$$\frac{L_2L_3'}{M_1'M_2} = \frac{L_1L_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3}\ . \tag{2}$$

Substituting (2) in (1) yields

$$x\frac{M_1'M_2}{L_2L_3'} = \frac{M_1M_2}{L_2L_3}\ . \tag{3}$$

Since $M_1M_2 = M_1M_1' + M_1'M_2$, (3) can be rewritten as

$$x\frac{L_2L_3}{L_2L_3'} = \frac{M_1M_1'}{M_1'M_2} + 1\ . \tag{4}$$

We have

$$\frac{L_2L_3}{L_2L_3'} = \frac{QP_3''}{QP_3'' + P_3P_3'} = \frac{1}{1 + \dfrac{P_3P_3'}{QP_3''}}\ ; \tag{5}$$

furthermore, it is clear that

$$\frac{P_3P_3'}{QP_3''} = \frac{M_1M_1'}{M_2M_1''}\ . \tag{6}$$

Combining (4)–(6), we obtain

$$\begin{aligned} x &= \left(\frac{M_1M_1'}{M_1'M_2} + 1\right)\left(\frac{M_1M_1'}{M_2M_1''} + 1\right) \\ &= M_1M_1'\left(\frac{M_1M_1' + M_1'M_2 + M_2M_1''}{M_1'M_2 \cdot M_2M_1''}\right) + 1. \end{aligned} \tag{7}$$

From (7) it follows that $x = 1$ if, and only if,

$$M_1M_1' = 0 \text{ or } M_1M_1'\left(\frac{M_1M_1' + M_1'M_2 + M_2M_1''}{M_1'M_2 \cdot M_2M_1''}\right) = 0.$$

If $M_1M_1' = 0$, then P_3 must lie on ℓ_1^*. On the other hand,

$$0 = M_1M_1' + M_1'M_2 + M_2M_1'' = M_1M_1''$$

implies that P_3 must lie on ℓ_2. Thus, the only points P_3 for which the claim holds must lie on ℓ_1^* or ℓ_2.

Also solved by Volkhard Schindler (Germany).

Maximize $\alpha(G)\chi(G)$ **February 2023**

2165. *Proposed by Zion Hefty (student), Grinell College, Grinnell, IA, and Peter Johnson, Auburn University, Auburn, AL.*

Let G be a graph. We will denote the vertex set of G by $V(G)$. The independence number of G, denoted $\alpha(G)$, is the cardinality of the largest subset of $V(G)$ such that no two vertices of that subset are connected by an edge. The chromatic number of G, denoted $\chi(G)$, is the smallest number of colors needed to color each vertex of $V(G)$ so that no two vertices with the same color are connected by an edge.

If we let

$$g(n) = \min\left(\{\alpha(G)\chi(G) \mid |V(G)| = n\}\right),$$

it is well known that $g(n) = n$. This can be realized, for example, if G is the complete graph on n vertices.

Let

$$f(n) = \max\left(\{\alpha(G)\chi(G) \mid |V(G)| = n\}\right).$$

Determine $f(n)$.

Note: This problem and its solution arose during the 2022 Research Experience for Undergraduates in Algebra and Discrete Mathematics at Auburn University. This research was supported by NSF(DMS) grant no. 1950563.

Solution by Edward Schmeichel, San José State University, San José, CA.
We claim that

$$f(n) = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

Note first that for any graph G with n vertices,

$$\alpha(G) + \chi(G) \leq n + 1.$$

To prove this, let I be any maximum independent subset of $V(G)$ (so $|I| = \alpha(G)$). If we color the vertices of I with a first color, the remaining vertices in $V(G) - I$ can be colored using at most

$$|V(G)| - |I| = n - \alpha(G)$$

additional colors. Thus $\chi(G) \leq 1 + (n - \alpha(G))$.

By the AM-GM inequality, we have

$$\alpha(G)\chi(G) \leq \left(\frac{\alpha(G) + \chi(G)}{2}\right)^2 \leq \left(\frac{n+1}{2}\right)^2$$

for any graph G with n vertices. Thus

$$f(n) \leq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

For the reverse inequality, consider the n-vertex graph

$$G_n = K_{\lceil n/2 \rceil} \cup \overline{K}_{\lfloor n/2 \rfloor},$$

where K_m is the complete graph on m vertices and $\overline{K}_m$ is the graph on m vertices having no edges. We have

$$\alpha(G_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1,\ \chi(G_n) = \left\lceil \frac{n}{2} \right\rceil,$$

and therefore

$$f(n) \geq \alpha(G_n)\chi(G_n) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{n}{2} \right\rceil\right) = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

The claim now follows.

Also solved by José Nieto and the proposers.

Bitstrings starting with 1 can only be followed by a string starting with 0, hence

$$z_n = y_{n-1}.$$

This, together with the first equation, implies $y_n = y_{n-2} + y_{n-3}$ and $z_n = z_{n-2} + z_{n-3}$. Since $x_n = y_n + z_n$, we get the recursive formula:

$$x_n = x_{n-2} + x_{n-3}.$$

The initial conditions can be obtained by enumerating the bitstrings with the desired property. For $n = 1 : 0, 1$, for $n = 2 : 00, 01, 10$, and for $n = 3 : 001, 010, 100, 101$. Hence $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$.

Also solved by Ricardo Bittencourt (Brazil), Robert Calcaterra, Kyle Calderhead, Eagle Problem Solvers (Georgia Southern University), John Ferdinands, Eugene A. Herman, Aykhan Ismayilov (Azerbaijan), Walther Janous (Austria), Kee Wai Lau (Hong Kong), Kent E. Morrison, José Heber Nieto (Venezuela), Michelle Nogin, Shing Hin Jimmy Pa (China), Angel Plaza (Spain), Rob Pratt, Edward Schmeichel, Randy Schwartz, Albert Stadler (Switzerland), Paul Stockmeyer, and the proposers. There were two incomplete or incorrect solution.

An infinite series involving skew-harmonic numbers **February 2023**

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Prove that

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n(2n+1)4^n}\binom{2n}{n} = 4G + \frac{\pi^2}{12} - 2\pi \ln 2,$$

where

$$\overline{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$$

is the nth skew-harmonic number and

$$G = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2}$$

is Catalan's constant.

Solution by the proposer.
It is well known that

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n(2n+1)}\binom{2n}{n} = \arcsin x - x.$$

Dividing by x^2 and multiplying by $\ln(1+x)$ on both sides, gives us

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}\ln(1+x)}{4^n(2n+1)}\binom{2n}{n} = \frac{\arcsin x \ln(1+x)}{x^2} - \frac{\ln(1+x)}{x}. \tag{1}$$

Now

$$\int_0^1 \frac{1-x^{2n}}{1+x}\, dx = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} = \overline{H}_{2n}.$$

Integrating by parts yields

$$\int_0^1 \frac{1-x^{2n}}{1+x}\,dx = 2n\int_0^1 x^{2n-1}\ln(1+x)\,dx.$$

Therefore,

$$\int_0^1 x^{2n-1}\ln(1+x)\,dx = \frac{\overline{H}_{2n}}{2n}.$$

Integrating (1) from 0 to 1 and using the above, we obtain

$$\sum_{n=1}^{\infty}\frac{\overline{\mathcal{H}}_{2n}}{2n(2n+1)4^n}\binom{2n}{n} = \int_0^1 \frac{\arcsin x\ln(1+x)}{x^2}dx - \int_0^1 \frac{\ln(1+x)}{x}dx.$$

The second integral can be evaluated as

$$\mathcal{J} = \int_0^1 \frac{\ln(1+x)}{x}dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{k-1}dx = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}.$$

For the first integral $\mathcal{I}$, substituting $x=\sin y$ and applying integration by parts (with $u = y\ln(1+\sin y)$ and $dv = \cos y/\sin^2 y$), we get

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We have

$$A = \int_0^{\frac{\pi}{2}}\frac{\ln(1+\sin y)}{\sin y}dy \overset{y=2\arctan t}{=} \int_0^1 \frac{\ln\left(1+\frac{2t}{1+t^2}\right)}{\frac{2t}{1+t^2}}\frac{2tdt}{1+t^2}$$
$$= \int_0^1 \frac{2\ln(1+t)-\ln(1+t^2)}{t}dt = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$$
$$= \frac{3}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{8}$$

and

$$B = \int_0^{\frac{\pi}{2}}\frac{y\cos y}{1+\sin y}dy \overset{\text{IBP}}{=} \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\ln(1+\sin y)dy$$
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$$= \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\frac{y\tan y}{\sec y+1}dy = \frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{2}}\left(\frac{y}{\sin y}-\frac{y}{\tan y}\right)dy$$
$$= \frac{\pi}{2}\ln(2) - 2G + \int_0^{\frac{\pi}{2}}\frac{y}{\tan y}dy,$$

where we have used the well-known fact that $G = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{y}{\sin y} dy$. Therefore,

$$\mathcal{I} = -\frac{\pi}{2}\ln(2) + A - B + \int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx = -\frac{\pi}{2}\ln(2) + \frac{\pi^2}{8} + 2G - \frac{\pi}{2}\ln(2)$$

$$= 2G - \pi\ln(2) + \frac{\pi^2}{8},$$

and hence

$$\sum_{n=1}^{\infty} \frac{\overline{\mathcal{H}}_{2n}}{n(2n+1)4^n}\binom{2n}{n} = 2(\mathcal{I} - \mathcal{J}) = 4G + \frac{\pi^2}{12} - 2\pi\ln(2).$$

Also solved by Paul Bracken, Hongwei Chen, Walther Janous, and Albert Stadler (Switzerland). There was one incomplete or incorrect solution.

Sequences on which σ (respectively ϕ) is decreasing **February 2023**

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Recall that for a positive integer n, $\sigma(n)$ denotes the sum of the positive divisors of n, and $\phi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n. Show that the following hold.

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(b) There are arbitrarily long sequences $n_1 < n_2 < \cdots < n_k$ such that $\phi(n_1) > \phi(n_2) > \cdots > \phi(n_k)$.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela, (part (a)) and the proposer (part (b))

(a) We shall prove, by induction on k, that for any $k > 1$ there is a sequence

$$n_1 < n_2 < \cdots < n_k \text{ such that } \sigma(n_1) > \sigma(n_2) > \cdots > \sigma(n_k).$$

For $k = 2$, put $n_1 = 4$, $n_2 = 5$. Then $n_1 < n_2$ and $\sigma(n_1) = 7 > 6 = \sigma(n_2)$. Now assume that the result is true for k. Take a prime $r > n_k$. It is well known that, given $\epsilon > 0$, there exists an integer N such that, for any $n > N$, there is a prime p such that $n < p < (1+\epsilon)n$. Take $\epsilon = 1/r$ and a prime $q > r$ such that $q > N/r$ as well. Then $rq > N$ and there exists a prime p such that $rq < p < rq(1 + 1/r)$, Put $m_i = rqn_i$ for $i = 1, \ldots, k$ and $m_{k+1} = n_k p$. Clearly

$$m_1 < m_2 < \cdots < m_k < m_{k+1} \text{ and } \sigma(m_1) > \sigma(m_2) > \cdots > \sigma(m_k).$$

Moreover

$$\sigma(m_{k+1}) = \sigma(n_k)(p+1) < \sigma(n_k)(rq+q+1) < \sigma(n_k)(r+1)(q+1) = \sigma(rqn_k) = \sigma(m_k)$$

and we are done.

(b) Let us reason again by induction. For $k = 2$, put $n_1 = 5$ and $n_2 = 6$. Then $n_1 < n_2$ and $\phi(n_1) = 4 > 2 = \phi(n_2)$. Assume that we have a sequence

$$n_1 < n_2 < \cdots < n_k \text{ such that } \phi(n_1) > \phi(n_2) > \cdots > \phi(n_k).$$

We consider an integer y whose prime factors are strictly greater than all the prime factors of the n_i, $i = 1, 2, \ldots, k$. Since ϕ is multiplicative, $\phi(yn_i) = \phi(y)\phi(n_i)$, hence

$$yn_1 < yn_2 < \cdots < yn_k \text{ and } \phi(yn_1) > \phi(yn_2) > \cdots > \phi(yn_k).$$

We claim that there is an integer y having the properties above, which satisfies the relation $4\phi(yn_1) < yn_1$. To see this, note that if $q_1, \ldots, q_r$ are the prime factors of y, then

$$\frac{\phi(y)}{y} = \prod_{i=1}^{r} \left(1 - \frac{1}{q_i}\right)$$

and this product tends to zero as r goes to infinity. In particular, we can find a y such that $\phi(y)/y < n_1/(4\phi(n_1))$ and the claim follows.

Since, for such a y, $4\phi(yn_1) < yn_1$, there is an integer m such that $\phi(yn_1) < 2^m < 2^{m+1} < yn_1$. Taking $z = 2^{m+1}$, we have $z < yn_1$ and $\phi(z) = 2^m > \phi(yn_1)$.

Also solved by Robert Calcaterra (part (b)). There was one incomplete or incorrect solution.

A locus comprising (almost) two lines **February 2023**

2164. *Proposed by Dixon J. Jones, Coralville, IA.*

In the plane of a triangle ABC, let P_1 and P_2 be fixed points such that P_1P_2 is not perpendicular to AB, BC, or CA. Find the set of points P_3 for which

$$\frac{L_1L_2}{L_2L_3} \cdot \frac{M_1M_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3} = +1\ ,$$

where L_i, M_{i+1}, N_{i+2} are the feet of the perpendiculars from P_i to BC, CA, AB, respectively (indices taken modulo 3), and the quantities are directed distances.

Solution by the proposer.
The solution set comprises a line ℓ_1^* from which two points have been removed, along with a complete line ℓ_2.

First, let ℓ_1^* be the line through P_1 and P_2, but with those two points excluded. If P_3 lies on ℓ_1^*, then P_1L_1, P_2L_2, and P_3L_3 are parallel, distinct, and of nonzero length. Therefore,

$$\frac{L_1L_2}{L_2L_3} = \frac{P_1P_2}{P_2P_3}\ ,$$

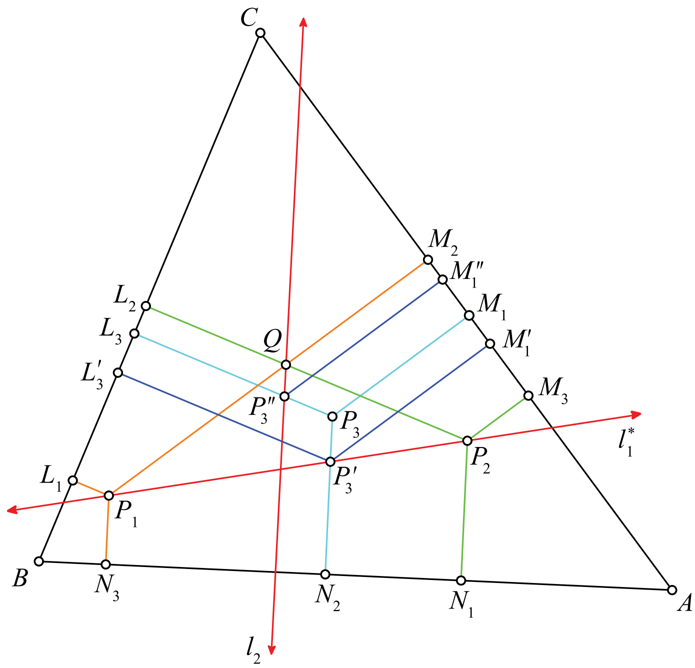
and similarly

$$\frac{M_1M_2}{M_2M_3} = \frac{P_3P_1}{P_1P_2}\ , \quad \text{and} \quad \frac{N_1N_2}{N_2N_3} = \frac{P_2P_3}{P_3P_1}\ .$$

Multiplying these equalities and canceling like terms yields the claim.

Next, let P_1M_2 meet P_2L_2 at Q. Let ℓ_2 be the perpendicular to AB through Q. We assert that if P_3 lies on ℓ_2, the claim follows. We have

$$L_1L_2 = QP_1 \sin\angle P_2QP_1\ .$$



Similarly

$$L_2L_3 = QP_3 \sin\angle P_3QP_2 \ ,$$

$$M_1M_2 = QP_3 \sin\angle P_3QP_1 \ , \quad M_2M_3 = QP_2 \sin\angle P_2QP_1 \ ,$$

$$N_1N_2 = QP_2 \sin\angle P_3QP_1 \ , \quad N_2N_3 = QP_1 \sin\angle P_3QP_2 \ .$$

Again, forming quotients and canceling produces the claim.

We now show that if the claim holds, then P_3 necessarily lies on ℓ_1^* or ℓ_2. Let P_3N_2 meet ℓ_1^* at P_3', and let the feet of the perpendiculars from P_3' to BC and CA be L_3' and M_1', respectively. Let P_3L_3 meet ℓ_2 at P_3'', and let M_1'' be the foot of the perpendicular from P_3'' to CA. Set

$$x = \frac{L_1L_2}{L_2L_3} \cdot \frac{M_1M_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3} \ . \tag{1}$$

Since P_3' lies on ℓ_1^*, the claim applies; that is,

$$\frac{L_1L_2}{L_2L_3'} \cdot \frac{M_1'M_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3} = +1 \ ,$$

which can be rearranged as

$$\frac{L_2L_3'}{M_1'M_2} = \frac{L_1L_2}{M_2M_3} \cdot \frac{N_1N_2}{N_2N_3} \ . \tag{2}$$

Substituting (2) in (1) yields

$$x\frac{M_1'M_2}{L_2L_3'} = \frac{M_1M_2}{L_2L_3} \ . \tag{3}$$

Since $M_1M_2 = M_1M_1' + M_1'M_2$, (3) can be rewritten as

$$x\frac{L_2L_3}{L_2L_3'} = \frac{M_1M_1'}{M_1'M_2} + 1\ . \tag{4}$$

We have

$$\frac{L_2L_3}{L_2L_3'} = \frac{QP_3''}{QP_3'' + P_3P_3'} = \frac{1}{1 + \dfrac{P_3P_3'}{QP_3''}}\ ; \tag{5}$$

furthermore, it is clear that

$$\frac{P_3P_3'}{QP_3''} = \frac{M_1M_1'}{M_2M_1''}\ . \tag{6}$$

Combining (4)–(6), we obtain

$$\begin{aligned} x &= \left(\frac{M_1M_1'}{M_1'M_2} + 1\right)\left(\frac{M_1M_1'}{M_2M_1''} + 1\right) \\ &= M_1M_1'\left(\frac{M_1M_1' + M_1'M_2 + M_2M_1''}{M_1'M_2 \cdot M_2M_1''}\right) + 1. \end{aligned} \tag{7}$$

From (7) it follows that $x = 1$ if, and only if,

$$M_1M_1' = 0 \text{ or } M_1M_1'\left(\frac{M_1M_1' + M_1'M_2 + M_2M_1''}{M_1'M_2 \cdot M_2M_1''}\right) = 0.$$

If $M_1M_1' = 0$, then P_3 must lie on ℓ_1^*. On the other hand,

$$0 = M_1M_1' + M_1'M_2 + M_2M_1'' = M_1M_1''$$

implies that P_3 must lie on ℓ_2. Thus, the only points P_3 for which the claim holds must lie on ℓ_1^* or ℓ_2.

Also solved by Volkhard Schindler (Germany).

Maximize $\alpha(G)\chi(G)$ **February 2023**

2165. *Proposed by Zion Hefty (student), Grinell College, Grinnell, IA, and Peter Johnson, Auburn University, Auburn, AL.*

Let G be a graph. We will denote the vertex set of G by $V(G)$. The independence number of G, denoted $\alpha(G)$, is the cardinality of the largest subset of $V(G)$ such that no two vertices of that subset are connected by an edge. The chromatic number of G, denoted $\chi(G)$, is the smallest number of colors needed to color each vertex of $V(G)$ so that no two vertices with the same color are connected by an edge.

If we let

$$g(n) = \min\left(\{\alpha(G)\chi(G) \mid |V(G)| = n\}\right),$$

it is well known that $g(n) = n$. This can be realized, for example, if G is the complete graph on n vertices.

Let

$$f(n) = \max\left(\{\alpha(G)\chi(G) \mid |V(G)| = n\}\right).$$

Determine $f(n)$.

Note: This problem and its solution arose during the 2022 Research Experience for Undergraduates in Algebra and Discrete Mathematics at Auburn University. This research was supported by NSF(DMS) grant no. 1950563.

Solution by Edward Schmeichel, San José State University, San José, CA.
We claim that

$$f(n) = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

Note first that for any graph G with n vertices,

$$\alpha(G) + \chi(G) \leq n + 1.$$

To prove this, let I be any maximum independent subset of $V(G)$ (so $|I| = \alpha(G)$). If we color the vertices of I with a first color, the remaining vertices in $V(G) - I$ can be colored using at most

$$|V(G)| - |I| = n - \alpha(G)$$

additional colors. Thus $\chi(G) \leq 1 + (n - \alpha(G))$.

By the AM-GM inequality, we have

$$\alpha(G)\chi(G) \leq \left(\frac{\alpha(G) + \chi(G)}{2}\right)^2 \leq \left(\frac{n+1}{2}\right)^2$$

for any graph G with n vertices. Thus

$$f(n) \leq \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

For the reverse inequality, consider the n-vertex graph

$$G_n = K_{\lceil n/2 \rceil} \cup \overline{K}_{\lfloor n/2 \rfloor},$$

where K_m is the complete graph on m vertices and $\overline{K}_m$ is the graph on m vertices having no edges. We have

$$\alpha(G_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1, \chi(G_n) = \left\lceil \frac{n}{2} \right\rceil,$$

and therefore

$$f(n) \geq \alpha(G_n)\chi(G_n) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{n}{2} \right\rceil\right) = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor.$$

The claim now follows.

Also solved by José Nieto and the proposers.

Solutions

The quadrilateral must be a square **December 2022**

2156. *Proposed by Cezar Lupu, Tsinghua University, Beijing, China.*

Let $ABCD$ be a convex quadrilateral in the plane with vertices having rational coordinates. Let P be a point in its interior having rational coordinates such that

$$m\angle PAB = m\angle PBC = m\angle PCD = m\angle PDA = q\pi, \text{ with } q \in \mathbb{Q}.$$

Show that $ABCD$ is a square. Give an example to show that the condition that $q \in \mathbb{Q}$ cannot be dropped.

Solution by Victor Pambuccian, Arizona State University, Phoenix, AZ.
Given that the coordinates of A, B, and P are rational, the distances AB, AP, and BP are all of the form $\sqrt{r}$, where $r \in \mathbb{Q}$. Therefore,

$$\cos q\pi = \cos \angle PAB = \frac{AP^2 + AB^2 - BP^2}{2 \cdot AB \cdot AP} = \sqrt{s},$$

where $q, s \in \mathbb{Q}$.

Although it is a well-known result, we will show that

$$\cos q\pi = \sqrt{s} \text{ with } q, s \in \mathbb{Q} \text{ if and only if } s = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \text{ or } 1.$$

To see this, note that if $q = m/n$ with $m, n \in \mathbb{Z}$, then $\exp(q\pi i)$ and $\exp(-q\pi i)$ are algebraic integers, being roots of $x^{2n} = 1$. Therefore, their sum, $2\cos q\pi = 2\sqrt{s}$, is also an algebraic integer. But then, $4s = \left(2\sqrt{s}\right)^2$ is a rational algebraic integer, hence an integer, and the candidates given are the only possibilities. To finish, we note that $s = 0, 1/4, 1/2, 3/4, 1$ give $q = 1/2, 1/3, 1/4, 1/6, 0$, respectively.

The case $s = 1$ is impossible since P would have to lie on all sides of the polygon. If $s = 0$, then $\angle PAB$ would be a right angle. But this would make all of the angles of $ABCD$ obtuse.

To deal with the cases $s = 1/4$ and $s = 3/4$, we need a well-known result: no triangle in the plane whose coordinates are rational can have an angle with measure $\pi/3$ or $\pi/6$. To see this, we let the vertices of the triangle be

$$A = (x_1, y_1), B = (x_2, y_2), \text{ and } C = (x_3, y_3),$$

where $x_i, y_i \in \mathbb{Q}$. If the side lengths of the triangle are a, b, and c, then $a^2, b^2, c^2 \in \mathbb{Q}$. Let θ denote the angle between the sides of length a and length b. If K is the area of the triangle, then

$$\frac{1}{2}ab\sin\theta = K = \frac{1}{2}\left|\det\begin{pmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{pmatrix}\right| \in \mathbb{Q}.$$

By the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$. Solving the displayed equation for ab and substituting, we have

$$c^2 = a^2 + b^2 - 4K\cot\theta \text{ or } \cot\theta = \frac{a^2 + b^2 - c^2}{4K} \in \mathbb{Q}.$$

But $\cot(\pi/3) = 1/\sqrt{3}$ and $\cot(\pi/6) = \sqrt{3}$, which gives a contradiction.

The result above eliminates the cases $s = 1/4$ and $s = 3/4$, so we are left with $s = 1/2$ and $q = 1/4$. We will invoke the following result of Dmitriev and Dynkin (see: Besenyei, Á. (2015). The Brocard angle and a geometrical gem from Dmitriev and Dynkin. *Amer. Math. Monthly* 122(5): 495–499. https://doi.org/10.4169/amer.math.monthly.122.5.495.)

Let P be an arbitrary point in the interior of a convex n-gon $A_1A_2 \ldots A_n$ and denote $A_{n+1} = A_1$. Then

$$\min_{k=1,\ldots,n} \angle PA_kA_{k+1} \le \frac{\pi}{2} - \frac{\pi}{n}.$$

Equality occurs if and only if $A_1A_2 \ldots A_n$ is a regular n-gon and P is its center.

Applying this result with $n = 4$ and $q = 1/4$ gives the conclusion we seek.

Here is a non-square parallelogram with $q = \arccos(3/\sqrt{10})/\pi \notin \mathbb{Q}$:

$$A = (0, 0), B = (2, 0), C = (6, 2), D = (4, 2), \text{ and } P = (3, 1).$$

Also solved by Robert Calcaterra and the proposer.

An application of the Rayleigh-Beatty theorem **December 2022**

2157. *Proposed by Philippe Fondanaiche, Paris, France.*

Consider two sequences. One is the number of digits in the base 2 representation of 10^k, $k = 1, 2, \ldots$, and the other is the number of digits in the base 5 representation of 10^k, $k = 1, 2, \ldots$. Show that every integer greater than 1 appears in exactly one of the two sequences. Which sequence contains 2023?

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC.

We denote the first sequence by a_k and the second sequence by b_k. Then for each positive integer k,

$$a_k = \lfloor \log_2(10^k) \rfloor + 1 = \lfloor k(\log_2 10) \rfloor + 1$$

and

$$b_k = \lfloor \log_5(10^k) \rfloor + 1 = \lfloor k(\log_5 10) \rfloor + 1.$$

Let $c = \log_2 10$ and $d = \log_5 10$. Then c and d are irrational with

$$\frac{1}{c} + \frac{1}{d} = \log_{10} 2 + \log_{10} 5 = 1.$$

Given an integer $n \ge 2$, we have

$$a_k = n \iff n < kc + 1 < n + 1 \iff \frac{n-1}{c} < k < \frac{n}{c}$$

and

$$b_k = n \iff n < kd + 1 < n + 1 \iff \frac{n-1}{d} < k < \frac{n}{d},$$

since each of $(n-1)/c$, n/c, $(n-1)/d$, and n/d is irrational.

(i) If there are positive integers i and j with $n = a_i = b_j$, then

$$\frac{n-1}{c} < i < \frac{n}{c} \text{ and } \frac{n-1}{d} < j < \frac{n}{d}.$$

Adding the inequalities yields the contradiction $n-1 < i+j < n$.

(ii) If neither sequence contains n, then there are positive integers p and q with

$$p-1 < \frac{n-1}{c} < \frac{n}{c} < p \text{ and } q-1 < \frac{n-1}{d} < \frac{n}{d} < q.$$

Once again, adding the inequalities produces a contradiction, namely

$$p+q-2 < n-1 < n < p+q.$$

(iii) For $n = 2023$, we have

$$\frac{2022}{c} \approx 608.683 \text{ and } \frac{2023}{c} \approx 608.984,$$

while

$$\frac{2022}{d} \approx 1413.317 \text{ and } \frac{2023}{d} \approx 1414.016.$$

Thus $b_{1414} = 2023$.

Editor's Note. Several solvers noted that the result follows from the Rayleigh-Beatty theorem. Let

$$N_\alpha = \{\lfloor n\alpha \rfloor : n \in \mathbb{N}\}.$$

If α and β are real numbers, then

$$N_\alpha \cup N_\beta = \mathbb{N} \text{ and } N_\alpha \cap N_\beta = \emptyset$$

if and only if α and β are irrational and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Jacob Boswell, Robert Calcaterra, Eagle Problem Solvers, Dmitry Fleischman, Walther Janous (Austria), José Heber Nieto (Venezuela), Michelle Nogin, Mariam Obeidallah, Shing Hin Jimmy Pa (China), Celia Schacht, Edward Schmeichel, Randy K. Schwartz, Paul K. Stockmeyer, Ertugrul Tarhan, Enrique Treviño, Edward White & Roberta White, and the proposer.

Adjacent numbers sum to a perfect square **December 2022**

2158. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

(a) Arrange the integers from 1 to 15 (inclusive) in a row so that the sum of any two adjacent numbers is a perfect square.

(b) Find the smallest positive integer n such that the integers from 1 to n can be arranged in a circle so that the sum of any two adjacent numbers is a perfect square. Justify your answer.

Solution by Jacob Boswell and Chip Curtis, Missouri Southern State University, Joplin, MO.

Let Γ_n denote the graph whose vertices are the integers from 1 to n with an edge between two vertices if their sum is a perfect square. Part (a) asks for a Hamiltonian path in Γ_{15} and part (b) askes for the smallest n such that Γ_n has a Hamiltonian cycle.

(a) Since 8 and 9 are the only vertices of degree 1, a Hamiltonian path must start at one of these and end at the other. It is not difficult to show that the following solution is unique (up to reversal).

$$8 - 1 - 15 - 10 - 6 - 3 - 13 - 12 - 4 - 5 - 11 - 14 - 2 - 7 - 9.$$

(b) We claim that the smallest possible integer with the given property is $n = 32$. To prove this, we first exhibit a solution for $n = 32$.

$$\begin{aligned} &22 - 27 - 9 - 16 - 20 - 29 - 7 - 18 - 31 - 5 - 11 - 25 - 24 - 12 \\ &- 13 - 3 - 6 - 30 - 19 - 17 - 32 - 4 - 21 - 28 - 8 - 1 - 15 \\ &- 10 - 26 - 23 - 2 - 14 - 22 \end{aligned}$$

We next show that no n with $3 \le n \le 30$ works. (We exclude $n = 1$ and $n = 2$ from consideration.) A necessary condition for Γ_n to have a Hamiltonian cycle is that every vertex must have degree at least two.

- For $3 \le n \le 4$, the only neighbor of 1 is 3.
- For $5 \le n \le 8$, the only neighbor of 4 is 5.
- For $9 \le n \le 15$, the only neighbor of 9 is 7.
- For $16 \le n \le 19$, the only neighbor of 16 is 9.
- For $20 \le n \le 30$, the only neighbor of 18 is 7.

Finally, we show that $n = 31$ does not work. Two edges emanating from a vertex of degree two must be part of any Hamiltonian cycle. This gives the following fragments.

$$22 - 27 - 9 - 16 - 20 - 29 - 7 - 18 - 31 - 5,$$

$$21 - 28 - 8 - 17 - 19 - 30 - 6,$$

$$10 - 26 - 23, \text{ and } 11 - 25 - 24$$

Consider a vertex of degree three. Suppose one of its neighbors already has two edges of the Hamiltonian cycle we are building emanating from it. Then the Hamiltonian cycle must contain the other two edges emanating from the given vertex. This gives the fragments

$$14 - 2 - 23,\ 3 - 1 - 24,\ \text{(and subsequently) } 10 - 15 - 24.$$

Finally, the only possible remaining edge emanating

$$\text{from 6 is } 6 - 3,\ \text{from 22 is } 22 - 14,\ \text{and from 11 is } 11 - 5.$$

However, this gives a cycle that does not contain all of the vertices, yielding a contradiction.

We note that this type of argument shows that there are two solutions when $n = 32$ (up to reversal).

Also solved by Carl Axness (Spain), Brett Chiodo, Keon Cruz, Eagle Problem Solvers, Dmitry Fleischman (part (a)), Evan Grahn, Shannon Heinig, Alyssa Janowski, Kelly McLenithan & Stephen Mortenson, José Heber Nieto (Venezuela), Michelle Nogin, Pittsburg State University Problem Solving Group, Rob Pratt, Zaur Rajabov (Azerbaijan), Mary Reil (part (a)), Edward Schmeichel, Randy Schwartz, Paul Stockmeyer, and the proposers.

The winner is the one whose roll occurs first **December 2022**

2159. *Proposed by George Stoica, Saint John, NB, Canada.*

Two players, A and B, alternately throw a pair of dice with A going first. Let $a, b \in \{2, 3, \dots, 12\}$ be fixed. Player A wins by having a roll worth a points before player B has a roll worth b points. Otherwise, player B wins.

What is the probability that player A wins?

Solution by Michelle Nogin (student), Clovis North High School, Fresno, CA.
Let $f(n)$ be the probability of having a roll worth n points. Observe that there are 36 possible outcomes when rolling two dice. Since there is only one way to get the value 2 $(1+1)$ and only one way to get the value 12 $(6+6)$, $f(2) = f(12) = 1/36$. Similarly, since there are two ways to get the value 3 $(1+2$ and $2+1)$ and two ways to get the value 11 $(5+6$ and $6+5)$, $f(3) = f(11) = 2/36$, and so forth until there are six ways to get the value 7, so $f(7) = 6/36$. From this, we can write an explicit formula for $f(n)$:

$$f(n) = \frac{6 - |7-n|}{36}.$$

The probability of player A winning on the first move is the probability of player A having a roll worth a points on their first move, that is, $f(a)$. The probability of player A winning on the second move is the probability of player A and player B not having rolls worth a and b points, respectively, on their first moves times the probability of player A having a roll worth a points on their second move, that is,

$$(1 - f(a))(1 - f(b))f(a).$$

Similarly, the probability of player A winning on the third move is

$$((1 - f(a))(1 - f(b)))^2 f(a),$$

and so forth. Thus, the probability of player A winning is the sum of the geometric series

$$f(a) + (1 - f(a))(1 - f(b))f(a) + ((1 - f(a))(1 - f(b)))^2 f(a) + \cdots$$

which is equal to

$$\frac{f(a)}{1 - (1 - f(a))(1 - f(b))}.$$

Also solved by Robert A. Agnew, Jacob Boswell & Chip Curtis, Brian Bradie, Cal Poly Pomona Problem Solving Group, Robert Calcaterra, Eagle Problem Solvers, Eugene A. Herman, Stephen Herschkorn, Walther Janous (Austria), Kenneth Levasseur, Northwestern University Math Problem Solving Group, Pittsburgh State University Problem Solving Group, Rob Pratt, Gary Radmus, Celia Schacht, Edward Schmeichel, Randy K. Schwartz, and the proposer. There were two incomplete or incorrect solutions.

Find the area of the checkerboard pattern **December 2022**

2160. *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA.*

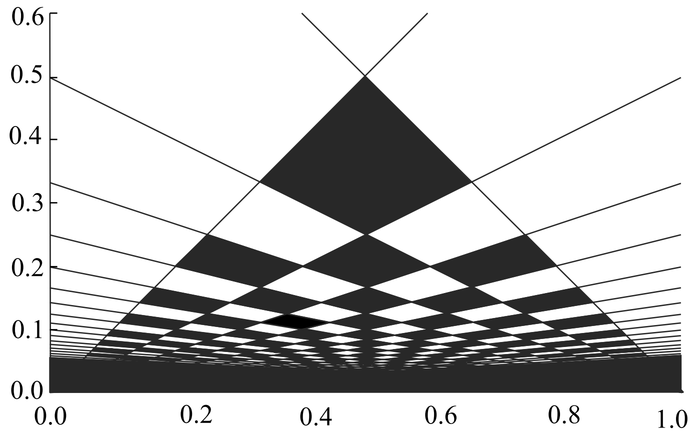
Consider the lines

$$y = x/1,\ y = x/2,\ y = x/3,\ y = x/4, \ldots$$

and the lines

$$y = (1-x)/1,\ y = (1-x)/2,\ y = (1-x)/3,\ y = (1-x)/4, \ldots,$$

which intersect to form an infinite number of quadrilaterals. Starting with the lozenge at the top, shade every other quadrilateral, as shown in the figure.



Find the total area of all the shaded quadrilaterals.

Solution by Clayton Coe (student), Cal Poly Pomona, Pomona, CA.
The total area is $2\ln 2 - \frac{5}{4}$.

Let $y = x/n$ be the equation of line L_n, and $y = (1-x)/k$ be the equation of line M_k, where $n, k \geq 1$. Observe that

$$L_n \cap M_k = \left(\frac{n}{n+k}, \frac{1}{n+k}\right).$$

The set of vertices of any of these tiles is

$$\{L_n \cap M_k, L_n \cap M_{k+1}, L_{n+1} \cap M_{k+1}, L_{n+1} \cap M_k\}.$$

Note that $L_{n+1} \cap M_k$ and $L_n \cap M_{k+1}$ have the same y-coordinate. Therefore, we can calculate the area of a quadrilateral to be the sum of the area of two triangles with horizontal bases. Doing so, we find the area to be $hw/2$, where h is the difference between the y-coordinates of $L_n \cap M_k$ and $L_{n+1} \cap M_{k+1}$, and w is the difference between the x-coordinates of $L_n \cap M_{k+1}$ and $L_{n+1} \cap M_k$.

Let $A(n,k)$ denote the area of a single tile, with uppermost vertex $L_n \cap M_k$. We therefore have

$$A(n,k) = \frac{1}{2}\left(\frac{1}{n+k} - \frac{1}{n+k+2}\right)\left(\frac{n+1}{n+k+1} - \frac{n}{n+k+1}\right)$$

$$= \frac{1}{2}\left(\frac{2}{(n+k)(n+k+2)}\right)\left(\frac{1}{n+k+1}\right)$$

$$= \frac{1}{(n+k)(n+k+1)(n+k+2)}.$$

Note that each black quadrilateral has uppermost vertex $L_n \cap M_k$ with $n+k$ even. Letting $n+k=2m$,

$$A(n,k) = \frac{1}{2m(2m+1)(2m+2)}.$$

In each horizontal row of quadrilaterals, $n+k$ is constant, and there are $n+k-1 = 2m-1$ quadrilaterals in that row. Consequently, the sum of the areas of all black quadrilaterals is

$$S = \sum_{m=1}^{\infty} \frac{2m-1}{2m(2m+1)(2m+2)}$$

Observe that the above sum is absolutely convergent because it is comparable to a p-series, with $p=2$. We perform a partial fraction decomposition, yielding

$$S = \sum_{m=1}^{\infty}\left(\frac{-1/2}{2m} + \frac{2}{2m+1} + \frac{-3/2}{2m+2}\right)$$

Because of absolute convergence, we may shift the index of the first term of the summand, and obtain

$$\begin{aligned}
S &= -\frac{1}{4} + \sum_{m=1}^{\infty}\left(\frac{-1/2}{2(m+1)} + \frac{2}{2m+1} + \frac{-3/2}{2m+2}\right)\\
&= -\frac{1}{4} + \sum_{m=1}^{\infty}\left(\frac{2}{2m+1} - \frac{2}{2m+2}\right)\\
&= -\frac{1}{4} + 2\sum_{m=0}^{\infty}\left(\frac{1}{2m+1} - \frac{1}{2m+2}\right) - 2\left(1 - \frac{1}{2}\right)\\
&= -\frac{5}{4} + 2\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\\
&= -\frac{5}{4} + 2\ln 2
\end{aligned}$$

Also solved by Farrukh Rakhimjanovich Ataev (Uzbekistan), Chip Curtis, Eagle Problem Solvers, Dmitry Fleischman, Eugene A. Herman, Walther Janous (Austria), Do Hyun Lee (South Korea), Chrysostom G. Petalas (Greece), William Reil, Volkhard Schindler (Germany), Edward Schmeichel, Paul K. Stockmeyer, Maria van der Walt, and the proposer.

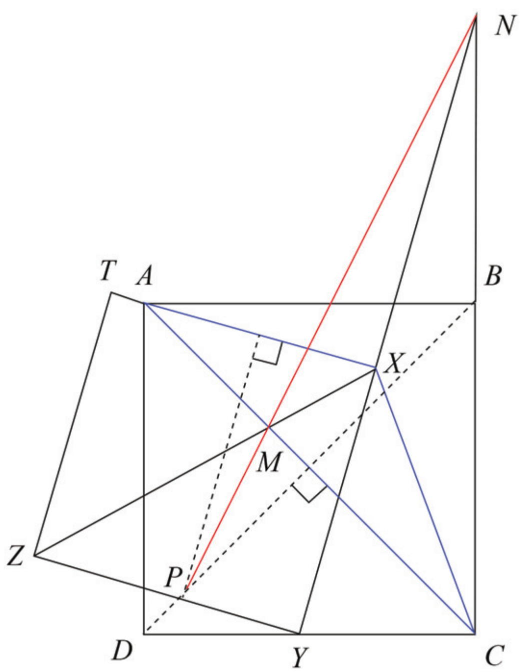
Solutions

Two directly similar squares **October 2022**

2151. *Proposed by Tran Quang Hung, Hanoi, Vietnam.*

Let $ABCD$ and $XYZT$ be two directly similar squares such that A and Y lie on the lines XT and CD, respectively. Let M be the intersection of lines XZ and AC, and let N be the intersection of lines XY and BC. Prove that the circumcenter of $\triangle XAC$ lies on the line MN.

Solution by Katherine Nogin, Clovis North High School (student), Fresno, CA.
Position the squares in the coordinate plane, so that D is at the origin and $ABCD$ is a unit square with $A(0, 1)$, $B(1, 1)$, and $C(1, 0)$. Let the coordinates of X be (r, s) and let P be the circumcenter of $\triangle XAC$.



We will make calculations in terms of r and s assuming that none of the resulting denominators are zero. We will address the question of what happens if some of the denominators are zero later.

It is straightforward to determine that the equation of line XY is

$$y = \frac{r}{1-s}(x-r) + s.$$

Since N lies on XY, we have

$$N = \left(1, \frac{r}{1-s}(1-r) + s\right) = \left(1, \frac{r+s-r^2-s^2}{1-s}\right).$$

Since the y-coordinate of Y is 0, we can solve the equation

$$0 = \frac{r}{1-s}(x-r)+s$$

to obtain

$$Y = \left(\frac{r^2+s^2-s}{r}, 0\right),$$

therefore

$$\overrightarrow{XY} = \left(\frac{s^2-s}{r}, -s\right).$$

Since $\overrightarrow{XT}$ is perpendicular to $\overrightarrow{XY}$ and has the same length,

$$\overrightarrow{XT} = \left(-s, \frac{s-s^2}{r}\right).$$

As $\overrightarrow{TZ} = \overrightarrow{XY}$, we have

$$\overrightarrow{XZ} = \overrightarrow{XT} + \overrightarrow{TZ} = \left(\frac{s^2-s}{r} - s, \frac{s-s^2}{r} - s\right).$$

The slope of line XZ is thus

$$m_{XZ} = \frac{\frac{s-s^2}{r} - s}{\frac{s^2-s}{r} - s} = \frac{1-s-r}{s-1-r},$$

and its equation is therefore

$$y = \frac{1-s-r}{s-1-r}(x-r)+s.$$

The equation of line AC is $y = -x+1$. Since M is the intersection of XZ and AC, we can solve the corresponding system of linear equations to obtain

$$M = \left(\frac{r^2+s^2-2s+1}{2r}, \frac{2r+2s-r^2-s^2-1}{2r}\right).$$

In order to find the coordinates of P, we will find the intersection of the perpendicular bisectors of AC and AX. The equation of BD, which is the perpendicular bisector of AC, is $y = x$. The midpoint of AX is

$$\left(\frac{r}{2}, \frac{1+s}{2}\right)$$

and the slope of the perpendicular bisector of AX, which is also the slope of XY, is $r/(1-s)$. Thus, the equation of the perpendicular bisector of AX is

$$y = \frac{r}{1-s}\left(x - \frac{r}{2}\right) + \frac{1+s}{2}.$$

Setting these two expressions for y equal to each other and solving, we find

$$P = \left(\frac{r^2+s^2-1}{2(r+s-1)}, \frac{r^2+s^2-1}{2(r+s-1)}\right).$$

In order to prove that P lies on line MN, we will show that the slopes of line NP and line MN are equal. We have

$$\begin{aligned} m_{MN} &= \frac{\frac{2r+2s-r^2-s^2-1}{2r} - \frac{r+s-r^2-s^2}{1-s}}{\frac{r^2+s^2-2s+1}{2r} - 1} \\ &= \frac{(2r+2s-r^2-s^2-1)(1-s)-(r+s-r^2-s^2)2r}{(r^2+s^2-2r-2s+1)(1-s)}. \end{aligned}$$

Then

$$\begin{aligned} m_{NP} &= \frac{\frac{r^2+s^2-1}{2(r+s-1)} - \frac{r+s-r^2-s^2}{1-s}}{\frac{r^2+s^2-1}{2(r+s-1)} - 1} \\ &= \frac{(r^2+s^2-1)(1-s)-(r+s-r^2-s^2)2(r+s-1)}{(r^2+s^2-2r-2s+1)(1-s)} \\ &= \frac{(r^2+s^2-1)(1-s)-(r+s-r^2-s^2)2r+(r+s-r^2-s^2)2(1-s)}{(r^2+s^2-2r-2s+1)(1-s)} \\ &= \frac{(2r+2s-r^2-s^2-1)(1-s)-(r+s-r^2-s^2)2r}{(r^2+s^2-2r-2s+1)(1-s)} \\ &= m_{MN}. \end{aligned}$$

It follows that the three points N, P, and M lie on the same line, thus the circumcenter of $\triangle XAC$ lies on the line MN.

We now consider the cases where the denominators in the above expressions are zero. Note that $r \neq 0$, otherwise $XZ \parallel AC$, so point M would not be defined. Also, $1 - s \neq 0$, otherwise $XY \parallel BC$, so point N would not be defined. Next, $r + s - 1 \neq 0$, otherwise point X lies on AC, so $\triangle XAC$ is degenerate and does not have a circumcenter.

The case $s - 1 - r = 0$ is possible, however. In that case,

$$N = (1, 2r)\,, \; M = (r, 1-r)\,, \text{ and } P = \left(\frac{r+1}{2}, \frac{r+1}{2}\right),$$

which are collinear.

Finally, $r^2 + s^2 - 2r - 2s + 1 = 0$ is possible. This equation is equivalent to $(r - 1)^2 + (s - 1)^2 = 1$, so point X lies on a circle of radius 1 centered at B. It follows that $P = B$, $M = C$, and N lies on BC. Thus, in all possible cases P lies on MN.

Also solved by Robert Calcaterra, Ivko Dimitrič, Walther Janous (Austria), Michael Vowe (Switzerland), and the proposer. There was one incomplete or incorrect solution.

Evaluate the double integral **October 2022**

2152. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.*

Evaluate

$$\int_0^1 \int_0^1 \frac{dy\,dx}{\sqrt{1-x^2}\sqrt{1-y^2}\,(1+xy)}.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

Denote the proposed integral by I. The change of variables $x = \sin\theta$, $y = \sin\varphi$ shows that

$$I = \int_0^{\pi/2}\int_0^{\pi/2} \frac{d\theta\, d\varphi}{1+\sin\varphi\sin\theta}.$$

For $\varphi \in (-\pi/2, \pi/2)$, we define $F(\varphi)$ by

$$F(\varphi) = \int_0^{\pi/2} \frac{d\theta}{1+\sin\varphi\sin\theta}$$

Clearly, for $0 < \varphi < \pi/2$ we have

$$\begin{aligned}
F(\varphi) + F(-\varphi) &= 2\int_0^{\pi/2} \frac{d\theta}{1-\sin^2\varphi\sin^2\theta} = 2\int_0^{\pi/2} \frac{1}{1+\cot^2\theta-\sin^2\varphi}\,\frac{d\theta}{\sin^2\theta}\\
&= 2\int_0^{\infty} \frac{du}{\cos^2\varphi+u^2} = \frac{\pi}{\cos\varphi}\\
F(\varphi) - F(-\varphi) &= \int_0^{\pi/2} \frac{-2\sin\varphi\sin\theta}{1-\sin^2\varphi\sin^2\theta}d\theta = \int_0^{\pi/2} \frac{-2\sin\varphi\sin\theta}{\cos^2\varphi+\sin^2\varphi\cos^2\theta}d\theta\\
&= \frac{2}{\cos\varphi}\int_{\tan\varphi}^{0} \frac{du}{1+u^2} = -\frac{2\varphi}{\cos\varphi}\ (u = \tan\varphi\cos\theta).
\end{aligned}$$

Thus

$$F(\varphi) = \frac{\pi-2\varphi}{2\cos\varphi}, \quad 0 < \varphi < \frac{\pi}{2}.$$

Integrating and using the change of variables $\varphi = \frac{\pi}{2} - 2x$, we get

$$\begin{aligned}
I &= \int_0^{\pi/2} F(\varphi)\, d\varphi\\
&= \int_0^{\pi/4} \frac{4x}{\sin 2x}\, dx = 2\int_0^{\pi/4} \frac{x}{\sin x\cos x}\, dx\\
&= \Big[2x\ln(\tan x)\Big]_0^{\pi/4} - 2\int_0^{\pi/4} \ln(\tan x)\, dx = -2\int_0^1 \frac{\ln(u)}{1+u^2}\, du\\
&= -\sum_{n=0}^{\infty}(-1)^n\int_0^1 u^{2n}\ln(u)\, du = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 2G.
\end{aligned}$$

where G is the Catalan number.

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Carl Axness, Michel Bataille (France), Khristo N. Boyadzhiev, Brian Bradie, James Brewer, Charles Burnette, Robert Calcaterra, Hongwei Chen, Bruce Davis, Cal Poly Pomona Problem Solving Group, Eagle Problem Solvers, Fejéntalàltuka Szeged Problem Solving Group (Hungary), Jan Grzesik, Walther Janous (Austria), Stephen Kaczkowski, Kee-Wai Lau (Hong Kong, China), Muzahim Mamedov (Azerbaijan), Moubinool Omarjee (France), Shing Hin Jimmy Pa (China), Paolo Perfetti, Volkhard Schindler (Germany), Seán Stewart (Saudi Arabia), Michael Vowe (Switzerland), Haohao Wang, and the proposer. There were two incomplete or incorrect solutions.

Series involving Fibonacci and Lucas numbers **October 2022**

2153. *Proposed by Rex H. Wu, New York, NY.*

Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Evaluate the following for $k \geq 0$.

(a) $\displaystyle\sum_{n=0}^{\infty} \arctan \frac{F_{2k}}{F_{2n+1}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \arctan \frac{L_{2k+1}}{L_{2n}}$

Solution by Russell Gordon, Whitman College, Walla Walla, WA.

We first establish some common notation and make note of a simple trigonometric identity. Let $\alpha = \phi$ and $\beta = -1/\phi$, where ϕ is the golden ratio. By the Binet formulas for the Lucas and Fibonacci numbers, we have

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ and } L_n = \alpha^n + \beta^n$$

for all nonnegative integers n. In addition, the trigonometric identity

$$\arctan u - \arctan v = \arctan\left(\frac{u-v}{1+uv}\right),$$

is valid for all positive real numbers u and v.

When $k = 0$, the Fibonacci series clearly sums to 0. For the other series involving the Fibonacci numbers, we begin by noting that

$$\frac{\alpha^{2n+2k+1} - \alpha^{2n-2k+1}}{1+\alpha^{4n+2}} = \frac{-\alpha^{2k} + \alpha^{-2k}}{\beta^{2n+1} - \alpha^{2n+1}} = \frac{\alpha^{2k} - \beta^{2k}}{\alpha^{2n+1} - \beta^{2n+1}} = \frac{F_{2k}}{F_{2n+1}}$$

for all positive integers k and all nonnegative integers n. It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \arctan\left(\frac{F_{2k}}{F_{2n+1}}\right) &= \lim_{N\to\infty} \sum_{n=0}^{N} \Big(\arctan\big(\alpha^{2n+2k+1}\big) - \arctan\big(\alpha^{2n-2k+1}\big)\Big) \\
&= \lim_{N\to\infty} \left(\sum_{n=N-2k+1}^{N} \arctan\big(\alpha^{2n+2k+1}\big) - \sum_{n=0}^{2k-1} \arctan\big(\alpha^{2n-2k+1}\big) \right) \\
&= \lim_{N\to\infty} \left(\sum_{\ell=1}^{2k} \arctan\big(\alpha^{2N-2k+2\ell+1}\big) \right) - \sum_{n=0}^{2k-1} \arctan\big(\alpha^{2n-2k+1}\big) \\
&= 2k \cdot \frac{\pi}{2} - \sum_{n=0}^{k-1} \arctan\big(\alpha^{-2(k-n-1)-1}\big) - \sum_{n=k}^{2k-1} \arctan\big(\alpha^{2(n-k)+1}\big) \\
&= 2k \cdot \frac{\pi}{2} - \sum_{j=0}^{k-1} \arctan\big(\alpha^{-(2j+1)}\big) - \sum_{j=0}^{k-1} \arctan\big(\alpha^{2j+1}\big) \\
&= 2k \cdot \frac{\pi}{2} - \sum_{j=0}^{k-1} \Big(\arctan\big(\alpha^{2j+1}\big) + \arctan\big(\alpha^{-(2j+1)}\big)\Big)
\end{aligned}$$

$$= 2k \cdot \frac{\pi}{2} - k \cdot \frac{\pi}{2}$$
$$= \frac{k\pi}{2}.$$

This gives the sums of the series involving the Fibonacci numbers.

For the series involving the Lucas numbers, we first observe that

$$\frac{\alpha^{2n+2k+1} - \alpha^{2n-2k-1}}{1 + \alpha^{4n}} = \frac{\alpha^{2k+1} - \alpha^{-2k-1}}{\beta^{2n} + \alpha^{2n}} = \frac{\alpha^{2k+1} + \beta^{2k+1}}{\alpha^{2n} + \beta^{2n}} = \frac{L_{2k+1}}{L_{2n}}$$

for all positive integers k and all nonnegative integers n. It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \arctan\Big(\frac{L_{2k+1}}{L_{2n}}\Big) &= \lim_{N\to\infty} \sum_{n=0}^{N} \Big(\arctan\big(\alpha^{2n+2k+1}\big) - \arctan\big(\alpha^{2n-2k-1}\big)\Big) \\
&= \lim_{N\to\infty} \Bigg(\sum_{n=N-2k}^{N} \arctan\big(\alpha^{2n+2k+1}\big) - \sum_{n=0}^{2k} \arctan\big(\alpha^{2n-2k-1}\big)\Bigg) \\
&= \lim_{N\to\infty} \Bigg(\sum_{\ell=0}^{2k} \arctan\big(\alpha^{2N-2k+2\ell+1}\big)\Bigg) - \sum_{n=0}^{2k} \arctan\big(\alpha^{2n-2k-1}\big) \\
&= (2k+1)\cdot\frac{\pi}{2} - \sum_{n=0}^{k} \arctan\big(\alpha^{-2(k-n+1)+1}\big) - \sum_{n=k+1}^{2k} \arctan\big(\alpha^{2(n-k)-1}\big) \\
&= (2k+1)\cdot\frac{\pi}{2} - \arctan\big(\alpha^{-(2k+1)}\big) - \sum_{j=1}^{k} \arctan\big(\alpha^{-(2j-1)}\big) - \sum_{j=1}^{k} \arctan\big(\alpha^{2j-1}\big) \\
&= (2k+1)\cdot\frac{\pi}{2} - \arctan\big(\alpha^{-(2k+1)}\big) - \sum_{j=1}^{k} \Big(\arctan\big(\alpha^{2j-1}\big) + \arctan\big(\alpha^{-(2j-1)}\big)\Big) \\
&= 2k\cdot\frac{\pi}{2} + \Big(\frac{\pi}{2} - \arctan\big(\alpha^{-(2k+1)}\big)\Big) - k\cdot\frac{\pi}{2} \\
&= \frac{k\pi}{2} + \arctan\big(\alpha^{2k+1}\big) \\
&= \frac{k\pi}{2} + \arctan\big(\phi^{2k+1}\big).
\end{aligned}$$

We have thus found the sums of all of the series involving the Lucas numbers.

Also solved by Michel Bataille (France), Brian Bradie, Hongwei Chen, Michael Goldenberg & Mark Kaplan, Walther Janous (Austria), Won Kyun Jeong (South Korea), Omran Kouba (Syria), Angel Plaza (Spain), Albert Stadler (Switzerland), and the proposer.

Ordered partitions with all parts odd **October 2022**

2154. *Proposed by the Columbus State University Problem Solving Group, Columbus, GA.*

Let $f(n)$ denote the number of ordered partitions of a positive integer n such that all of the parts are odd. For example, $f(5) = 5$, since 5 can be written as $5, 3+1+1, 1+3+1, 3+1+1$, and $1+1+1+1+1$. Determine $f(n)$.

Solution by Dominic Kozlowski and Anayelli Vigo (students), Seton Hall University, South Orange, NJ.

We claim that $f(n) = F_n$, where F_n is the nth Fibonacci number. We will write individual partitions as ordered k-tuples and denote the set of odd partitions of n by A_n. For example,

$$A_5 = \{(5), (3, 1, 1), (1, 3, 1), (1, 1, 3), (1, 1, 1, 1, 1)\}.$$

It will suffice to show that

$$|A_1| = 1, |A_2| = 1, \text{ and } |A_n| = |A_{n-1}| + |A_{n-2}| \text{ for all } n > 2.$$

That $|A_1| = |A_2| = 1$ can be calculated immediately. For $n > 2$, define the maps $p_1 : A_{n-1} \to A_n$ and $p_2 : A_{n-2} \to A_n$ as

$$p_1(a_1, a_2, \dots, a_k) = (1, a_1, a_2, \dots, a_k),$$
$$p_2(a_1, a_2, \dots, a_k) = (a_1 + 2, a_2, \dots, a_k).$$

Now let $p : A_{n-1} \cup A_{n-2} \to A_n$ be the map consisting of the application of p_1 to the elements of A_{n-1} and the application of p_2 to the elements of A_{n-2}.

To show p is onto, let $b \in A_n$, with $b = (b_1, b_2, \dots, b_k)$. We have two cases.

Case 1: $b_1 = 1$. In this case $(b_2, b_3, \dots, b_k)$ is an odd partition of $n - 1$, and so $(b_2, b_3, \dots, b_k) \in A_{n-1}$. But then

$$p(b_2, b_3, \dots, b_k) = p_1(b_2, b_3, \dots, b_k) = b.$$

Case 2: $b_1 > 1$. In this case since b_1 is odd we must have $b_1 \geq 3$ and odd. But then $b_1 - 2 \geq 1$ and is odd, and so $(b_1 - 2, b_2, \dots, b_k)$ is an odd partition of $n - 2$ and so $(b_1 - 2, b_2, \dots, b_k) \in A_{n-2}$. But then

$$p(b_1 - 2, b_2, \dots, b_k) = p_2(b_1 - 2, b_2, \dots, b_k) = b.$$

To show that p is one-to-one, we note that both of the individual maps p_1 and p_2 are clearly one-to-one. We need only check $a \neq a'$ implies $p(a) \neq p(a')$ for all $a \in A_{n-1}$, $a' \in A_{n-2}$. But $p(a)$ starts with a 1, and $p(a')$ starts with a number greater than 1, so $p(a) \neq p(a')$.

Since p is a bijection between $A_{n-1} \cup A_{n-2}$ and A_n, $|A_{n-1}| + |A_{n-2}| = |A_n|$ for all $n > 2$ and the claim follows.

Also solved by Alrich Abel & Vitaliy Kushnirevych (Germany), Ashland University Problem Solving Group, McCrea Black & the Texas State University Problem Solvers Group, Saham Bhadra (India), Ricardo Bittencourt (Brazil), Charles Burnette, Robert Calcaterra, Rohan Dalal, Eagle Problem Solvers, Fejéntaláltuka Szeged Problem Solving Group (Hungary), Haydn Gwyn, Brian Hopkins, Walther Janous (Austria), Kenneth Klinger, Kee-Wai Lau (Hong Kong, China), S. C. Locke, Samuel Lucas Mazariegos, Katherine Nogin, Northwestern University Math Problem Solving Group, Rob Pratt, Edward Schmeichel, Albert Stadler (Switzerland), Paul K. Stockmeyer, and the proposers. There was one incomplete or incorrect solution.

A problem from ring theory **October 2022**

2155. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let R be a ring with identity and U a subset of the units of R with $|U| = p$, where p is an odd prime. Suppose that for all $a \in R$, there is a $u \in U$ and a $k \in \mathbb{Z}^+$ such that $ua^k = a^{k+1}$. Show that

(a) For all $a \in R$, there is a $u \in U$ such that $ua = a^2$.

(b) The ring R is commutative.

Solution by Robert Calcaterra, University of Wisconsin - Platteville, Platteville, WI.
The condition that p be prime is unnecessary; it suffices to assume that p is odd.

Suppose v is an invertible element of R. Then $uv^k = v^{k+1}$ for some $u \in U$ and $k \in \mathbb{Z}^+$ and so $v = u \in U$. Therefore, U must be the group of all the units of R. The order of -1 in the group U is 1 as it is a divisor of both 2 and p. Hence, $-1 = 1$ and R has characteristic 2.

Next we claim $a^2 = 0$ in the ring R if and only if $a = 0$ and that every finite subring of R containing 1 is the direct sum of fields. To prove this, suppose $a \in R$ and $a^2 = 0$. Then $(a+1)^2 = a^2 + 1 = 1$. Thus, $a + 1$ is its own inverse in the group U and so $a + 1 = 1$, and $a = 0$ and the first part of the claim is validated. Note that this implies every finite subring of R containing 1 is semiprime. Since such a ring must also be artinian, The Wedderburn-Artin theorem implies it is isomorphic to the direct sum

$$M_1 \oplus M_2 \oplus \cdots \oplus M_k,$$

where each M_i is a full matrix ring over a division ring. Since these division rings are necessarily finite, another theorem of Wedderburn implies they are fields. Finally, note that the square of any $n \times n$ matrix with $n > 1$ that has a 1 as the upper right entry and zeros elsewhere is the zero matrix. This is impossible by the first part of the claim, so each M_i must consist of 1×1 matrices and therefore be isomorphic to its component field. This verifies the second part of the claim.

Let $a \in R$ and let S_a be the set of finite sums in R in which each term is either 1 or a raised to a positive integer. Then S_a is a subring of R. Since $ua^k = a^{k+1}$ for some $u \in U$ and $k \in \mathbb{Z}^+$, it follows that

$$u^2a^k = ua^{k+1} = a^{k+2}, u^3a^k = ua^{k+2} = a^{k+3},$$

and so on. Therefore,

$$a^k = u^pa^k = a^{k+p}, a^{k+1} = a^{k+p+1},$$

and so on. Consequently, S_a is finite, so by the claim above

$$S_a \cong F_1 \oplus F_2 \oplus \cdots \oplus F_n,$$

where the F_i are finite fields. Let $a \in S_a$ be represented by $(a_1, a_2, \ldots, a_n)$ in that direct sum and let u_j be the identity of F_j if $a_j = 0$ and a_j if $a_j \neq 0$. Then

$$(u_1, u_2, \ldots, u_n)(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n)^2,$$

where $(u_1, u_2, \ldots, u_n)$ is a unit in $F_1 \oplus F_2 \oplus \cdots \oplus F_n$. Hence, there is a unit $u \in S_a$ such that $ua = a^2$. Since S_a and R share the same identity, a unit of S_a is also invertible in R. This proves statement (a).

Suppose $a \in R$. Let T_a be the set of all finite sums in which each term is of the form u or uav for some $u, v \in U$. Observe that if $x, y, s, t \in U$ then

$$(xay)(sat) = x(ays)(ays)s^{-1}y^{-1}t = x(uays)s^{-1}y^{-1}t = xuat$$

for some $u \in U$ by (a). Hence, T_a is a subring of R and is finite because U is finite. Therefore, T_a is isomorphic to the direct sum of fields and is commutative. Since a is an arbitrary element of R, U must be in the center of R.

Lastly, let a and b be elements of R. Let Q_{ab} be the set of all finite sums in which each term is of the form u, ua, ub, uab, uba, $uaba$, or $ubab$ for some $u \in U$. Note that whenever two elements of these types are multiplied, their product is also of one of these types. For example, if $x, y \in U$, then there exists $u, v, w \in U$ such that

$$(xa)(yab) = xya^2b = xy(ua)b,$$

$$(xb)(yaba) = xy(ba)^2 = xy(vba), \text{ and,}$$

$$(xab)(yaba) = xy(ab)^2a = xy(wab)a.$$

But Q_{ab} is finite, hence $ab = ba$ for all $a, b \in R$ by the claim above. This verifies statement (b).

Also solved by the proposer.

Solutions

Good and bad integer pairs **June 2022**

2146. *Proposed by Kenneth Fogarty, Bronx Community College (emeritus), Bronx, NY.*

Let a and d be integers with $d > 0$. We say that (a, d) is *good* if there is an arithmetic sequence with initial term a and difference d that can be split into two sequences of consecutive terms with the same sum. In other words, there exist integers k and n with $0 < k < n$ such that

$$\sum_{i=0}^{k-1} (a + di) = \sum_{i=k}^{n-1} (a + di) .$$

If there is no such arithmetic sequence, we say that (a, d) is *bad*.

(a) Show that if $2a > d$, then (a, d) is good.
(b) Show that if $2a = d$, then (a, d) is bad.
(c) Show that if $a = 0$ (and hence $2a < d$), then (a, d) is good.
(d) Show that if $2a < d$ and $a \neq 0$, then there is a d such that (a, d) is good and a d such that (a, d) is bad.

Solution by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.
It is straightforward to verify that

$$\sum_{i=0}^{k-1}(a + di) = \frac{k(2a + d(k - 1))}{2} \quad \text{and} \quad \sum_{i=k}^{n-1}(a + di) = \frac{(n - k)(2a + d(k + n - 1))}{2}.$$

One then readily deduces that (a, d) is good if and only if there exist integers k and n with $0 < k < n$ such that

$$(2a - d)(2k - n) = d(n^2 - 2k^2). \tag{1}$$

For part (a), if $2a > d$, then $2a - d \in \mathbb{N}$. Thus, to show (a, d) is good, it suffices to show there is a solution to $(2k - n) = d(n^2 - 2k^2)$ since if (k_1, n_1) is a solution to $(2k - n) = d(n^2 - 2k^2)$ with $0 < k_1 < n_1$, then $((2a - d)k_1, (2a - d)n_1)$ is a solution to $(2a - d)(2k - n) = d(n^2 - 2k^2)$ with $0 < (2a - d)k_1 < (2a - d)n_1$. Multiplying both sides of $(2k - n) = d(n^2 - 2k^2)$ by $4d$ and rearranging gives

$$\begin{aligned} 4d^2n^2 + 4dn - 2(4d^2k^2 + 4dk) &= 0 \\ (2dn + 1)^2 - 2(2dk + 1)^2 &= -1 \\ x^2 - 2y^2 &= -1, \end{aligned}$$

where $x = 2dn + 1$ and $y = 2dk + 1$. Positive integer solutions (x, y) to Pell's equation $x^2 - 2y^2 = -1$ are given by

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^j \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where j is a nonnegative integer. Since $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponds to $k = n = 0$, we seek a positive integer j such that $x_j \equiv 1 \equiv y_j \pmod{2d}$. Since $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ has determinant 1, it follows that $A \in SL_2(\mathbb{Z}_{2d})$, which has finite order for each positive integer d. Thus, there exists $j \in \mathbb{N}$ such that $A^j \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2d}$, and

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^j \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pmod{2d}.$$

So, $x_j = 2dn + 1$ and $y_j = 2dk + 1$ for positive integers k and n, and (k, n) is a solution to $(2k - n) = d(n^2 - 2k^2)$. An easy induction argument shows that $x_j > y_j$ for $j > 9$ and hence, $n > k > 0$. Thus, (a, d) is good for all integers a and d with $2a > d > 0$.

For part (b), if $2a = d$, then we would need $n^2 = 2k^2$ or $\sqrt{2} = n/k$, which is impossible for positive integers n and k since $\sqrt{2}$ is irrational. Thus, (a, d) is bad if $2a = d$.

For part (c), if $a = 0$, then equation (1) becomes $2k - n = 2k^2 - n^2$, or $n^2 - n = 2(k^2 - k)$, which is satisfied for $k = 3$ and $n = 4$ (for all positive integers d). Thus (a, d) is good if $a = 0$.

For part (d), we show that for each nonzero integer a, there exist positive integers $d_1 > 2a$ and $d_2 > 2a$ for which (a, d_1) is good and (a, d_2) is bad. If $2a < d$, then equation (1) becomes

$$(d - 2a)(2k - n) = d(2k^2 - n^2).$$

We consider two cases, depending on the sign of a.

CASE 1: $a > 0$. Let $d_1 = 3a > 0$. Letting $k = 5$ and $n = 7$,

$$(d_1 - 2a)(2k - n) = a(10 - 7) = 3a = d_1 = d_1(2k^2 - n^2),$$

so that $(a, 3a)$ is good for every positive integer a.

Let $d_2 = 4a > 0$. Then equation (1) becomes $2k - n = 2(2k^2 - n^2)$. Multiplying both sides by 8 and rearranging gives us

$$\begin{aligned} (16n^2 - 8n) - 2(16k^2 - 8k) &= 0 \\ (16n^2 - 8n + 1) - 2(16k^2 - 8k + 1) &= -1 \\ x^2 - 2y^2 &= -1, \end{aligned}$$

where $x = 4n - 1 \equiv 3 \pmod 4$ and $y = 4k - 1 \equiv 3 \pmod 4$. From our earlier discussion of solutions to $x^2 - 2y^2 = -1$ in part (a), we see that $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (mod 4). So, if j is odd, then

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^j \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 1 \end{pmatrix} \pmod 4;$$

meanwhile, if j is even, then

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^j \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pmod 4.$$

Thus, there are no solutions (x, y) to $x^2 - 2y^2 = -1$ with $x \equiv 3 \equiv y \pmod 4$. Hence, $(a, 4a)$ is bad for every positive integer a.

CASE 2: $a < 0$. Then $2a < d$ for any positive integer d. Let $d_1 = 1$. Then $d_1 - 2a = 1 - 2a > 0$. Letting $k = 3(1 - 2a)$ and $n = 4(1 - 2a)$, we see that $0 < k < n$ and

$$(d_1 - 2a)(2k - n) = (1 - 2a)^2 \cdot 2 = 2\,(3(1 - 2a))^2 - (4(1 - 2a))^2 = d_1(2k^2 - n^2),$$

and $(a, 1)$ is good for every negative integer a.

Let $d_2 = -4a > 0$. Then $d_2 - 2a = -6a$ and equation (1) becomes

$$\begin{aligned} -6a(2k - n) &= -4a(2k^2 - n^2)\\ 3(2k - n) &= 2(2k^2 - n^2)\\ (16n^2 - 24n + 9) - 2(16k^2 - 24k + 9) &= -9\\ x^2 - 2y^2 &= -9, \end{aligned}$$

where $x = 4n - 3 \equiv 1 \pmod 4$ and $y = 4k - 3 \equiv 1 \pmod 4$. Positive solutions to $x^2 - 2y^2 = -9$ are given by

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^j \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

where j is a nonnegative integer. If j is odd, then

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^j \begin{pmatrix} 3 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 3 \end{pmatrix} \pmod 4;$$

meanwhile, if j is even, then

$$\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^j \begin{pmatrix} 3 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 3 \end{pmatrix} \pmod 4.$$

Thus, there is no solution (x, y) to $x^2 - 2y^2 = -9$ with $x \equiv 1 \equiv y \pmod 4$; hence, $(a, -4a)$ is bad for every negative integer a.

Also solved by Eugene A. Herman, Dmitry Fleischman (partial solution), Fresno State Problem Solving Group (partial solution), William Boyd & Ernest James (partial solution), Fejéntaláltuka Szeged Problem Solving Group (Hungary) (partial solution), and the proposer.

Evaluate the infinite product **June 2022**

2147. *Proposed by Lokman Gökçe, Istanbul, Turkey.*

Evaluate

$$\prod_{n=2}^{\infty} \frac{n^4 + 4}{n^4 - 1}.$$

Solution by Michel Bataille, Rouen, France.

We prove that the value of the given infinite product is $\dfrac{2\sinh(\pi)}{5\pi}$.

From Sophie Germain's identity

$$x^4+4=((x+1)^2+1)((x-1)^2+1),$$

we deduce that for any integer $N \geq 3$, we have

$$P_N := \prod_{n=2}^{N} \frac{n^4+4}{n^4-1} = \prod_{n=2}^{N} \frac{((n+1)^2+1)((n-1)^2+1)}{(n^2-1)(n^2+1)} = \frac{2}{N^2+1} \cdot \frac{\prod\limits_{n=3}^{N+1}(n^2+1)}{\prod\limits_{n=2}^{N}(n^2-1)},$$

that is,

$$P_N = \frac{N^2+2N+2}{2(N^2+1)} \cdot \frac{\prod\limits_{n=3}^{N}\left(1+\frac{1}{n^2}\right)}{\prod\limits_{n=2}^{N}\left(1-\frac{1}{n^2}\right)}.$$

Now, from

$$\prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{N} \frac{(n-1)(n+1)}{n \cdot n} = \frac{1}{2} \cdot \frac{N+1}{N},$$

we obtain

$$\lim_{N\to\infty} \prod_{n=2}^{N}\left(1-\frac{1}{n^2}\right) = \frac{1}{2}$$

and from the well-known

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin(\pi z)}{\pi z} \quad (z \in \mathbb{C}, z \neq 0),$$

we deduce that

$$\lim_{N\to\infty} \prod_{n=1}^{N}\left(1+\frac{1}{n^2}\right) = \frac{\sin(\pi i)}{\pi i} = \frac{\sinh(\pi)}{\pi}.$$

Since

$$\lim_{N\to\infty} \frac{N^2+2N+2}{N^2+1} = 1,$$

it follows that

$$\prod_{n=2}^{\infty} \frac{n^4+4}{n^4-1} = \lim_{N\to\infty} P_N = \frac{\sinh(\pi)/\pi}{2(1+\frac{1}{1^2})(1+\frac{1}{2^2})} \cdot \frac{1}{1/2} = \frac{2\sinh(\pi)}{5\pi}.$$

Also solved by Anthony J. Bevelacqua, Ricardo Bittencourt (Brazil), Paul Bracken, Brian Bradie, Hongwei Chen, Junan Chen (China), Bruce E. Davis, Fejéntaláltuka Szeged Problem Solving Group (Hungary), Tasha Fellman, Shuyang Gao, Eugene A. Herman, Walther Janous (Austria), Warren P. Johnson, Sofia Lacerda (Brazil), Kee-Wai Lau (China), Isaac Venegas Macevschi, Donald Jay Moore, Raymond Mortini (Luxembourg) & Rudolf Rupp (Germany), Northwestern University Math Problem Solving Group, Peter Oman & Haohao Wang, Celia Schacht, Albert

Stadler (Switzerland), Seán M. Stewart (Saudi Arabia), Michael Vowe (Switzerland), Mark Wildon (UK), and the proposer. There were four incomplete or incorrect solutions.

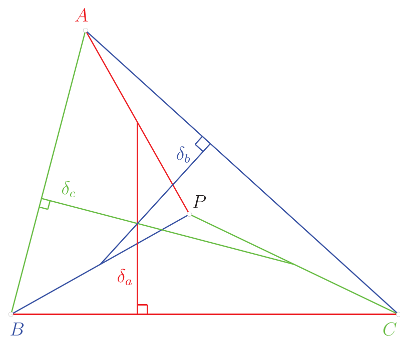
An application of the Erdős-Mordell inequality **June 2022**

2148. *Proposed by Quang Hung Tran, Hanoi, Vietnam.*

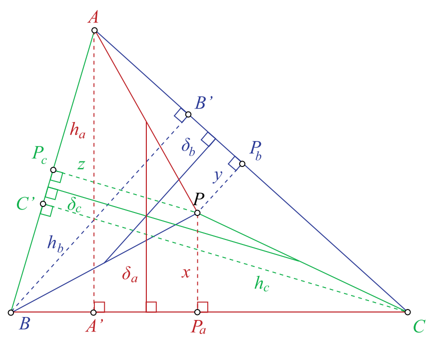
Let P be an interior point of triangle ABC. Denote by δ_a, δ_b, and δ_c the distances from the midpoints of segments PA, PB, and PC to the lines BC, CA, and AB. Prove that

$$PA + PB + PC \geq \delta_a + \delta_b + \delta_c.$$

Show that equality holds if and only if triangle ABC is equilateral and P is its center.



Solution by Fejéntaláltuka Szeged Problem Solving Group, University of Szeged, Szeged, Hungary.
Let h_a, h_b, and h_c be the altitudes of triangle ABC through the vertices A, B, and C respectively. Denote by x, y, and z the distances from P to the lines BC, CA, and AB.



Consider the trapezoid defined by the points A, A', P_a, P. The segment of length δ_a is the midline of this trapezoid, thus $\delta_a = (h_a + x)/2$, and $h_a = 2\delta_a - x$.

From the triangle inequality we have $PA + x \geq AP_a$. As the segment of length h_a is also the altitude of triangle ABP_a, we have $AP_a \geq h_a$. Thus,

$$PA + x \geq h_a$$

$$PA + x \geq 2\delta_a - x,$$

which implies

$$PA \geq 2\delta_a - 2x.$$

Similarly, it can be shown that

$$PB \geq 2\delta_b - 2y,$$
$$PC \geq 2\delta_c - 2z.$$

The sum of these inequalities gives

$$PA + PB + PC \geq 2\delta_a + 2\delta_b + 2\delta_c - 2x - 2y - 2z.$$

The Erdős-Mordell inequality states that

$$PA + PB + PC \geq 2(x + y + z).$$

Adding this to the inequality preceding it gives

$$2(PA + PB + PC) \geq 2(\delta_a + \delta_b + \delta_c).$$

After dividing both sides by 2, we obtain the desired inequality.

It is clear from the proof that if equality holds in the proposed inequality, then we must have equality in all the inequalities above. However, it is known that the Erdős-Mordell equality holds if and only if triangle ABC is equilateral and P is its center. One can easily show that if triangle ABC is equilateral and P is its center then equality holds in the desired inequality. This finishes the solution.

Also solved by Nandan Sai Dasireddy (India), Celia Schacht, Michael Vowe (Switzerland), and the proposer.

An irrational alternating sum **June 2022**

2149. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let $a_1, a_2, \ldots$ be a sequence of integers greater than 1. The series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\prod_{i=1}^{k} a_i} = 1 - \frac{1}{a_1} + \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} + \cdots$$

converges by the alternating series test.

(a) If the sequence $a_1, a_2, \ldots$ is unbounded, show that the sum of the series is irrational.

(b) Give an example of a bounded sequence of a_i's such that the sum of the series is irrational.

Solution by the Fresno State Journal Problem Solving Group, Fresno State University, CA.

(a) Suppose to the contrary that for some unbounded sequence $a_1, a_2, \ldots$, the sum of the series is rational, say,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\prod_{i=1}^{k} a_i} = \frac{m}{n},$$

where $m, n \in \mathbb{Z}$, $n > 0$. Since the sequence $\{a_i\}$ is unbounded, there exists $r \in \mathbb{N}$ such that $a_r > n$. Expanding the left-hand side of the above equation, we have

$$1 - \frac{1}{a_1} + \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} + \cdots + (-1)^{r-1} \frac{1}{a_1 a_2 \ldots a_{r-1}}$$

$$+ (-1)^r \frac{1}{a_1 a_2 \dots a_{r-1} a_r} + (-1)^{r+1} \frac{1}{a_1 a_2 \dots a_{r-1} a_r a_{r+1}} + \cdots$$
$$= \frac{m}{n}.$$

Multiplying both sides by $a_1 a_2 \dots a_{r-1} n$ gives

$$a_1 a_2 \dots a_{r-1} n \left(1 - \frac{1}{a_1} + \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} + \cdots + (-1)^{r-1} \frac{1}{a_1 a_2 \dots a_{r-1}}\right)$$
$$+ (-1)^r \frac{n}{a_r} + (-1)^{r+1} \frac{n}{a_r a_{r+1}} + \cdots$$
$$= a_1 a_2 \dots a_{r-1} m,$$

or, equivalently,

$$\underbrace{a_1 a_2 \dots a_{r-1} n \left(1 - \frac{1}{a_1} + \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} + \cdots + (-1)^{r-1} \frac{1}{a_1 a_2 \dots a_{r-1}}\right)}_{S}$$
$$+ (-1)^r \left(\underbrace{\frac{n}{a_r} - \frac{n}{a_r a_{r+1}} + \cdots}_{X}\right)$$
$$= a_1 a_2 \dots a_{r-1} m,$$

Observe that S and $a_1 a_2 \cdots a_{r-1} m$ are integers, therefore, X is an integer. However, X is the sum of an alternating series with decreasing terms approaching 0 (i.e., $\lim_{q\to\infty} \frac{n}{a_r \cdots a_q} = 0$), therefore, $0 < X < \frac{n}{a_r} < 1$. In this case X cannot be an integer, so we have a contradiction.

(b) Consider the following sequence:

$$2,\ 5,\ 5,\ 2,\ 2,\ 5,\ 2,\ 5,\ 5,\ 2,\ 2,\ 5,\ 2,\ 5,\ 2,\ 5,\ 5,\ 2,\ \dots,$$

where a block of 2,5 is followed by one block of 5,2, then two blocks of 2,5 are followed by one block of 5,2, then three blocks of 2,5 are followed by one block of 5,2, and so on.

In this case, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\prod_{i=1}^k a_i} = \left(1 - \frac{1}{a_1}\right) + \left(\frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3}\right) + \cdots$$
$$= \left(1 - \frac{1}{a_1}\right) + \frac{1}{a_1 a_2}\left(1 - \frac{1}{a_3}\right) + \frac{1}{a_1 a_2 a_3 a_4}\left(1 - \frac{1}{a_5}\right) + \cdots$$
$$= 0.5 + \frac{1}{10} \cdot 0.8 + \frac{1}{10^2} \cdot 0.5 + \frac{1}{10^3} \cdot 0.5 + \frac{1}{10^4} \cdot 0.8 + \cdots$$
$$= 0.585585558 \dots.$$

Since the blocks of 5 in the resulting number increase in length, this is a nonrepeating decimal, so it represents an irrational number.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Eugene A. Herman (partial solution), Evin Liang, Northwestern University Problem Solving Group, Ioana Mihaila & Ivan Ventura, Celia Schacht, and the proposer. There was one incomplete or incorrect solutions.

Maximize the area of a triangle in a cardioid **June 2022**

2150. *Proposed by Matthew McMullen, Otterbein University, Westerville, OH.*

Find the maximum area of a triangle whose vertices lie on the cardioid $r = 1 + \cos\theta$.

Editor's Note. Unfortunately, the argument that the triangle having maximal area is symmetric with respect to the x-axis is too long and involved to include here. The statement of the problem should have included the condition that this symmetry held in order to make the problem more tractable. We regret not having done so.

Solution by the proposer.
There are two cases to consider. First, assume the vertex lying on the x-axis is (0, 0). If the other vertices are $((1 + \cos\theta)\cos\theta, \pm(1 + \cos\theta)\sin\theta)$ with $0 \le \theta \le \pi$, the area of the triangle is $A(\theta) = |f(\theta)|$, where

$$f(\theta) = \sin\theta\cos\theta\,(1 + \cos\theta)^2 .$$

Now

$$f'(\theta) = \cos^2\theta\,(1 + \cos\theta)^2 - \sin^2\theta\,(1 + \cos\theta)^2 - \sin^2\theta\cos\theta\,(1 + \cos\theta) .$$

Letting $\sin^2\theta = 1 - \cos^2\theta$ and simplifying gives

$$f'(\theta) = (1 + \cos\theta)^2 \left(4\cos^2\theta - 2\cos\theta - 1\right) .$$

The only critical points for $A(\theta)$ having a nonzero area are when $\cos\theta = \left(1 \pm \sqrt{5}\right)/4$. The maximum value of $A(\theta)$ occurs when $\cos\theta = \left(1 + \sqrt{5}\right)/4$ and in that case

$$A(\theta) = \frac{5}{32}\sqrt{50 + 22\sqrt{5}} \approx 1.55619.$$

The second case is when the vertex lying on the x-axis is (2, 0). In this case,

$$\begin{aligned} A(\theta) &= \sin\theta(1 + \cos\theta)(2 - \cos\theta(1 + \cos\theta)) \\ &= (2 + \cos\theta)\sin^3\theta. \end{aligned}$$

Therefore,

$$A'(\theta) = (4\cos^2\theta + 6\cos\theta - 1)\sin^2\theta.$$

The only critical point for $A(\theta)$ having a nonzero area is when $\cos\theta = \left(\sqrt{13} - 3\right)/4$. In that case

$$A(\theta) = \frac{3}{32}\sqrt{105 + 39\sqrt{13}} \approx 2.07785.$$

This is therefore the overall maximum area.

Solutions

An improper logarithmic integral **April 2022**

2141. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.*

Evaluate

$$\int_0^\infty \ln\left(1+2x^{-2}\cos\varphi + x^{-4}\right)\, dx.$$

Solution by J. A. Grzesik, Torrance, CA.
Let

$$\begin{aligned} I(\varphi) &= \int_0^\infty \ln(1+2x^{-2}\cos\varphi + x^{-4})\, dx \\ &= \int_0^\infty \ln(x^4+2x^2\cos\varphi+1) - 4\ln x\, dx. \end{aligned}$$

Since I has period 2π, we may take $\varphi \in (-\pi, \pi]$. We claim that $I(\varphi) = 2\pi\cos(\varphi/2)$.

One obtains this by first noting that

$$x^4+2x^2\cos\varphi+1 = (x^2+e^{i\varphi})(x^2+e^{-i\varphi}),$$

whence

$$\begin{aligned} I(\varphi) &= 2\mathrm{Re}\int_0^\infty \ln(x^2+e^{i\varphi}) - 2\ln x\, dx \\ &= 2\mathrm{Re}\int_0^\infty \ln(x+e^{i(\pi+\varphi)/2}) + \ln(x-e^{i(\pi+\varphi)/2}) - 2\ln x\, dx \\ &= 2\mathrm{Re}\Big((x+e^{i(\pi+\varphi)/2})\ln(x+e^{i(\pi+\varphi)/2}) - x \\ &\qquad + (x-e^{i(\pi+\varphi)/2})\ln(x-e^{i(\pi+\varphi)/2}) - x - 2x\ln x + 2x\Big)\Big|_{x=0}^{x=\infty} \\ &= 2\pi\,\mathrm{Im}\left(e^{i(\pi+\varphi)/2}\right) = 2\pi\cos(\varphi/2). \end{aligned}$$

These manipulations hold so long as $\varphi \neq \pi$. When $\varphi = \pi$, there is a singularity when $x = 1$ and the integral must be split into two parts. Here one finds that

$$\begin{aligned} I(\pi) &= 2\int_0^\infty \ln\Big(|x-1|(x+1)\Big) - 2\ln x\, dx \\ &= 2\left[\int_0^1 \ln(1-x)dx + \int_1^\infty \ln(x-1)dx + \int_0^\infty \ln(x+1) - 2\ln x\, dx\right] \\ &= 2\Bigg[-(1-x)\ln(1-x) + (1-x)\Big|_0^1 + (x-1)\ln(x-1) - (x-1)\Big|_1^\infty \end{aligned}$$

$$(x+1)\ln(x+1)-(x+1)\Big|_0^\infty - 2\left(x\ln x - x\right)\Big|_0^\infty \Bigg]$$

$$= 0\,,$$

in agreement with the claimed value of $I(\varphi) = 2\pi\cos(\varphi/2)$.

Note that replacing the definite integrals with indefinite integrals in the second set of displayed equations allows us to find an elementary antiderivative

$$\int \ln\left(x^{-4}+2x^{-2}\cos\varphi+1\right)dx = x\ln\left(x^{-4}+2x^{-2}\cos\varphi+1\right) + \sin\left(\frac{\varphi}{2}\right)\ln\left(\frac{x^2+2x\sin(\varphi/2)+1}{x^2-2x\sin(\varphi/2)+1}\right) + 2\cos\left(\frac{\varphi}{2}\right)\arctan\left(\frac{2x\cos(\varphi/2)}{1-x^2}\right).$$

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Carl Axness (Spain), Michel Bataille (France), Robert Benim, Khristo N. Boyadzhiev, Brian Bradie, Bruce S. Burdick, Hongwei Chen, Bruce E. Davis, John N. Fitch, Fatima Gulieva (Azerbaijan), Eugene A. Herman, Walther Janous (Austria), Warren P. Johnson, Stephen Kaczkowski, Omran Kouba (Syria), James Magliano, Kelly D. McLenithan, Raymond Mortini (France) & Rudolph Rupp (Germany), Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Shing Hin Jimmy Pa (China), Paolo Perfetti (Italy), Didier Pinchon (France) Albert Stadler (Switzerland), Seán M. Stewart (Saudi Arabia), Michael Vowe (Switzerland), and the proposer.

Constructing the axis and focus of a parabola **April 2022**

2142. *Proposed by Roger Izard, Dallas, TX.*

Given a parabola in the plane, find its axis and focus using compass and straightedge.

Solution by Michelle Nogin (student), Clovis North High School, Fresno, CA.
We will use the following facts about parabolas.

(1) If points A, B, C, and D lie on the parabola with $AB \parallel CD$, then the line through the midpoints of segments AB and CD is parallel to the axis of symmetry.

Proof. Choose the coordinate system so that the vertex of the parabola is at the origin and the axis of symmetry is the y-axis. Then the parabola is given by $y = ax^2$. Let the lines AB and CD be given by $y = mx + b_1$ and $y = mx + b_2$, respectively. The x-coordinates of points A and B are the roots of $ax^2 = mx + b_1$. By Vieta's formulas, their sum is m/a. Thus, the x-coordinate of the midpoint of AB is $m/(2a)$. Similarly, the x-coordinate of the midpoint of CD is also $m/(2a)$. Therefore, the line going through the midpoints of AB and CD is parallel to the y-axis, which is the axis of symmetry.

(2) If points G and H lie on the parabola and line GH is perpendicular to the axis of symmetry, then the axis goes through the midpoint of segment GH.

(3) For the parabola $y = ax^2$, the line $y = x$ meets the parabola at the origin and another point, whose y-coordinate is four times larger than the y-coordinate of the focus. Note that the line $y = x$ forms a $45°$ angle with the axis of the parabola.

Proof. The x-coordinates of the intersection points satisfy the equation $ax^2 = x$. Therefore, the points of intersection are $(0, 0)$ and $(1/a, 1/a)$. Since the focus is at $(0, 1/(4a))$, the result follows.

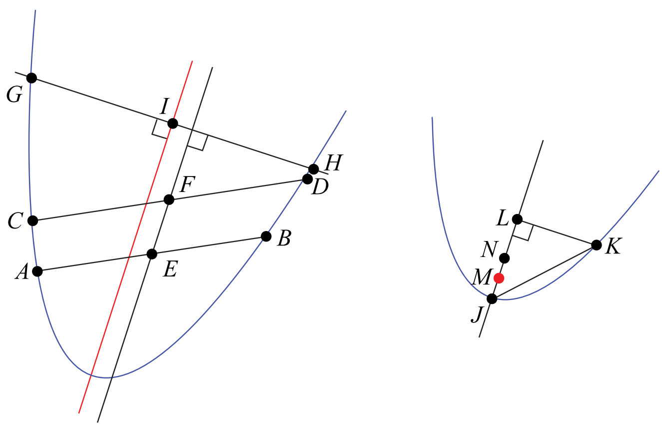
We will also use the following well-known constructions using compass and straightedge:

(a) Construct a line through a given point parallel to a given line.

(b) Construct a line through a given point perpendicular to a given line.

(c) Construct the midpoint of a given line segment.

(d) Given a point that lies on a line, construct a line through the given point that forms a 45° angle with the given line.



We first give the construction of the axis of the parabola. Take any two points A and B on the given parabola and draw a line through them. Take another point C on the parabola and draw a second line through C parallel to line AB. Let D be the other intersection point of this line and the parabola. (If the line through C happens to be tangent to the parabola, choose another point C.) Next, take points E and F, the midpoints of line segments AB and CD, respectively, and draw line EF. By Fact 1, this line is parallel to the axis of symmetry. Next, pick a point G on the parabola. Draw a line perpendicular to EF through G and call the other intersection point of that line and the parabola H. (If the line through G happens to be tangent to the parabola, choose another point G.) Let I be the midpoint of GH. By Fact 2, the line parallel to EF that goes through I is the parabola's axis of symmetry.

We now construct the focus of the parabola. The vertex of the parabola is J, the intersection point of the axis of symmetry and the parabola. Through J, draw a line that forms a 45° angle with the axis of symmetry. Call the second intersection point of this line and the parabola K. Next, draw a line perpendicular to the axis of symmetry through K. Call the intersection point of that line and the axis of symmetry L. Construct N, the midpoint of segment JL and M, the midpoint of segment JN. By Fact 3, M is the focus of the parabola.

Also solved by Michel Bataille (France), Bruce S. Burdick, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Micah Fogel, Michael Goldenberg & Mark Kaplan, Shing Hin Jimmy Pa (China), Randy K. Schwartz, and the proposer.

The limit of a binomial sum **April 2022**

2143. *Proposed by Florin Stănescu, Şerban Cioculescu School, Găeşti, Romania.*

Evaluate

$$\lim_{n\to\infty}\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(-1)^{i+j}}{2i+2j+1}\binom{n+i}{n-i}\binom{n+j}{n-j}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.
Let

$$S_n=\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(-1)^{i+j}}{2i+2j+1}\binom{n+i}{n-i}\binom{n+j}{n-j}.$$

The Chebyshev polynomials of the second kind, $U_n(x)$, are given by

$$U_n(x)=\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}(2x)^{n-2j},$$

so

$$\begin{aligned}U_{2n}(x)&=\sum_{j=0}^{n}(-1)^j\binom{2n-j}{j}(2x)^{2n-2j}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n+j}{n-j}(2x)^{2j}\\&=(-1)^n\sum_{j=0}^{n}(-1)^j\binom{n+j}{n-j}(2x)^{2j}.\end{aligned}$$

Write

$$\frac{1}{2i+2j+1}=\int_0^1 x^{2i+2j}\,dx.$$

Then

$$\begin{aligned}S_n&=\int_0^1\left(\sum_{i=0}^{n}(-1)^i\binom{n+i}{n-i}x^{2i}\right)\left(\sum_{j=0}^{n}(-1)^j\binom{n+j}{n-j}x^{2j}\right)dx\\&=\int_0^1 U_{2n}^2\left(\frac{x}{2}\right)dx.\end{aligned}$$

With the substitution $x=2\cos\theta$, we get

$$S_n=2\int_{\pi/3}^{\pi/2}U_{2n}^2(\cos\theta)\sin\theta\,d\theta=2\int_{\pi/3}^{\pi/2}\frac{\sin^2(2n+1)\theta}{\sin\theta}\,d\theta.$$

Now,

$$\begin{aligned}\sin\theta\sum_{j=0}^{2n}\sin(2j+1)\theta&=\frac{1}{2}\sum_{j=0}^{2n}[\cos 2j\theta-\cos(2j+2)\theta]\\&=\frac{1}{2}(1-\cos(4n+2)\theta)=\sin^2(2n+1)\theta,\end{aligned}$$

so

$$\begin{aligned}S_n&=2\int_{\pi/3}^{\pi/2}\sum_{j=0}^{2n}\sin(2j+1)\theta\,d\theta=-2\sum_{j=0}^{2n}\frac{\cos(2j+1)\theta}{2j+1}\bigg|_{\pi/3}^{\pi/2}\\&=2\sum_{j=0}^{2n}\frac{\cos(2j+1)\frac{\pi}{3}}{2j+1}=2\sum_{j=0}^{2n}\frac{\cos((2j+1)\arccos\frac{1}{2})}{2j+1}=2\sum_{j=0}^{2n}\frac{T_{2j+1}\left(\frac{1}{2}\right)}{2j+1},\end{aligned}$$

where $T_n(x)$ is a Chebyshev polynomial of the first kind. Next, the generating function for the Chebyshev polynomials of the first kind is

$$\sum_{j=0}^{\infty} T_j(x)t^j = \frac{1-xt}{1-2xt+t^2}.$$

Separating the $j=0$ term from the series, dividing by t, and integrating yields

$$\sum_{j=1}^{\infty} \frac{T_j(x)}{j}t^j = \ln \frac{1}{\sqrt{1-2xt+t^2}},$$

from which it follows that

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{T_{2j+1}(x)}{2j+1} &= \frac{1}{2}\left(\ln\frac{1}{\sqrt{1-2xt+t^2}} - \ln\frac{1}{\sqrt{1+2tx+t^2}}\right)\Bigg|_{t=1}\\ &= \frac{1}{2}\ln\frac{\sqrt{2+2x}}{\sqrt{2-2x}} = \frac{1}{4}\ln\frac{1+x}{1-x}.\end{aligned}$$

Finally,

$$\lim_{n\to\infty} S_n = 2\sum_{j=0}^{\infty}\frac{T_{2j+1}\left(\frac{1}{2}\right)}{2j+1} = \frac{1}{2}\ln\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2}\ln 3.$$

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Omran Kouba (Syria), Didier Pinchon (France), Albert Stadler (Switzerland) Séan M. Stewart (Saudi Arabia) Michael Vowe (Switzerland) and the proposer.

A ring with distinct ideals having distinct orders April 2022

2144. *Proposed by Souvik Dey (graduate student), University of Kansas, Lawrence, KS.*

Let R be a finite commutative ring with unity such that distinct ideals of R have distinct orders. Show that R is a principal ideal ring.

Solution by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.

We will define a finite commutative ring with unity such that distinct ideals have distinct orders to be *distinctive*. It is well known that any finite ring R is a direct sum of finite local rings, that ideals of R correspond to direct sums of ideals of the finite local rings, and that a direct sum of principal ideals is principal. Clearly, any summand of a distinctive ring must be distinctive. Therefore, it suffices to prove the result for R local with maximal ideal $\mathfrak{m}$. Suppose, to the contrary, that $a, b \in \mathfrak{m}$ are distinct elements of a minimal generating set for $\mathfrak{m}$, and let $I = (a, \mathfrak{m}^2)$ and $J = (b, \mathfrak{m}^2)$. Then I and J are distinct ideals. Both $I/\mathfrak{m}^2$ and $J/\mathfrak{m}^2$ are one-dimensional vector spaces over $R/\mathfrak{m}$, implying that $\left|I/\mathfrak{m}^2\right| = |R/\mathfrak{m}| = \left|J/\mathfrak{m}^2\right|$. Since R is finite, $\left|I/\mathfrak{m}^2\right| = |I|/\left|\mathfrak{m}^2\right|$, hence $|I| = |R/\mathfrak{m}|\,|\mathfrak{m}^2|$. Similarly, $|J| = |R/\mathfrak{m}|\,|\mathfrak{m}^2|$, which contradicts the fact that I and J are distinct ideals. Thus, $\mathfrak{m}$ is principal. It is well known that if the maximal ideal of a finite local ring R is principal, then R is a principal ideal ring, and the result follows.

We note that if R is a finite principal ideal ring, then for R to be distinctive, it is necessary that the cardinalities of all its summands are distinct, and it is sufficient for the cardinalities of its summands to be pairwise relatively prime. The problem of completely characterizing distinctive rings seems to be complicated.

Also solved by the proposer.

Determine $L(L(S))$ **April 2022**

2145. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

Given a set of points S, let $L(S)$ be the set of all points lying on any line connecting two distinct points in S. For example, if S is the disjoint union of a closed line segment and a point not lying on the line containing the segment, then $L(S)$ consists of two vertical angles, their interiors, and the line containing the segment. In this case, $L(L(S))$ is the entire plane.

Determine $L(L(S))$ when S consists of the vertices of a regular tetrahedron.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
Without loss of generality, we may assume that the vertices of the tetrahedron are $S = \{A, B, C, D\}$, where

$$A = (1, 1, 1),\ B = (1, -1, -1),\ C = (-1, 1, -1),\ \text{and } D = (-1, -1, 1).$$

Let

$$A' = (-1, -1, -1),\ B' = (-1, 1, 1),\ C' = (1, -1, 1),\ \text{and } D' = (1, 1, -1).$$

Note that A', B', C', and D' are the reflections of A, B, C, and D through the origin, which is the centroid of the tetrahedron. We claim that

$$L(L(S)) = \mathbb{R} - \{A', B', C', D'\}.$$

Clearly, $L(S)$ consists of the lines through the vertices. The points on line AB are of the form $(1, s, s)$ and those on line CD are of the form $(-1, t, -t)$. Given any point (x, y, z) with $x \neq \pm 1$, we have $(x, y, z) = \lambda(1, s, s) + (1 - \lambda)(-1, t, -t)$, where

$$\lambda = \frac{x+1}{2}, s = \frac{y+z}{x+1}, \text{ and } t = \frac{y-z}{1-x}.$$

Therefore, $L(L(S))$ contains all points with $x \neq \pm 1$. Similar arguments using the other pairs of skew lines shows that $L(L(S))$ contains all points with $y \neq \pm 1$ and all points with $z \neq \pm 1$. Hence,

$$L(L(S)) \supseteq \mathbb{R} - \{A, B, C, D, A', B', C', D'\},$$

but $L(L(S))$ clearly contains A, B, C, and D, so $L(L(S)) \supseteq \mathbb{R} - \{A', B', C', D'\}$. It only remains to prove that A', B', C', and D' do not belong to $L(L(S))$. Suppose, on the contrary, that A' is on the line between P and Q with $P, Q \in L(S)$. Then P must be on the line determined by two vertices and Q on the line determined by the other two (if P and Q were on two intersecting lines A' would lie on the plane of a face, and it does not). For example, suppose that P lies on line AB and Q lies on line CD. But then the line through Q and A' lies in the plane $x = -1$, which does not meet line

AB, whch lies in the plane $x = 1$. This gives a contradiction. Examining the other two pairs of skew lines, shows that $A' \notin L(L(S))$. Similar arguments show the same for B', C', and D'. Therefore, $L(L(S)) = \mathbb{R} - \{A', B', C', D'\}$, as claimed.

Also solved by Robert Calcaterra, Eugene A. Herman, Didier Pinchon (France), and the proposers. There were three incomplete or incorrect solutions.

Solutions

A formula for $\zeta(3)$ **February 2022**

2136. *Proposed by Necdet Batir, Nevşehir HBV University, Nevşehir, Turkey.*

Evaluate

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{n}\frac{H_k^2}{k}\right)-\frac{H_n^3}{3}\right),$$

where $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number.

Solution by Kelly D. McLenithan, Los Alamos, NM.
The desired limit is

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{n}\frac{H_k^2}{k}\right)-\frac{H_n^3}{3}\right)=\frac{5}{3}\zeta(3)\,,$$

where $\zeta(3)$ is Apéry's constant given by

$$\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^3}=1.20205\,69031\,59594\ldots.$$

This follows from an application of the summation-by-parts formula

$$\sum_{k=1}^{n}(a_{k+1}-a_k)b_k\ =\ a_{n+1}b_{n+1}-a_1b_1-\sum_{k=1}^{n}a_{k+1}(b_{k+1}-b_k)\,.$$

Letting $a_1=0$, $a_{k+1}-a_k=1/k$, and $b_k=H_k^2$, we find that $a_k=H_{k-1}$ and

$$\begin{aligned}b_{k+1}-b_k&=H_{k+1}^2-H_k^2\\&=(H_{k+1}-H_k)(H_{k+1}+H_k)\\&=\frac{1}{k+1}\left(\frac{1}{k+1}+2H_k\right)\\&=\frac{2H_k}{k+1}+\frac{1}{(k+1)^2}\,.\end{aligned}$$

By summation by parts, we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{H_k^2}{k}&=H_nH_n^2-0-\sum_{k=1}^{n}H_k\left(\frac{2H_k}{k+1}+\frac{1}{(k+1)^2}\right)\\&=H_n^3-2\sum_{k=1}^{n}\frac{H_k^2}{k+1}-\sum_{k=1}^{n}\frac{H_k}{(k+1)^2}\\&=H_n^3-2\sum_{k=1}^{n}\frac{H_{k-1}^2}{k}-\sum_{k=1}^{n}\frac{H_{k-1}}{k^2}\end{aligned}$$

$$= H_n^3 - 2\sum_{k=1}^{n} \frac{1}{k}\left(H_k - \frac{1}{k}\right)^2 - \sum_{k=1}^{n} \frac{1}{k^2}\left(H_k - \frac{1}{k}\right)$$

$$= H_n^3 - 2\sum_{k=1}^{n} \frac{H_k^2}{k} + 4\sum_{k=1}^{n} \frac{H_k}{k^2} - 2\sum_{k=1}^{n} \frac{1}{k^3} - \sum_{k=1}^{n} \frac{H_k}{k^2} + \sum_{k=1}^{n} \frac{1}{k^3}.$$

Collecting terms and rearranging, it follows that

$$\sum_{k=1}^{n} \frac{H_k^2}{k} - \frac{H_n^3}{3} = \sum_{k=1}^{n} \frac{H_k}{k^2} - \frac{1}{3}\sum_{k=1}^{n} \frac{1}{k^3}.$$

After taking the limit, we obtain

$$\lim_{n\to\infty} \left(\left(\sum_{k=1}^{n} \frac{H_k^2}{k}\right) - \frac{H_n^3}{3}\right) = \sum_{k=1}^{\infty} \frac{H_k}{k^2} - \frac{1}{3}\sum_{k=1}^{\infty} \frac{1}{k^3}$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{k^2} - \frac{1}{3}\zeta(3).$$

In 1775, Euler showed that for integers $q \geq 2$

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^q} = (q+2)\zeta(q+1) - \sum_{m=1}^{q-2} \zeta(m+1)\zeta(q-m).$$

When $q = 2$, this gives

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3).$$

Therefore, our desired limit is

$$\lim_{n\to\infty} \left(\left(\sum_{k=1}^{n} \frac{H_k^2}{k}\right) - \frac{H_n^3}{3}\right) = \sum_{k=1}^{\infty} \frac{H_k}{k^2} - \frac{1}{3}\zeta(3)$$

$$= 2\zeta(3) - \frac{1}{3}\zeta(3) = \frac{5}{3}\zeta(3),$$

as claimed.

Also solved by Michel Bataille (France), Jake Boswell & Chip Curtis, Paul Bracken, Brian Bradie, Bruce S. Burdick, Hongwei Chen, Robert L. Doucette, Russell Gordon, Lixing Han, Eugene A. Herman, Walther Janous (Austria), Kee-Wai Lau (Hong Kong, China), Shing Hin Jimmy Pa (Canada), Paolo Perfetti (Italy), Didier Pinchon (France), Albert Stadler (Switzerland), Séan M. Stewart (Saudi Arabia), and the proposer.

The gcd of terms in a recursive sequence **February 2022**

2137. *Proposed by the Columbus State University Problem Solving Group, Columbus State University, Columbus, GA.*

For a positive integer n, let a_n and b_n be the unique integers such that

$$(5 + \sqrt{3})^n = a_n + b_n\sqrt{3}.$$

Find $\gcd(a_n, b_n)$ as a function of n. Solve the analogous problem when $5 + \sqrt{3}$ is replaced by $3 + \sqrt{5}$.

Solution by Jacob Boswell and Chip Curtis, Missouri Southern State University, Joplin, MO.

For the first version of the problem, we claim that

$$\gcd(a_n, b_n) = 2^{\lfloor n/2 \rfloor}.$$

To see this, set $v_n = [a_n, b_n]^{\mathrm{T}}$. The sequence v_n satisfies the recurrence

$$v_{n+1} = M v_n, \quad \text{and} \quad v_0 = [1, 0]^{\mathrm{T}}.$$

where $M = \begin{bmatrix} 5 & 3 \\ 1 & 5 \end{bmatrix}$. We note that

$$M^2 = \begin{bmatrix} 28 & 30 \\ 10 & 28 \end{bmatrix} \quad \text{and} \quad M^3 = \begin{bmatrix} 170 & 234 \\ 78 & 170 \end{bmatrix}.$$

Solving $v_{n+1} = M v_n$ for a_n and b_n gives

$$22a_n = 5a_{n+1} - 3b_{n+1}$$
$$22b_n = -a_{n+1} + 5b_{n+1}.$$

Set $d_n = \gcd(a_n, b_n)$. Thus, any factor that divides a_{n+1} and b_{n+1} must also divide $22d_n$. Noting that

$$v_1 = [5, 1]^{\mathrm{T}}, v_2 = [28, 10]^{\mathrm{T}}, \text{ and } M^3 \cong M \mod 11,$$

we find that 11 is not a factor of $\gcd(a_n, b_n)$ for any n. From $v_{n+2} = M^2 v_n$, we see that $2d_n$ divides d_{n+2}, but $4d_n$ does not divide d_{n+2}. Hence, $d_{n+2} = 2d_n$. Consider the subsequences of $\{d_n\}$ of even index and odd index separately, and note that

$$v_1 = [5, 1]^{\mathrm{T}} \text{ and } v_2 = [28, 10]^{\mathrm{T}},$$

so $d_1 = 1$ and $d_2 = 2$. A simple induction completes the proof.

For the second case, we claim that $\gcd(a_n, b_n) = 2^{n - \alpha(n)}$, where $\alpha(n)$ is 0 if n is a multiple of 3 and 1 otherwise.

Here

$$M = \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix}, \quad M^2 = \begin{bmatrix} 14 & 30 \\ 6 & 14 \end{bmatrix}, \quad \text{and} \quad M^3 = \begin{bmatrix} 72 & 160 \\ 32 & 72 \end{bmatrix}.$$

From $v_{n+3} = M^3 v_n$, we find that $8d_n$ divides d_{n+3}, and from

$$8a_n = 9a_{n+3} - 20b_{n+3}$$
$$8b_n = -4a_{n+3} + 9b_{n+3},$$

obtained by solving $v_{n+3} = M^3 v_n$ for a_n and b_n, we find that d_{n+3} divides $8d_n$. Hence, $d_{n+3} = 8d_n$. Since $v_1 = [3, 1]^{\mathrm{T}}$, $v_2 = [14, 6]^{\mathrm{T}}$, and $v_3 = [72, 32]^{\mathrm{T}}$, we have $d_1 = 1$, $d_2 = 2$, and $d_3 = 8$. The claim again follows by induction.

Also solved by Michel Bataille (France), Anthony J. Bevelacqua, Robert Calcaterra, Hongwei Chen, John Christopher, Rohan Dalal, John Ferdinands, Michael Goldenberg & Mark Kaplan, Russell Gordon, Eugene A. Herman, Northwestern University Math Problem Solving Group, Michael Reid, Albert Stadler (Switzerland), and the proposers.

Find the locus of the circumcenter **February 2022**

2138. *Proposed by Alexandru Girban, Constanta, Romania.*

Let $\triangle ABC$ be a triangle with circumcircle ω and let D be a fixed point on side BC. Let E be a point on ω and let AE meet line BC at F. Find the locus of the circumcenter of $\triangle DEF$ as E varies along ω.

Solution by Michel Bataille, Rouen, France.
In what follows, the line AE will be taken to be the tangent line to ω at A when $A = E$.

Let the parallel to BC through A intersect ω again at X and let the line XD intersect ω again at Y (see the figure). We show that the required locus is the perpendicular bisector of DY with three points removed.

Let $\angle(\ell, \ell')$ denote the directed angle from line ℓ to line ℓ'.

Let E be a point of ω, with $E \neq X$ (so that AE does intersect BC). Assuming that $\triangle DEF$ is not degenerate, we have

$$\begin{aligned}\angle(YE, YD) &= \angle(YE, YX) = \angle(AE, AX) \quad \text{(since } A, Y, E, X \text{ are concyclic)}\\ &= \angle(AF, AX) = \angle(FA, FD) \quad \text{(since } FD \parallel AX\text{)},\end{aligned}$$

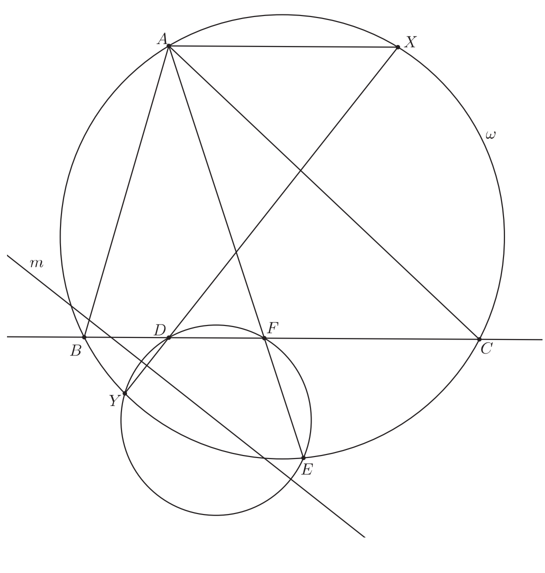
hence $\angle(YE, YD) = \angle(FE, FD)$. Therefore, Y lies on the circumcircle of $\triangle DEF$. The circumcenter of $\triangle DEF$ is on the perpendicular bisector m of DY, so the locus we seek is a subset of m.

Conversely, Let U be any point of m and let γ be the circle with center U and radius $UD = UY$. Let BC intersect γ again at F and ω intersect γ again at E. Then U is a point of the locus if A, E, F are collinear and $\triangle DEF$ is not degenerate.

Now, we have

$$\begin{aligned}\angle(AF, AE) &= \angle(AF, AX) + \angle(AX, AE) = \angle(FA, FD) + \angle(YX, YE)\\ &= \angle(FA, FD) + \angle(YD, YE) = \angle(FA, FD) + \angle(FD, FE)\\ &= \angle(AF, FE).\end{aligned}$$

Therefore, A, E, and F are collinear. Since $\triangle DEF$ is degenerate if and only if $F = D$, B, or C, the centers P, Q, and R of the circle tangent to BC at D, of the circumcircle of $\triangle BDY$, and of the circumcircle of $\triangle CDY$ (respectively) must be excluded. Finally, the desired locus is $m - \{P, Q, R\}$.



Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Eugene A. Herman, Walther Janous (Austria), Albert Stadler (Switzerland), and the proposer.

Infinitely many "very good" Pythagorean triples **February 2022**

2139. *Proposed by Philippe Fondanaiche, Paris, France.*

Recall that a Pythagorean triple is a triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. We say that a Pythagorean triple is *good* if adding the same single digit to the front of the decimal representations of a, b, and c yields another Pythagorean triple. We will call a Pythagorean triple *very good* if it is good and it is not a nontrivial scalar multiple of another good Pythagorean triple. For example (50, 120, 130) is good, since (150, 1120, 1130) is also a Pythagorean triple, but it is not very good since it is a scalar multiple of the very good triple (5, 12, 13).

Show that there are infinitely many very good Pythagorean triples.

Solution by Michael Reid, University of Central Florida, Orlando, FL.
Let $n \geq 2$ be an integer, and put

$$\begin{aligned} a &= 5 \cdot 10^n, \\ b &= 125 \cdot 10^{2n-2} - 5, \quad \text{and} \\ c &= 125 \cdot 10^{2n-2} + 5. \end{aligned}$$

We have

$$\begin{aligned} c^2 - b^2 &= (c - b)(c + b) \\ &= (10)(250 \cdot 10^{2n-2}) \\ &= 25 \cdot 10^{2n} = a^2, \end{aligned}$$

so (a, b, c) is a Pythagorean triple. Let A, B, C be the integers obtained by prepending the digit 1 to the decimal representations of a, b, c. Then

$$\begin{aligned} A &= 15 \cdot 10^n, \\ B &= 1125 \cdot 10^{2n-2} - 5, \quad \text{and} \\ C &= 1125 \cdot 10^{2n-2} + 5. \end{aligned}$$

Hence,

$$C^2 - B^2 = (C - B)(C + B) = (10)(2250 \cdot 10^{2n-2}) = 225 \cdot 10^{2n} = A^2,$$

so (A, B, C) is a Pythagorean triple. Thus, (a, b, c) is a good Pythagorean triple.

The good Pythagorean triples above are all very good, as we now show. Note that the only prime divisors of $a = 5 \cdot 10^n$ are 2 and 5. Also, b and c are divisible by 5 but not by 5^2. Since $n \geq 2$, b and c are odd, so $\gcd(a, b, c) = 5$. Thus, if (a, b, c) is not very good, it is 5 times a good Pythagorean triple. Let

$$\begin{aligned} x &= a/5 = 10^n, \\ y &= b/5 = 25 \cdot 10^{2n-2} - 1, \quad \text{and} \\ z &= c/5 = 25 \cdot 10^{2n-2} + 1, \end{aligned}$$

and let X, Y, Z be the numbers obtained by prepending the nonzero digit d to x, y, z. Since x, y, and z have $n + 1$, $2n$, and $2n$ digits, respectively,

$$\begin{aligned} X &= d \cdot 10^{n+1} + x, \\ Y &= d \cdot 10^{2n} + y, \quad \text{and} \\ Z &= d \cdot 10^{2n} + z. \end{aligned}$$

The equation $X^2 + Y^2 = Z^2$ yields a quadratic equation in d whose roots are $d = 0$ and $d = -4/25$. This contradiction shows that (x, y, z) is not good, so (a, b, c) is indeed very good.

Also solved by Arya Gupta & Amishi Gupta & Ethan Strubbe, and the proposer.

Minimize the exponential sum **February 2022**

2140. *Proposed by Antonio Garcia, Strasbourg, France.*

For a fixed integer $n \geq 2$, find the minimum value of

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \exp\left(x_i^2\right) + \exp\left(\sum_{1 \leq i < j \leq n} -x_i x_j\right).$$

Solution by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Friedberg, Germany.
Application of the AGM inequality

$$(a_1 \cdots a_n)^{1/n} \leq (a_1 + \cdots + a_n)/n$$

for positive reals a_i, yields

$$\sum_{i=1}^{n} \exp\left(x_i^2\right) \geq n \left(\prod_{i=1}^{n} \exp\left(x_i^2\right)\right)^{1/n} = n \exp\left(\sum_{i=1}^{n} x_i^2/n\right).$$

By the Cauchy–Schwarz inequality, it follows that

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i \cdot 1\right)^2 \leq n \sum_{i=1}^{n} x_i^2,$$

which implies

$$2 \sum_{1 \leq i < j \leq n} x_i x_j = \left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2 \leq (n-1) \sum_{i=1}^{n} x_i^2.$$

Combining both inequalities leads to

$$f(x_1, \ldots, x_n) \geq n \exp\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right) + \exp\left(-\frac{n-1}{2} \sum_{i=1}^{n} x_i^2\right),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and equality holds if and only if $x_1 = \cdots = x_n$. Putting $t = \sum_{i=1}^{n} x_i^2$, we want to find the minimum of

$$g(t) = n \exp\left(\frac{1}{n} t\right) + \exp\left(-\frac{n-1}{2} t\right)$$

for $t \geq 0$. We have

$$g'(t) = \exp\left(\frac{1}{n}t\right) - \frac{n-1}{2}\exp\left(-\frac{n-1}{2}t\right) = 0$$

if and only if

$$\exp\left(\left(\frac{n-1}{2} + \frac{1}{n}\right)t\right) = \frac{n-1}{2},$$

which occurs at

$$t_0 = \frac{\ln((n-1)/2)}{(n-1)/2 + 1/n}.$$

We also have

$$g''(t) = \frac{1}{n}\exp\left(\frac{1}{n}t\right) + \left(\frac{n-1}{2}\right)^2 \exp\left(-\frac{n-1}{2}t\right) > 0,$$

so g has an absolute minimum at t_0.

For $n = 2$, we have $t_0 < 0$. Since t is restricted to nonnegative values, the minimum occurs when $t = 0$ giving 3 as the minimum value in this case.

For $n \geq 3$, we have $t_0 \geq 0$ and

$$g(t_0) = \left(n + \frac{2}{n-1}\right)\exp\left(\frac{1}{n}t_0\right) = \frac{n^2-n+2}{n-1}\left(\frac{n-1}{2}\right)^{\frac{2}{n^2-n+2}}$$

is the minimum value.

Also solved by Carl Axness (Spain), Jacob Boswell & Chip Curtis, Robert Calcaterra, Hongwei Chen, Lixing Han, Eugene A. Herman, Kelly D. McLenithan, Michael Reid, Edward Schmeichel, Albert Stadler (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

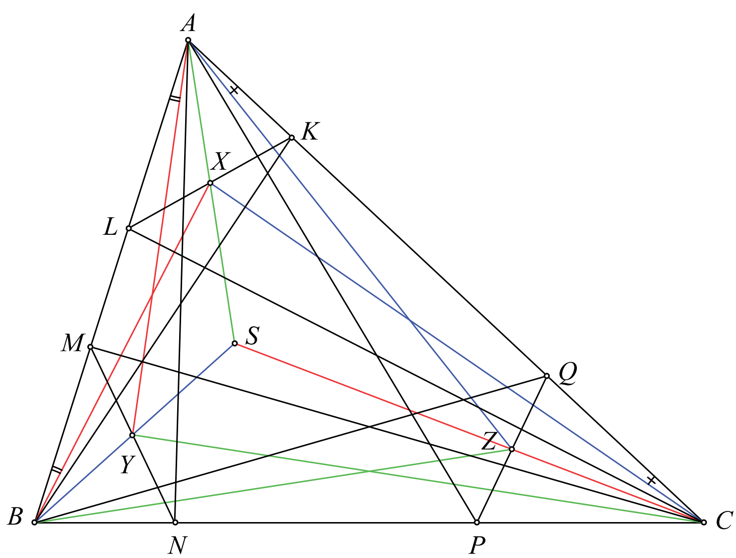
Solutions

A property of the symmedian point **December 2021**

2131. *Proposed by Tran Quang Hung, Hanoi, Vietnam.*

Recall that a symmedian is the reflection of a median through a vertex across the angle bisector passing through that vertex. The three symmedians of a triangle meet in a point known as the symmedian (or Lemoine or Grebe) point. Let ABC be a triangle with symmedian point S. Let X, Y, and Z be points lying on segments SA, SB, and SC, respectively, such that $\angle XBA \cong \angle YAB$ and $\angle XCA \cong \angle ZAC$. Prove that $\angle ZBC \cong \angle YCB$.

Solution by Do Van Quyet, Vinh Phuc, Vietnam.
Recall that line ℓ_1 is said to be anti-parallel to line ℓ_2 with respect to lines m_1 and m_2 if the opposite angles in the quadrilateral formed by the four lines are supplementary.



Let the anti-parallel line to BC with respect to sides AC and AB passing through X meet those sides at K and L, respectively.

Let the anti-parallel line to AC with respect to sides BA and BC passing through Y meet those sides at M and N, respectively.

Let the anti-parallel line to AB with respect to sides CB and CA passing through Z meet those sides at P and Q, respectively.

Note that quadrilaterals $BCKL$, $CAMN$, and $ABPQ$ are cyclic.

A key property of a symmedian through a vertex is that it bisects any anti-parallel to the opposite side with respect to the adjacent sides. Therefore, X, Y, and Z are the midpoints of segments KL, MN, and PQ, respectively.

We have

$$\angle ALK \cong \angle ACB \text{ (since } BCKL \text{ is cyclic)},$$

and

$$\angle ACB \cong \angle BMN \text{ (since } CAMN \text{ is cyclic)}.$$

Therefore, $\angle ALK \cong \angle BMN$ and consequently, $\angle BLX \cong \angle AMY$ (supplementary angles). We are given that $\angle XBA \cong \angle YAB$, so

$$\triangle AMY \sim \triangle BLX \text{ by the AA criterion.}$$

Since X and Y are the midpoints of KL and MN, respectively, we deduce that $\triangle AMN \sim \triangle BLK$. Therefore, $\angle LBK \cong \angle MAN$. Now

$$\angle MAN \cong \angle MCN$$

since $CAMN$ is cyclic and the angles are subtended by the same arc. Therefore, $\angle LBK \cong \angle MCN$.

A similar argument shows that $\angle LCK \cong \angle QBP$.

We have

$$\angle LBK \cong \angle LCK,$$

since $BCKL$ is cyclic and the angles are subtended by the same arc.

From the three congruences directly above, we obtain $\angle MCN \cong \angle QBP$. Now

$$\angle QPC \cong \angle BAC \text{ (because } CAMN \text{ is cyclic)}$$

and

$$\angle BAC \cong \angle MNB \text{ (because } ABPQ \text{ is cyclic)}.$$

Thus

$$\angle QPC \cong \angle MNB, \text{ and therefore, } \angle BPQ \cong \angle MNC \text{ (supplementary angles)}.$$

Hence,

$$\triangle CMN \sim \triangle BQP \text{ by the AA criterion.}$$

Since Y and Z are the midpoints of MN and PQ, respectively, $\triangle BZP \sim \triangle CYN$. Therefore, $\angle ZBC = \angle YCB$, as we wished to show.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Nandan Sai Dasireddy (India), Michael Goldenberg & Mark Kaplan, Volkhard Schindler (Germany), and the proposer.

Buffon's tetrahedron **December 2021**

2132. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

A regular tetrahedral die with sides of length 1 is tossed onto a floor having a family of parallel lines spaced 1 unit apart. What is the probability that the die lands on a line?

Solution by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

Since the tetrahedron is regular, every configuration of the bottom triangular face on the floor is equally likely. In other words, the probability we seek is the same as the probability of a randomly tossed equilateral triangle landing on a line. Orient the parallel lines horizontally and use the usual cartesian coordinate system. We can give the vertical coordinate of any point on the floor as a real number in the interval $[0, 1)$, representing the distance to the closest horizontal line below, or passing through, the given point. Let y represent the vertical coordinate of the lowest point of the triangular face. Let θ represent the angle with smallest nonnegative measure between the sides of the triangle containing the lowest point and the positive x-axis. Then

$$0 \le y < 1 \qquad \text{and} \qquad 0 \le \theta < \frac{2\pi}{3}.$$

Thus, a random toss of the equilateral triangle corresponds to a random selection of a point (θ, y) from the rectangle

$$\left[0, \frac{2\pi}{3}\right) \times [0, 1).$$

If we rotate around a vertex fixed on a horizontal line, then the vertical coordinate of the highest vertex will be

$$\sin\left(\frac{\pi}{3} + \theta\right) \text{ for } 0 \le \theta \le \frac{\pi}{3}, \quad \text{and} \quad \sin\theta \text{ for } \frac{\pi}{3} \le \theta < \frac{2\pi}{3}.$$

Thus, the triangle will miss all horizontal lines if and only if

$$0 < y < 1 - \sin\left(\frac{\pi}{3} + \theta\right)$$

for $0 \le \theta \le \frac{\pi}{3}$ and

$$0 < y < 1 - \sin\theta$$

for $\frac{\pi}{3} \le \theta < \frac{2\pi}{3}$.

The area of this region in the rectangle is given by

$$\int_0^{\pi/3} \left[1 - \sin\left(\frac{\pi}{3} + \theta\right)\right] d\theta + \int_{\pi/3}^{2\pi/3} (1 - \sin\theta)\, d\theta = \frac{2\pi}{3} - 2.$$

Thus, the probability that the tetrahedral die misses all lines is

$$\frac{\frac{2\pi}{3} - 2}{\frac{2\pi}{3}} = 1 - \frac{3}{\pi},$$

and the probability that the die lands on a line is

$$\frac{3}{\pi} \approx 0.95493.$$

Editor's Note. Michael Vowe points out that a more general result is known (and has been rediscovered multiple times): if d is the distance between the lines, and p is the perimeter of a convex polygon, then the probability the polygon lands on a line is $p/(\pi d)$ as long as the diameter of the polygon is less than or equal to d. See: Uspensky, J. V. (1937). *Introduction to Mathematical Probability*. New York: McGraw-Hill, pp. 251–255.

Also solved by Jacob Boswell & Chip Curtis, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Owen Byer and the Calculus II class at Eastern Mennonite University, Robert Calcaterra, Stephen J. Herschkorn, José Heber Nieto (Venezuela), Didier Pinchon (France), Volkhard Schindler (Germany), Randy K. Schwartz, Michael Vowe (Switzerland), and the proposers. There were three incomplete or incorrect solutions.

An infinite series involving the tangent function **December 2021**

2133. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.*

Evaluate the infinite sum

$$\sum_{k=1}^{\infty} 2^{-k} \tan\left(2^{-k}\right).$$

Solution by Seán M. Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.
Observe that for $x \in (0, \pi/2)$ we have

$$2\cot(2x) - \cot(x) = 2 \cdot \frac{\cot^2(x) - 1}{2\cot(x)} - \cot(x) = -\frac{1}{\cot(x)} = -\tan(x).$$

Setting $x = 2^{-k}$ in this trigonometric identity and multiplying both sides by 2^{-k} we obtain

$$\frac{1}{2^k} \tan\left(\frac{1}{2^k}\right) = -\frac{1}{2^k} \cot\left(\frac{1}{2^k}\right) - \frac{1}{2^{k-1}} \cot\left(\frac{1}{2^{k-1}}\right).$$

Consider the nth partial sum

$$S_n = \sum_{k=1}^{n} 2^{-k} \tan\left(2^{-k}\right).$$

From the equation above, we can write this partial sum as

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left[\frac{1}{2^k} \cot\left(\frac{1}{2^k}\right) - \frac{1}{2^{k-1}} \cot\left(\frac{1}{2^{k-1}}\right)\right] \\ &= -\cot(1) + 2^{-n} \cot\left(2^{-n}\right) \end{aligned}$$

since the sum telescopes. Therefore, the required sum is

$$\sum_{k=1}^{\infty} 2^{-k} \tan\left(2^{-k}\right) = \lim_{n\to\infty} S_n = -\cot(1) + \lim_{n\to\infty} 2^{-n} \cot\left(2^{-n}\right).$$

Letting $u = 2^{-n}$, we have

$$\lim_{n\to\infty} 2^{-n} \cot\left(2^{-n}\right) = \lim_{u\to 0^+} u\cot(u) = \lim_{u\to 0^+} \frac{u}{\tan(u)} = 1.$$

Therefore,

$$\sum_{k=1}^{\infty} 2^{-k} \tan\left(2^{-k}\right) = 1 - \cot(1).$$

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Paul Bracken, Brian Bradie, Robert Calcaterra, Hongwei Chen, CMC 328, Bruce Davis, Prithwijit De (India), Noah Garson (Canada), Subhankar Gayen (India), G. Greubel, Lixing Han, Mark Kaplan, Kelly McLenithan, Albert Natian, José Nieto (Venezuela), Northwestern University Math Problem Solving Group, Shing Hin Jimmy Pa (China), Didier Pinchon (France), Angel Plaza & Francisco Perdomo (Spain), Michael Reid, Henry Ricardo, Celia Schacht, Volkhard Schindler (Germany), Vishwesh Ravi Shrimali (India), Albert Stadler (Switzerland), Michael Vowe (Switzerland), and the proposer. There were three incomplete or incorrect solutions.

Questions about nilpotent matrices **December 2021**

2134. *Proposed by Antonio Garcia, Strasbourg, France.*

Let $N \in M_n(\mathbb{R})$ be a nilpotent matrix. In what follows, $X \in M_n(\mathbb{R})$.

(a) Show that there is always an X such that $N = X^2 + X - I$.

(b) Show that if n is odd, there is no X such that $N = X^2 + X + I$.

(c) Show that if $n = 2$ and $N \neq 0$, there is no X such that $N = X^2 + X + I$.

(d) Give examples, when $n = 4$, of an $N \neq 0$ and an X such that $N = X^2 + X + I$ and of an N with no X such that $N = X^2 + X + I$.

Solution by the Case Western Reserve University Problem Solving Group, Case Western Reserve University, Cleveland, OH.

(a) We claim that if M is a nilpotent matrix, then $I + M$ has a square root. Consider the formal power series

$$\sqrt{1+x} = \sum_{i=0}^{\infty} \binom{1/2}{i} x^i.$$

If $M^k = 0$, we set $x = M$ and obtain

$$\sqrt{1+M} = \sum_{i=0}^{k-1} \binom{1/2}{i} M^i.$$

Returning to the problem, we may rewrite the condition as

$$I + \frac{4}{5}N = \left(\frac{2\sqrt{5}}{5}X + \frac{\sqrt{5}}{5}I\right)^2.$$

Since $\frac{4}{5}N$ is nilpotent, we can solve for X using the claim above.

(b) Assume to the contrary that there exists such an X. Since N is nilpotent,

$$N^k = (X^2 + X + I)^k = 0$$

for some k. This implies that $(\lambda^2 + \lambda + 1)^k$ is a polynomial multiple of the minimal polynomial of X. Therefore X cannot have any real eigenvalues, since the eigenvalues of X are the roots of the minimal polynomial, and $(\lambda^2 + \lambda + 1)^k$ has no real roots. However, n is odd, which guarantees that X has a real eigenvalue. This is a contradiction.

(c) Suppose there exists such an X. Since we are in dimension two,

$$N^2 = (X^2 + X + I)^2 = 0.$$

This implies that $(\lambda^2 + \lambda + 1)^2$ is a polynomial multiple of the minimal polynomial of X. Since $N = X^2 + X + I \neq 0$, $(\lambda^2 + \lambda + 1)^2$ must be the minimal polynomial of X. The characteristic polynomial of X must have degree 2, and also must be a multiple of the minimal polynomial. But the minimal polynomial has degree 4. This is a contradiction.

(d) Let

$$X = \begin{bmatrix} -2 & -3 & -2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is straightforward to verify that the characteristic polynomial of X is

$$\lambda^4 + 2\lambda^3 + 3\lambda^2 + 2\lambda + 1 = \left(\lambda^2 + \lambda + 1\right)^2.$$

Let $N = X^2 + X + I$. One readily verifies that $N \neq 0$ and by the Cayley-Hamilton theorem, $N^2 = 0$. This solves the first part of the problem.

For the second part of the problem, let

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and assume there were an X such that $N = X^2 + X + I$. Since $N^4 = 0$, the minimal polynomial of X must be a polynomial multiple of $(\lambda^2 + \lambda + 1)$. Because $N^k \neq 0$ for $k < 4$, $(\lambda^2 + \lambda + 1)^4$ must be the minimal polynomial. The characteristic polynomial of X must be a multiple of the minimal polynomial and also must have degree 4. But $(\lambda^2 + \lambda + 1)^4$ has degree 8. This is a contradiction.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Jacob Boswell & Chip Curtis, Paul Budney, Robert Calcaterra, Lixing Han, Eugene A. Herman, Sonebi Omar (Morroco), Didier Pinchon (France), Michael Reid, and the proposer.

An exponential generating function **December 2021**

2135. *Proposed by Băetu Ioan, "Mihai Eminescu" National College, Botoşani, Romania.*

For $k \in \mathbb{Z}^+$, let $a_n(k)$ denote the number of elements $\sigma \in S_n$, the group of all permutations on an n-element set, such that $\sigma^k = e$, the identity element. We take $a_0(k) = 1$

by convention. Find a closed form for the exponential generating function

$$f_k(x) = \sum_{n=0}^{\infty} \frac{a_n(k)x^n}{n!}.$$

Solution by Jacob Boswell and Chip Curtis, Missouri Southern State University, Joplin, MO.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. A permutation σ satisfies $\sigma^k = e$ if and only if all of its disjoint cycles have lengths which are factors of k. Let $k_1, k_2, \ldots, k_r$ be the distinct factors of k. We note that the number of permutations of jk objects that are a product of j k-cycles is given by $(jk)!/k^j j!$. Breaking permutations with $\sigma^k = e$ into a product having j_i k_i-cycles, we see that

$$\begin{aligned} a_n(k) &= \sum_{\substack{(j_i)\in\mathbb{N}^r \\ \sum j_i k_i = n}} \binom{n}{j_1k_1,\, j_2k_2,\, \ldots,\, j_rk_r} \frac{(j_1k_1)!}{k_1^{j_1} j_1!} \cdots \frac{(j_rk_r)!}{k_r^{j_r} j_r!} \\ &= \sum_{\substack{(j_i)\in\mathbb{N}^r \\ \sum j_i k_i = n}} \frac{n!}{k_1^{j_1} \cdots k_r^{j_r} \cdot j_1! \cdots j_r!}, \end{aligned}$$

where

$$\binom{n}{i_1, i_2, \ldots, i_r} = \frac{n!}{i_1! i_2! \cdots i_r!}$$

is a multinomial coefficient. Thus,

$$\begin{aligned} f_k(x) &= \sum_{n=0}^{\infty} \left(\sum_{\substack{(j_i)\in\mathbb{N}^r \\ \sum j_i k_i = n}} \frac{1}{k_1^{j_1} \cdots k_r^{j_r} j_1! \cdots j_r!} \right) x^n \\ &= \left(\sum_{j_1}^{\infty} \frac{x^{j_1 k_1}}{k_1^{j_1} j_1!} \right) \cdots \left(\sum_{j_r}^{\infty} \frac{x^{j_r k_r}}{k_r^{j_r} j_r!} \right) \\ &= \prod_{i=1}^{r} \exp\left(\frac{x^{k_i}}{k_i} \right) = \exp\left(\sum_{d|k} \frac{x^d}{d} \right). \end{aligned}$$

Editor's Note. Albert Stadler notes that this result appears in an old paper of Chowla, Herstein, and Scott: Chowla, S., Herstein, I. N., Scott, W. R. (1952). The solutions of $x^d = 1$ in symmetric groups. *Norske Vid. Selsk.* 25: 29–31.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), CMC 328, Reiner Martin (Germany), José Heber Nieto (Venezuela), Michael Reid, and the proposer.

Solutions

Minimize the length of the tangent segment **October 2021**

2126. *Proposed by M. V. Channakeshava, Bengaluru, India.*

A tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the x-axis and y-axis at the points A and B, respectively. Find the minimum value of AB.

Solution by Kangrae Park (student), Seoul National University, Seoul, Korea.
We may assume that $a, b > 0$ and that the point of tangency $P = (\alpha, \beta)$ lies in the first quadrant. One readily verifies that the tangent line to the ellipse at P is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 1.$$

Therefore, A and B are $(a^2/\alpha, 0)$ and $(0, b^2/\beta)$, respectively. Note that

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$$

since the point P is on the ellipse. Applying the Cauchy-Schwarz inequality with

$$\mathbf{u} = \left(\frac{a^2}{\alpha}, \frac{b^2}{\beta}\right) \quad \text{and} \quad \mathbf{v} = \left(\frac{\alpha}{a}, \frac{\beta}{b}\right),$$

we obtain

$$\frac{a^4}{\alpha^2} + \frac{b^4}{\beta^2} = \left(\frac{a^4}{\alpha^2} + \frac{b^4}{\beta^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right) = (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) \geq (\mathbf{u}\cdot\mathbf{v})^2 = (a+b)^2.$$

It follows that

$$AB = \sqrt{\frac{a^4}{\alpha^2} + \frac{b^4}{\beta^2}} \geq a + b.$$

This lower bound is attained if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. A straightforward calculation shows that this occurs if and only if

$$\alpha^2 = \frac{a^3}{a+b} \quad \text{and} \quad \beta^2 = \frac{b^3}{a+b}.$$

This gives the esthetically pleasing result that when AB attains its minimum value of $a+b$, we have $PB = a$ and $PA = b$.

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Yagub Aliyev (Azerbaijan), Michel Bataille (France), Bejmanin Bittner, Khristo Boyadzhiev, Paul Bracken, Brian Bradie, Robert Calcaterra, Hongwei Chen, Joowon Chung (South Korea), Robert Doucette, Rob Downes, Eagle Problem Solvers (Georgia Southern University), Habib Y. Far, John Fitch, Dmitry Fleischman, Noah Garson (Canada), Kyle Gatesman, Subhankar Gayen (India), Jan Grzesik, Emmett Hart, Eugene A. Herman, David Huckaby, Tom Jager, Walther Janous (Austria), Mark Kaplan & Michael Goldenberg, Kee-Wai Lau (Hong Kong), Lucas Perry & Alexander Perry, Didier Pinchon (France), Ivan Retamoso, Celia Schacht, Randy Schwartz, Ioannis Sfikas (Greece), Vishwesh Ravi Shrimali (India), Albert Stadler (Switzerland), Seán M. Stewart (Saudi Arabia), David Stone & John Hawkins, Nora Thornber, R. S. Tiberio, Michael Vowe (Switzerland), Lienhard Wimmer (Germany), and the proposer. There were seventeen incomplete or incorrect solutions.

Two idempotent matrices **October 2021**

2127. *Proposed by Jeff Stuart, Pacific Lutheran University, Tacoma, WA and Roger Horn, Tampa, FL.*

Suppose that $A, B \in M_{n\times n}(\mathbb{C})$ is such that $AB = A$ and $BA = B$. Show that

(a) A and B are idempotent and have the same null space.

(b) If $1 \leq \operatorname{rank} A < n$, then there are infinitely many choices of B that satisfy the hypotheses.

(c) $A = B$ if and only if $A - I$ and $B - I$ have the same null space.

Solution by Michel Bataille, Rouen, France.
(a) The fact that $A^2 = A$ and $B^2 = B$ follows from:

$$A^2 = (AB)A = A(BA) = AB = A, \qquad B^2 = (BA)B = B(AB) = BA = B.$$

In addition, if X is a column vector and $AX = 0$, then $BAX = 0$, that is, $BX = 0$. Thus, $\ker A \subseteq \ker B$. Similarly, if $BX = 0$, then $ABX = 0$. Hence $AX = 0$ so that $\ker B \subseteq \ker A$. We conclude that $\ker A = \ker B$.

(b) Let $r = \operatorname{rank}(A)$. Since A is idempotent, we have $\operatorname{range}(A) \oplus \ker A = \mathbb{C}^n$. Since $AX = X$ if $X \in \operatorname{range}(A)$ and $\dim(\operatorname{range}(A)) = r$, it follows that $A = PJ_rP^{-1}$ for some invertible $n \times n$ matrix P and

$$J_r = \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right),$$

where I_r denotes the $r \times r$ unit matrix and O a null matrix of the appropriate size. Consider the matrices $B = PB'P^{-1}$ with

$$B' = \left(\begin{array}{c|c} I_r & O \\ \hline C & O \end{array}\right),$$

where C is an arbitrary $(n-r) \times r$ matrix with complex entries. There are infinitely many such matrices B, and we calculate

$$AB = PJ_rP^{-1}PB'P^{-1} = PJ_rB'P^{-1} = PJ_rP^{-1} = A,$$

and

$$BA = PB'P^{-1}PJ_rP^{-1} = PB'J_rP^{-1} = PB'P^{-1} = B.$$

(c) Clearly, $A - I$ and $B - I$ have the same null space if $A = B$. Conversely, suppose that $\ker(A - I) = \ker(B - I)$. Let X be a column vector. Since $(A - I)A = O$, the vector AX is in $\ker(A - I)$, hence is in $\ker(B - I)$. This means that $(B - I)AX = 0$, that is, $BX = AX$ (since $BA = B$). Since X is arbitrary, we can conclude that $A = B$.

Also solved by Paul Budney, Robert Calcaterra, Hongwei Chen, Robert Doucette, Dmitry Fleischman, Kyle Gatesman, Eugene A. Herman, Tom Jager, Rachel McMullan, Thoriq Muhammad (Indonesia), Didier Pinchon (France), Michael Reid, Randy Schwartz, Omar Sonebi (Morroco), and the proposer. There was one incomplete or incorrect solution.

Two exponential inequalities **October 2021**

2128. *Proposed by George Stoica, Saint John, NB, Canada.*

Let $0 < a < b < 1$ and $\epsilon > 0$ be given. Prove the existence of positive integers m and n such that $(1 - b^m)^n < \epsilon$ and $(1 - a^m)^n > 1 - \epsilon$.

Solution by Robert Doucette, McNeese State University, Lake Charles, LA.
It is well known that

$$\lim_{x\to 0}(1 - x)^{1/x} = e^{-1}.$$

Suppose $0 < \alpha < 1$. Then, since $\alpha^x \to 0^+$ as $x \to \infty$,

$$\lim_{x\to\infty}(1 - \alpha^x)^{\alpha^{-x}} = e^{-1}.$$

Hence,

$$\lim_{x\to\infty}(1 - \alpha^x)^{\beta^{-x}} = \lim_{x\to\infty}\left[(1 - \alpha^x)^{\alpha^{-x}}\right]^{(\beta/\alpha)^{-x}} = \begin{cases} 0, & \text{if } 0 < \beta < \alpha < 1 \\ 1, & \text{if } 0 < \alpha < \beta < 1 \end{cases}.$$

Choose c and d such that $0 < a < c < d < b < 1$. Note that $c^{-x} - d^{-x} \to \infty$ as $x \to \infty$.

By the limits established above, there exists a positive integer m such that

$$(1-b^m)^{d^{-m}} < \epsilon,\ (1-a^m)^{c^{-m}} > 1-\epsilon, \text{ and } c^{-m} - d^{-m} > 1.$$

There also exists a positive integer n such that $d^{-m} < n < c^{-m}$. Therefore,

$$(1-b^m)^n < (1-b^m)^{d^{-m}} < \epsilon \text{ and } (1-a^m)^n > (1-a^m)^{c^{-m}} > 1-\epsilon.$$

Also solved by Levent Batakci, Michel Bataille (France), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Bruce Burdick, Michael Cohen, Dmitry Fleischman, Kyle Gatesman, Michael Goldenberg & Mark Kaplan, Eugene Herman, Miguel Lerma, Reiner Martin (Germany), Raymond Mortini (France), Michael Nathanson, Moubinool Omajee (France), Didier Pinchon (France), Albert Stadler (Switzerland), Omar Sonebi (Morroco), and the proposer.

Two improper integrals **October 2021**

2129. *Proposed by Vincent Coll and Daniel Conus, Lehigh University, Bethlehem, PA and Lee Whitt, San Diego, CA.*

Determine whether the following improper integrals are convergent or divergent.

(a) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{2^k}\right) dx$

(b) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{3^k}\right) dx$

Solution by Gerald A. Edgar, Denver, CO.
(a) The integral diverges. For $0 < x < 1$ we have

$$\begin{aligned}\log\frac{1}{1-x} &= \sum_{n=1}^{\infty}\frac{1}{n}x^n = \sum_{k=0}^{\infty}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{n}x^n\right)\\ &\le \sum_{k=0}^{\infty}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{2^k}x^{2^k}\right) = \sum_{k=0}^{\infty}\left(\frac{2^k}{2^k}x^{2^k}\right) = \sum_{k=0}^{\infty}x^{2^k}.\end{aligned}$$

Therefore,

$$\exp\left(\sum_{k=0}^{\infty}x^{2^k}\right) \ge \frac{1}{1-x}.$$

The integral (a) diverges by comparison with the divergent integral $\int_0^1 dx/(1-x)$.

(b) The integral converges. We will need an estimate for a harmonic sum. The function $1/x$ is decreasing, so for $k \ge 1$

$$\sum_{n=3^{k-1}}^{3^k-1}\frac{1}{n} > \int_{3^{k-1}}^{3^k}\frac{dx}{x} = \log 3.$$

Now, for $0 < x < 1$ we have

$$\log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{1}{n} x^n = \sum_{k=1}^{\infty} \left(\sum_{n=3^{k-1}}^{3^k - 1} \frac{1}{n} x^n \right)$$

$$> \sum_{k=1}^{\infty} \left(\sum_{n=3^{k-1}}^{3^k - 1} \frac{1}{n} \right) x^{3^k} > \sum_{k=1}^{\infty} (\log 3) x^{3^k}.$$

Let $r = 1/\log 3$, so that $0 < r < 1$. Then

$$r \log \frac{1}{1-x} > \sum_{k=1}^{\infty} x^{3^k},$$

$$\log \frac{1}{(1-x)^r} + 1 > \sum_{k=0}^{\infty} x^{3^k},$$

$$\frac{e}{(1-x)^r} > \exp\left(\sum_{k=0}^{\infty} x^{3^k} \right).$$

The integral (b) converges by comparison with the convergent integral

$$\int_0^1 \frac{e}{(1-x)^r}\, dx.$$

Editor's Note. A more detailed analysis shows that

$$\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{\alpha^k} \right) dx$$

converges if $\alpha > e$ and diverges if $1 \le \alpha \le e$.

Also solved by Michael Bataille (France), Robert Calcaterra, Dmitry Fleischman, Eugene A. Herman, Walther Janous (Austria), Albert Natian, Moubinool Omarjee (France), Didier Pinchon (France), Albert Stadler (Switzerland), and the proposers. There was one incomplete or incorrect solution.

When does the circumcenter lie on the incircle? October 2021

2130. *Proposed by Florin Stanescu, Şerban Cioculescu School, Găeşti, Romania.*

Given the acute $\triangle ABC$, let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Choose $P, R \in \overleftrightarrow{AB}$, $S, T \in \overleftrightarrow{BC}$, $Q, U \in \overleftrightarrow{AC}$ so that

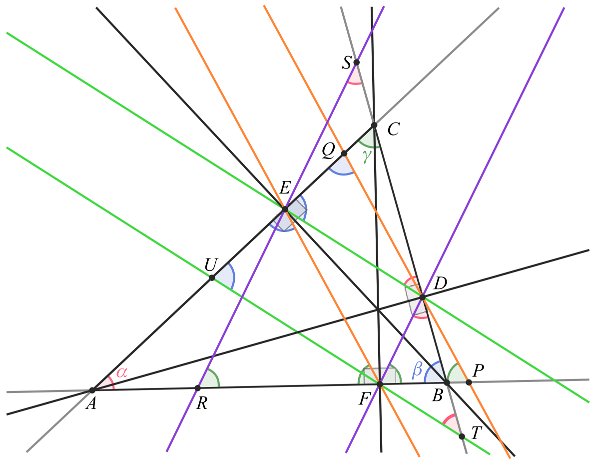
$$D \in \overleftrightarrow{PQ},\ E \in \overleftrightarrow{RS},\ F \in \overleftrightarrow{TU} \ \text{ and } \ \overleftrightarrow{PQ} \parallel \overleftrightarrow{EF},\ \overleftrightarrow{RS} \parallel \overleftrightarrow{DF},\ \overleftrightarrow{TU} \parallel \overleftrightarrow{DE}.$$

Show that

$$\frac{PQ + RS - TU}{AB} + \frac{RS + TU - PQ}{BC} + \frac{TU + PQ - RS}{AC} = 2\sqrt{2}$$

if and only if the circumcenter of $\triangle ABC$ lies on the incircle of $\triangle ABC$.

Solution by the Fejéntaláltuka Szeged Problem Solving Group, University of Szeged, Szeged, Hungary.



Let O and I be the circumcenter and the incenter of $\triangle ABC$. Then Euler's theorem states that $OI^2 = R(R-2r)$, where R and r are the circumradius and the inradius of the triangle, respectively. Now O lies on the incircle if and only if $R(R-2r) = r^2$, which is equivalent to $\left(\frac{r}{R}\right)^2 + 2\frac{r}{R} - 1 = 0$. Therefore, $\frac{r}{R} = \sqrt{2} - 1$ since $\frac{r}{R} > 0$. Since $\cos\alpha + \cos\beta + \cos\gamma = 1 + \frac{r}{R}$ in any triangle, we can reduce the original condition to $\cos\alpha + \cos\beta + \cos\gamma = \sqrt{2}$ where α, β and γ are the angles of $\triangle ABC$.

We have

$$\begin{aligned} DE^2 &\overset{(1)}{=} CD^2 + CE^2 - 2CD \cdot CE \cos\gamma \\ &\overset{(2)}{=} (CA\cos\gamma)^2 + (BC\cos\gamma)^2 - 2(CA\cos\gamma)(BC\cos\gamma)\cos\gamma \\ &= (CA^2 + BC^2 - 2CA \cdot BC\cos\gamma)\cos^2\gamma \overset{(3)}{=} AB^2\cos^2\gamma, \end{aligned}$$

where (1) and (3) are the result of the law of cosines applied to $\triangle CDE$ and $\triangle ABC$, respectively, and (2) follows from the fact that CD and CE are altitudes. Since $\triangle ABC$ is acute, $\cos\alpha > 0$, so

$$DE = AB\cos\gamma, \text{ and similarly } EF = BC\cos\alpha \text{ and } FD = CA\cos\beta. \tag{1}$$

Because $\angle BFC$ and $\angle BEC$ are right angles, E and F lie on the circle with diameter BC, thus $BCEF$ is a cyclic quadrilateral. Hence, $m\angle EFA = 180^\circ - m\angle BFE = m\angle ECB = \gamma$ and $m\angle AEF = 180^\circ - m\angle FEC = m\angle CBF = \beta$. We can similarly see that $m\angle FDB = m\angle CDE = \alpha$, $m\angle DEC = \beta$ and $m\angle BFD = \gamma$. Since $PQ \parallel EF$, $RS \parallel FD$ and $TU \parallel DE$ we have

$$\begin{aligned} m\angle RSB &= m\angle FDB = \alpha = m\angle CDE = m\angle CTU, \\ m\angle AQP &= m\angle AEF = \beta = m\angle DEC = m\angle TUC, \\ m\angle BRS &= m\angle BFD = \gamma = m\angle EFA = m\angle QPA. \end{aligned}$$

Therefore, the following triangles are all isosceles (because they all have two congruent angles): $\triangle DQE$, $\triangle EDS$, $\triangle ERF$, $\triangle FEU$, $\triangle FTD$, and $\triangle DFP$. Therefore,

$$DQ = DE = ES, \; RE = EF = FU, \text{ and } TF = FD = PD,$$

which (by (1)) leads to

$$\begin{aligned} PQ &= PD + DQ = FD + DE = CA\cos\beta + AB\cos\gamma, \\ RS &= RE + ES = EF + DE = BC\cos\alpha + AB\cos\gamma, \\ TU &= TF + FU = FD + EF = CA\cos\beta + BC\cos\alpha. \end{aligned}$$

Substituting these into our original statement, we get that

$$\frac{PQ + RS - TU}{AB} + \frac{RS + TU - PQ}{BC} + \frac{TU + PQ - RS}{CA} = 2\left(\cos\gamma + \cos\alpha + \cos\beta\right).$$

In the first paragraph, we showed that the right side of the last equation equals $2\sqrt{2}$ if and only if the circumcenter lies on the incircle, which is exactly what we wanted to prove.

Also solved by Michel Bataille (France), Kyle Gatesman, Volkhard Schindler (Germany), Albert Stadler (Switzerland), and the proposer.

Answers

Solutions to the Quickies from page 407.

A1123. We will need the fact that if f satisfies P_2, then

$$f\left(\frac{n}{n+1}A_1 + \frac{1}{n+1}A_2\right) = \frac{n}{n+1}f(A_1) + \frac{1}{n+1}(A_2). \tag{1}$$

We proceed by induction. When $n = 1$ this is just condition P_2. Let

$$X = \frac{n+1}{n+2}A_1 + \frac{1}{n+2}A_2 \quad \text{and} \quad Y = \frac{1}{n+2}A_1 + \frac{n+1}{n+2}A_2.$$

We have

$$X = \frac{n}{n+1}A_1 + \frac{1}{n+1}Y \quad \text{and} \quad Y = \frac{1}{n+1}X + \frac{n}{n+1}A_2,$$

so, by the induction hypothesis,

$$f(X) = \frac{n}{n+1}f(A_1) + \frac{1}{n+1}f(Y) \quad \text{and} \quad f(Y) = \frac{1}{n+1}f(X) + \frac{n}{n+1}f(A_2).$$

Eliminating $f(Y)$ gives the desired result.

We will now use induction to show that $P_2 \Rightarrow P_n$ for all $n \geq 2$, the case $n = 2$ being immediate. Let

$$G = \frac{1}{n+1}\sum_{i=1}^{n+1} A_i \quad \text{and} \quad G' = \frac{1}{n}\sum_{i=1}^{n} A_i.$$

Hence,

$$G = \frac{n}{n+1}G' + \frac{1}{n+1}A_{n+1}.$$

Therefore,

$$f(G) = \frac{n}{n+1}f(G') + \frac{1}{n+1}f(A_{n+1}) \text{ (by (1))}$$

$$= \frac{n}{n+1}\left(\frac{1}{n}\sum_{i=1}^{n} f(A_i)\right) + \frac{1}{n+1} f(A_{n+1}) \text{ (by induction)}$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1} f(A_i),$$

as desired.

To show that $P_n \Rightarrow P_2$, let $M = (A_1 + A_2)/2$. Then,

$$f\left(\frac{1}{n}\left(M + M + \sum_{i=3}^{n} A_i\right)\right) = f\left(\frac{1}{n}\left(A_1 + A_2 + \sum_{i=3}^{n} A_i\right)\right)$$

$$\frac{1}{n}\left(2f(M) + \sum_{i=3}^{n} f(A_i)\right) = \frac{1}{n}\left(f(A_1) + f(A_2) + \sum_{i=3}^{n} f(A_i)\right) \text{ (by } P_n\text{)},$$

so $f(M) = \left(f(A_1) + f(A_2)\right)/2$ as we wished to show.

A1124. The answer is yes. Note that if $1/F_n < x \le 1/F_{n-1}$ with $n \ge 3$, then

$$0 < x - \frac{1}{F_n} \le \frac{1}{F_{n-1}} - \frac{1}{F_n} \le \frac{2}{F_n} - \frac{1}{F_n} = \frac{1}{F_n}.$$

For $y \le 1$, let $g(y)$ denote the unique positive integer m such that

$$\frac{1}{F_m} < y \le \frac{1}{F_{m-1}}.$$

The relation above shows that $g(x - 1/F_n) > n$. Now take $x_1 = 1$, $n_1 = 3$ and recursively define

$$x_{k+1} = x_k - \frac{1}{F_{n_k}} \quad \text{and} \quad n_{k+1} = g(x_{k+1}).$$

This gives

$$1 = \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_6} + \frac{1}{F_9} + \frac{1}{F_{11}} + \frac{1}{F_{21}} + \frac{1}{F_{23}} + \cdots.$$

Note that the analogous result holds for any a such that

$$0 < a \le \sum_{n=1}^{\infty} \frac{1}{F_n} = 3.35988\ldots.$$

Solutions

Evaluate the definite integral **June 2021**

2121. *Proposed by Seán M. Stewart, Bomaderry, Australia.*

Evaluate

$$\int_0^{\frac{1}{2}} \frac{\arctan x}{x^2 - x - 1}\, dx.$$

Solution by Lixing Han, University of Michigan-Flint, Flint, MI and Xinjia Tang, Changzhou University, Changzhou, China.
Using the substitution

$$x = \frac{\frac{1}{2} - t}{1 + \frac{1}{2}t} = \frac{1 - 2t}{2 + t},$$

we obtain

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{\arctan x}{x^2 - x - 1} dx &= \int_{\frac{1}{2}}^0 \frac{\arctan\left(\frac{\frac{1}{2} - t}{1 + \frac{1}{2}t}\right)}{\left(\frac{1-2t}{2+t}\right)^2 - \frac{1-2t}{2+t} - 1} \cdot \frac{-5}{(2+t)^2} dt \\
&= \int_0^{\frac{1}{2}} \frac{\arctan\left(\frac{1}{2}\right) - \arctan t}{t^2 - t - 1} dt \\
&= \int_0^{\frac{1}{2}} \frac{\arctan\left(\frac{1}{2}\right)}{t^2 - t - 1} dt - \int_0^{\frac{1}{2}} \frac{\arctan t}{t^2 - t - 1} dt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{\arctan x}{x^2 - x - 1} dx &= \frac{1}{2} \arctan\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{dt}{t^2 - t - 1} \\
&= \frac{1}{2} \arctan\left(\frac{1}{2}\right) \frac{1}{\sqrt{5}} \ln\left(\left|\frac{2t - \sqrt{5} - 1}{2t + \sqrt{5} - 1}\right|\right)\Bigg|_0^{1/2} \\
&= -\frac{1}{2\sqrt{5}} \arctan\left(\frac{1}{2}\right) \ln\left(\frac{\sqrt{5} + 1}{\sqrt{5} - 1}\right) \\
&= -\frac{1}{\sqrt{5}} \arctan\left(\frac{1}{2}\right) \ln\left(\frac{\sqrt{5} + 1}{2}\right).
\end{aligned}$$

Also solved by Brian Bradie, Hongwei Chen, Herevé Grandmontagne (France), Eugene A. Herman, Omran Kouba (Syria), Kee-Wai Lau (China), Albert Natian, Moobinool Omarjee (France), Didier Pichon (France), Albert Stadler (Switzerland), Fejéntaláltuka Szöged (Hungary), and the proposer. There were four incomplete or incorrect solutions.

Find the maximum gcd **June 2021**

2122. *Proposed by Ahmad Sabihi, Isfahan, Iran.*

Let

$$G(m,k) = \max\{\gcd((n+1)^m + k, n^m + k) | n \in \mathbb{N}\}.$$

Compute $G(2,k)$ and $G(3,k)$.

Solution by Michael Reid, University of Central Florida, Orlando, FL.
We show that for $k \in \mathbb{Z}$, $G(2,k) = |4k+1|$, and

$$G(3,k) = \begin{cases} 27k^2+1 & \text{if } k \text{ is even,} \\ \left(27k^2+1\right)/4 & \text{if } k \text{ is odd.} \end{cases}$$

The polynomial identity

$$(2n+3)(n^2+k) - (2n-1)((n+1)^2+k) = 4k+1$$

shows that

$$\gcd((n+1)^2+k, n^2+k) \text{ divides } 4k+1,$$

and thus is at most $|4k+1|$. Hence, $G(2,k) \leq |4k+1|$.

Suppose $k > 0$, and let $n = 2k \in \mathbb{N}$. We have

$$n^2 + k = k(4k+1) \text{ and } (n+1)^2 + k = (k+1)(4k+1),$$

both of which are divisible by $4k+1$. Thus

$$\gcd((n+1)^2+k, n^2+k) = 4k+1 = |4k+1|,$$

so $G(2,k) = |4k+1|$ in this case.

For $k=0$, we have $\gcd((n+1)^2, n^2) = 1$ for all $n \in \mathbb{N}$, so $G(2,0) = 1 = |4k+1|$ in this case.

Suppose $k < 0$, and consider $n = -(2k+1) \in \mathbb{N}$. Then

$$n^2 + k = (k+1)(4k+1) \text{ and } (n+1)^2 + k = k(4k+1)$$

are each divisible by $4k+1$. Thus

$$\gcd((n+1)^2+k, n^2+k) = |4k+1|,$$

so $G(2,k) = |4k+1|$ in this case as well.

Now we consider $G(3,k)$. The polynomial identity

$$\begin{aligned}(6n^2 - 9nk - 3n + 9k + 1)&((n+1)^3 + k) \\ &- (6n^2 - 9nk + 15n - 18k + 10)(n^3 + k) = 27k^2 + 1\end{aligned}$$

shows that

$$\gcd((n+1)^3 + k, n^3 + k) \text{ divides } 27k^2 + 1. \tag{1}$$

For all n, $(n+1)^3+k$ and n^3+k have opposite parity, so their greatest common divisor is odd. If k is odd, then $27k^2+1 = 4\big((27k^2+1)/4\big)$ is a product of two integers. Since the greatest common divisor is odd, and divides this product,

$$\gcd((n+1)^3+k, n^3+k) \text{ divides } \frac{27k^2+1}{4}. \tag{2}$$

For $k=0$, we have $\gcd((n+1)^3, n^3) = 1$ for all n, so $G(3,0) = 27k^2+1 = 1$.

For nonzero k, take $n = 3k(9k-1)/2$, which is a positive integer. We calculate

$$n^3+k = (27k^2+1)\left(\frac{(729k^3-243k^2+8)k}{8}\right)$$

and

$$(n+1)^3+k = (27k^2+1)\left(\frac{729k^4-243k^3+162k^2-28k+8}{8}\right).$$

If k is even, each factor above is an integer, which shows that

$$27k^2+1 \text{ divides } \gcd((n+1)^3+k, n^3+k).$$

With (1), we have

$$\gcd((n+1)^3+k, n^3+k) = 27k^2+1,$$

so $G(3,k) = 27k^2+1$ when k is even.

If k is odd, rewrite the above factorizations as

$$n^3+k = \left(\frac{27k^2+1}{4}\right)\left(\frac{(729k^3-243k^2+8)k}{2}\right)$$

and

$$(n+1)^3+k = \left(\frac{27k^2+1}{4}\right)\left(\frac{729k^4-243k^3+162k^2-28k+8}{2}\right),$$

again, all factors being integers. Therefore

$$\frac{27k^2+1}{4} \text{ divides } \gcd((n+1)^3+k, n^3+k).$$

With (2), we conclude that

$$\gcd((n+1)^3+k, n^3+k) = \frac{27k^2+1}{4},$$

so $G(3,k) = (27k^2+1)/4$ when k is odd.

Also solved by Hongwei Chen, Eagle Problem Solvers (Georgia Southern University), Dmitry Fleischman, George Washington University Math Problem Solving Group, Eugene A. Herman, Walther Janous (Austria), Didier Pinchon (France), Albert Stadler (Switzerland), Enrique Treviño, and the proposer. There were two incomplete or incorrect solutions.

Find the expected winnings **June 2021**

2123. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

An urn contains n balls. Each ball is labeled with exactly one number from the set

$$\{a_1, a_2, \ldots, a_n\}, a_1 > a_2 > \cdots > a_n$$

(so no two balls have the same number). Balls are randomly selected from the urn and discarded. At each turn, if the number on the ball drawn was the largest number remaining in the urn, you win the dollar amount of that ball. Otherwise, you win nothing. Find the expected value of your total winnings after n draws.

Solution by Enrique Treviño, Lake Forest College, Lake Forest, IL.
Let X be the random variable described. Then $X = a_{i_1} + a_{i_2} + \cdots + a_{i_j}$ with $1 = i_1 < i_2 < \cdots < i_j \leq n$. Therefore, the expected value will be

$$\mathbb{E}[X] = \sum_{k=1}^{n} c_k a_k,$$

where c_k is the probability that the summand a_k appears in X. For a_k to appear, the ball labeled a_k must be drawn after those labeled $a_1, a_2, \ldots, a_{k-1}$, but this only happens if the permutation of $\{a_1, \ldots, a_k\}$ ends in a_k. This occurs with probability $1/k$. Therefore

$$\mathbb{E}[X] = a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \cdots + \frac{1}{n}a_n.$$

Also solved by Robert A. Agnew, Alan E. Berger, Brian Bradie, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Paul Budney, Michael P. Cohen, Eagle Problem Solvers (Georgia Southern University), John Fitch, Dmitry Fleischman, Fresno State Journal Problem Solving Group, GWstat Problem Solving Group, George Washington University Problems Group, Victoria Gudkova (student) (Russia), Stephen Herschkorn, Shing Hin Jimmy Pa (Canada), David Huckaby, Walther Janous (Austria), Omran Kouba (Syria), Ken Levasseur, Reiner Martin (Germany), Kelly D. McLenithan, José Nieto (Venezuela), Didier Pinchon (France), Michael Reid, Edward Schmeichel, Albert Stadler (Switzerland), Fejéntaláltuka Szöged, and the proposer. There were two incomplete or incorrect solutions.

A sum over the partitions of *n* **June 2021**

2124. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.*

For a positive integer n, prove that

$$\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_k=n \\ \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k>0}} (-1)^{n-\lambda_1} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_k}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}} = \frac{1}{n!},$$

where the sum runs over all the partitions of n.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
Put $s_1 = \lambda_1 - \lambda_2$, $s_2 = \lambda_2 - \lambda_3, \ldots, s_{k-1} = \lambda_{k-1} - \lambda_k$, $s_k = \lambda_k$. Clearly, we have $s_i \geq 0$, $s_1 + s_2 + \cdots + s_k = \lambda_1$, and $s_1 + 2s_2 + 3s_3 + \cdots + ks_k = n$. Moreover, for fixed λ_1, if we vary k and $\lambda_2, \lambda_3, \ldots, \lambda_k$ satisfying the conditions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$

and $\lambda_1+\lambda_2+\cdots+\lambda_k=n$, we obtain all the sequences of s_i's satisfying $s_i\geq 0$, $s_1+s_2+\cdots+s_k=\lambda_1$ and $s_1+2s_2+3s_3+\cdots+ks_k=n$.

Now

$$\frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_k}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}}=\frac{\lambda_1!}{s_1!s_2!\cdots s_k!(1!)^{s_1}(2!)^{s_2}\cdots(k!)^{s_k}}.$$

We note that

$$\frac{n!}{s_1!s_2!\cdots s_k!(1!)^{s_1}(2!)^{s_2}\cdots(k!)^{s_k}}$$

is the number of partitions of the set $\{1,2,\ldots,n\}$ into s_i blocks of size i, for $i=1,2,\ldots,k$. For fixed λ_1, if we sum these expressions for all values of the s_i's and k such that $s_i\geq 0$, $s_1+s_2+\cdots+s_k=\lambda_1$ and $s_1+2s_2+3s_3+\cdots+ks_k=n$, we obtain the number of partitions of the set $\{1,2,\ldots,n\}$ into λ_1 blocks, that is the Stirling number of second kind $\left\{{n \atop \lambda_1}\right\}$. Therefore

$$\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_k=n\\ \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k>0}}(-1)^{n-\lambda_1}\frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_k}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}}=\frac{1}{n!}\sum_{\lambda_1=1}^{n}(-1)^{n-\lambda_1}\lambda_1!\left\{{n \atop \lambda_1}\right\}. \tag{1}$$

It is well known that

$$\sum_{\lambda_1=1}^{n}\left\{{n \atop \lambda_1}\right\}x(x-1)(x-2)\cdots(x-\lambda_1+1)=x^n.$$

Substituting $-x$ for x we obtain

$$\sum_{\lambda_1=1}^{n}(-1)^{n-\lambda_1}\left\{{n \atop \lambda_1}\right\}x(x+1)(x+2)\cdots(x+\lambda_1-1)=x^n.$$

For $x=1$, we have

$$\sum_{\lambda_1=1}^{n}(-1)^{n-\lambda_1}\left\{{n \atop \lambda_1}\right\}\lambda_1!=1,$$

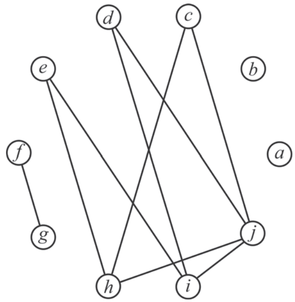
hence the right-hand side of (1) is $1/n!$ and we are done.

Also solved by Albert Stadler (Switzerland) and the proposer.

A graph involving a partition of 100 into ten parts **June 2021**

2125. *Proposed by Freddy Barrera, Colombia Aprendiendo, and Bernardo Recamán, Universidad Sergio Arboleda, Bogotá, Colombia.*

Given a collection of positive integers, not necessarily distinct, a graph is formed as follows. The vertices are these integers and two vertices are connected if and only if they have a common divisor greater than 1. Find an assignment of ten positive integers totaling 100 that results in the graph shown below.



Solution by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

With the labeling above,

$$(a, b, c, d, e, f, g, h, i, j) = (1, 1, 7, 9, 10, 11, 11, 14, 15, 21)$$

is a solution. Note that each of e, h, i, and j must have at least two prime divisors, since each is adjacent to two vertices that are not adjacent to each other. The simplest option is $e = pq$, $h = qr$, $i = rs$, and $j = ps$ with p, q, r, and s prime. Assuming $\{p, q, r, s\} = \{2, 3, 5, 7\}$, the vertices e, h, i, and j must consist of two of the three pairs $(6, 35)$, $(10, 21)$, and $(14, 15)$. The possibility with the smallest sum is $\{e, h, i, j\} = \{10, 14, 15, 21\}$. If we take $a = b = 1$ and $f = g = 11$, this forces $c + d = 16$. Assuming that c and d are powers of distinct primes from $\{2, 3, 5, 7\}$, we must have $(c, d) = (7, 9)$ or $(c, d) = (9, 7)$. The former forces $(e, h, i, j) = (10, 14, 15, 21)$, which yields the solution above. The latter gives a solution with $(e, h, i, j) = (10, 15, 14, 21)$.

A more detailed analysis shows that, in fact, these are the only solutions.

Also solved by Brian D. Beasley, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Dmitry Fleischman, George Washington University Problems Group, Kelly D. McLenithan & Stephen C. Mortenson, Lane Nielsen, José Heber Nieto (Venezuela), Didier Pinchon (France), Randy K. Schwartz, Albert Stadler (Switzerland), and the proposers.

Solutions

Evaluating an improper integral **April 2021**

2116. *Proposed by Fook Sung Wong, Temasek Polytechnic, Singapore.*

Evaluate

$$\int_0^\infty \frac{e^{\cos x}\cos(\alpha x+\sin x)}{x^2+\beta^2}\,dx,$$

where α and β are positive real numbers.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

We claim the answer is $\dfrac{\pi}{2\beta}\exp(e^{-\beta}-\alpha\beta)$.

Consider the meromorphic function

$$F(z) = \frac{g(z)}{z^2 + \beta^2}, \quad \text{where } g(z) = \exp(e^{iz} + i\alpha z).$$

If $z = x + iy$ with $x, y \in \mathbb{R}$ and $y \geq 0$, then

$$|g(z)| = \exp\left(\operatorname{Re}(e^{iz} + i\alpha z)\right) = \exp\left(e^{-y}\cos(x) - \alpha y\right) \leq \exp\left(e^{-y} - \alpha y\right) \leq e.$$

For $R > \beta$, consider the closed contour Γ_R consisting of the line segment $[-R, R]$ followed by the semicircle γ_R parametrized by $\theta \mapsto Re^{i\theta}$ for $\theta \in [0, \pi]$. The only singularity that F has inside the domain bounded by Γ_R is a simple pole at $z = i\beta$ with residue

$$\operatorname{Res}(F, i\beta) = \frac{g(i\beta)}{2i\beta} = \frac{\exp(e^{-\beta} - \alpha\beta)}{2i\beta}.$$

By the residue theorem we have

$$\int_{\Gamma_R} F(z)dz = 2i\pi \operatorname{Res}(F, i\beta) = \frac{\pi}{\beta}\exp(e^{-\beta} - \alpha\beta).$$

But

$$\begin{aligned}\int_{\Gamma_R} F(z)dz &= \int_{-R}^{R} F(x)dx + \int_{\gamma_R} F(z)dz \\ &= 2\int_0^R \frac{e^{\cos x}\cos(\alpha x + \sin x)}{x^2 + \beta^2}dx + \epsilon_R,\end{aligned}$$

where

$$\epsilon_R = \int_{\gamma_R} F(z)dz.$$

Since $R > \beta$, we have

$$|\epsilon_R| \leq \pi R \sup_{\theta \in [0,\pi]} |F(Re^{i\theta})| \leq \pi e \frac{R}{R^2 - \beta^2}.$$

Thus $\lim\limits_{R\to\infty} \epsilon_R = 0$. Therefore

$$2\int_0^\infty \frac{e^{\cos x}\cos(\alpha x + \sin x)}{x^2 + \beta^2}dx = \frac{\pi}{\beta}\exp(e^{-\beta} - \alpha\beta),$$

as claimed.

Also solved by Khristo N. Boyadzhiev, Hongwei Chen, John Fitch, G. C. Greubel, Eugene A. Herman, Rafe Jones, Kee-Wai Lau (Hong Kong), Kelly D. McLenithan, Raymond Mortini (France) & Rudolf Rupp (Germany), Moubinool Omarjee (France), Didier Pinchon (France), Ahmad Sabihi (Iran), Albert Stadler (Switzerland), Seán M. Stewart (Australia), and the proposer. There were two incomplete or incorrect solutions.

A factorial Diophantine equation **April 2021**

2117. *Proposed by Ahmad Sabihi, Isfahan, Iran.*

Find all positive integer solutions to the equation

$$(m+1)^n = m! + 1.$$

Solution by Michael Kardos (student), East Carolina University, Greenville, NC.
Note that for any solution with $m \geq 2$, we have m even. This follows from the fact that $m!$ is even for $m \geq 2$ and so $m! + 1 = (m+1)^n$ is odd. Thus $(m+1)$ must be odd and m even.

We can reduce the pool of possible solutions by showing that $m \leq 4$. Clearly a solution with $m > 4$ and m even implies $2 < m/2 < m$, so

$$2\left(\frac{m}{2}\right)m \big| m! \Rightarrow m^2 | \left((m+1)^n - 1\right) \Rightarrow m^2 \big| m \sum_{k=1}^{n} \binom{n}{k} m^{k-1}.$$

Thus

$$m \Big| \left(n + m \sum_{k=2}^{n} \binom{n}{k} m^{k-2}\right),$$

so m divides n and hence $n \geq m$.

We will now show that $n \geq m$ and $m > 4$ yields no solutions. In that case,

$$m! + 1 < m^{m-1} + 1 < (m+1)m^{m-1} < (m+1)^m \leq (m+1)^n.$$

Thus, there are no positive integer solutions with $m > 4$. Now we can find all solutions using the previously gathered information about m. For each possible m we have the following.

$$m = 1 \Rightarrow 2^n = 2, \text{ so } n = 1,$$
$$m = 2 \Rightarrow 3^n = 3, \text{ so } n = 1,$$
$$m = 4 \Rightarrow 5^n = 25, \text{ so } n = 2.$$

Also solved by John Christopher, Michael P. Cohen, Charles Curtis & Jacob Boswell, Eagle Problem Solvers (Georgia Southern University), John Fitch, Khaled Halaoua (Syria), Walther Janous (Austria), Rafe Jones, Koopa Tak Lun Koo (Hong Kong), Seungheon Lee (South Korea), Graham Lord, Kelly D. McLenithan, Stephen Meskin, Raymond Mortini (France) & Rudolf Rupp (Germany) & Amol Sasane (UK), Sonebi Omar (Morocco), Didier Pinchon (France), Henry Ricardo, Celia Schacht, Albert Stadler (Switzerland), Wong Fook Sung (Singapore), and the proposer. There were two incomplete or incorrect solutions.

Does the series converge or diverge? **April 2021**

2118. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

It is well known that the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}$$

converges. Does the series

$$\sum_{k=1}^{\infty} e^{-\lfloor \ln k \rfloor} \sin k$$

converge or diverge?

Solution by the Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.
Both series can be shown to be convergent using the following well-known result.

Dirichlet's test: If a_n is a monotonic sequence of real numbers that tends to zero, and b_n is a sequence of complex numbers such that, for some M, $\left|\sum_{n=1}^{N} b_n\right| \leq M$ for every positive integer N, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

For this problem, we use $a_k = e^{-\lfloor \ln k \rfloor}$, which tends monotonically to zero, and $b_k = \sin k$, whose partial sums are

$$\begin{aligned}
\sum_{k=1}^{N} \sin k &= \sum_{k=1}^{N} \frac{1}{2i}(e^{ik} - e^{-ik}) = \frac{1}{2i}\left(\frac{e^{i(N+1)} - e^{i}}{e^{i} - 1} - \frac{e^{-i(N+1)} - e^{-i}}{e^{-i} - 1}\right) \\
&= \frac{1}{2i}\left(\frac{e^{i(N+1/2)} - e^{i/2} + e^{-i(N+1/2)} - e^{-i/2}}{e^{i/2} - e^{-i/2}}\right) \\
&= \frac{1}{2}\left(\frac{\left(e^{i/2} + e^{-i/2}\right)/2 - \left(e^{i(N+1/2)} + e^{-i(N+1/2)}\right)/2}{\left(e^{i/2} - e^{-i/2}\right)/(2i)}\right) \\
&= \frac{1}{2\sin\frac{1}{2}}\left(\cos\tfrac{1}{2} - \cos\left(N + \tfrac{1}{2}\right)\right)
\end{aligned}$$

hence

$$\left|\sum_{k=1}^{N} \sin k\right| \leq \frac{1}{\sin\frac{1}{2}}$$

So, by Dirichlet's test, the series converges.

Also solved by Hongwei Chen, Richard Daquila, Eagle Problem Solvers (Georgia Southern University), John Fitch, Russell Gordon, Eugene A. Herman, Walther Janous (Austria), Mark Kaplan & Michael Goldenberg, Raymond Mortini (France), Didier Pinchon (France), Omar Sonebi (Morocco), Albert Stadler (Switzerland), Seán Stewart (Australia), and the proposer. There were three incomplete or incorrect solutions.

Find the side length of the regular n-simplex **April 2021**

2119. *Proposed by Viktors Berstis, Portland, OR.*

A point in the plane is a distance of a, b, and c units from the vertices of an equilateral triangle in the plane. Denote the side length of the equilateral triangle by s.

(a) Find a polynomial relation between a, b, c, and s.

(b) Give a simple compass and straightedge construction of a segment of length s given segments of lengths a, b, and c.

(c) Generalize part (a) to the case of a point at a distance of a_i units, $i = 1, \ldots, n+1$, from the vertices of a regular n-dimensional simplex having sides of length s.

Solution by Didier Pinchon, Toulouse, France.

(a) Given $s > 0$, the points

$$A = \left(0, \frac{\sqrt{3}}{3}s\right), \quad B = \left(-\frac{1}{2}s, -\frac{\sqrt{3}}{6}s\right), \text{ and } C = \left(\frac{1}{2}s, -\frac{\sqrt{3}}{6}s\right)$$

are the vertices of an equilateral triangle with side length s. For a point $P = (x, y)$, the relations $PA^2 = a^2$, $PB^2 = b^2$ and $PC^2 = c^2$ give the equations

$$E_1 : x^2 + \left(y - \frac{\sqrt{3}}{3}s\right)^2 - a^2 = 0,$$

$$E_2 : \left(x + \frac{1}{2}s\right)^2 + \left(y + \frac{\sqrt{3}}{6}s\right)^2 - b^2 = 0,$$

$$E_3 : \left(x - \frac{1}{2}s\right)^2 + \left(y + \frac{\sqrt{3}}{6}s\right)^2 - c^2 = 0.$$

From $E_2 - E_3$, we get $x = (b^2 - c^2)/(2s)$, and substituting this value into $E_1 - E_2$, we get $y = \sqrt{3}(b^2 + c^2 - 2a^2)/(6s)$. Finally, substituting these values into equation E_1, we obtain

$$s^4 - (a^2 + b^2 + c^2)s^2 + a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 = 0.$$

(b) Note that s^2 satisfies a second-degree polynomial with discriminant

$$\begin{aligned}\Delta &= (a^2 + b^2 + c^2)2 - 4(a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2)\\ &= 3(a + b + c)(a + b - c)(a + c - b)(b + c - a).\end{aligned}$$

Given positive real numbers a, b and c, $\Delta \geq 0$ if and only if $c \leq a + b$, $b \leq a + c$ and $a \leq b + c$. Indeed, when two factors of Δ are negative, say for example $a + c \leq b$, $b + c \leq a$, then $c \leq 0$, which is impossible. Hence, $\Delta \geq 0$ if and only a, b, and c are the lengths of the sides of a triangle. Note that the triangle is degenerate if and only if $\Delta = 0$. If a, b, and c are not all equal, then

$$a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 = \frac{1}{2}\left[(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2\right] > 0.$$

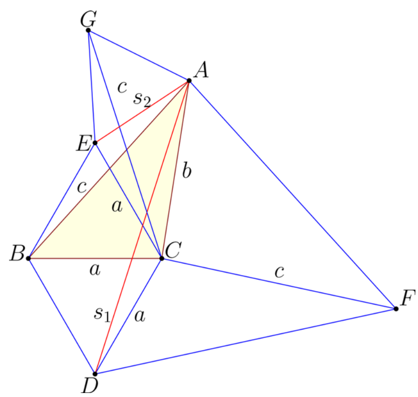
Therefore, the equation in s^2 has two different positive solutions, denoted by s_1 and s_2, if $\Delta > 0$ and a, b, and c are not all equal, and one positive solution otherwise.

The two solutions will now be constructed using a compass and straightedge. It is straightforward to construct a triangle ABC with side lengths a, b and c. The two circles of centers B and C and radius a intersect in two points D and E such that the triangles BCD and BCE are equilateral, with D and A being on opposite sides of line BC. Because ABC is not an equilateral triangle, point E is distinct from A.

We claim the lengths of the segments AD and AE are the solutions s_1 and s_2. The images of the points B and A by the rotation of center D and angle $-\pi/3$ are C and

F, and thus $BA = CF = c$. In a similar way, the images of A and B by the rotation of center E and angle $\pi/3$ are C and G, and thus $AB = c = CG$.

When $a = b = c$, then the first part of the construction is possible, and the unique solution is $s = DA = a\sqrt{3}$, and C is the center of equilateral triangle ADF. When $\Delta = 0$, A, B, and C are collinear, so A is equidistant from D and E and there is only one solution.



(c) *Editor's Note.* The solver uses the fact that if the distance between the ith and jth vertices of an n-simplex is $d_{i,j}$, then the volume of the simplex is

$$V^2 = \frac{(-1)^{n+1}}{2^n(n!)^2}\begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{1,2}^2 & d_{1,3}^2 & \dots & d_{1,n+1}^2 \\ 1 & d_{1,2}^2 & 0 & d_{2,3}^2 & \dots & d_{2,n+1}^2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & d_{1,n}^2 & d_{2,n}^2 & & 0 & d_{n,n+1}^2 \\ 1 & d_{1,n+1}^2 & d_{2,n+1}^2 & \dots & d_{n,n+1}^2 & 0 \end{vmatrix}.$$

He applies this formula to the degenerate $(n+1)$-simplex whose vertices are the vertices of the regular n-simplex along with the additional point and performs a series of row and column operations to derive the result.

Here is an alternative derivation. Let the vertices of the regular n-simplex be

$$(s/\sqrt{2}, 0, 0, \dots, 0), (0, s/\sqrt{2}, 0, \dots, 0), \dots, (0, 0, \dots, s/\sqrt{2})$$

in $\mathbb{R}^{n+1}$. Note that these vertices lie in the hyperplane whose equation is

$$\sum_{i=1}^{n+1} x_i = \frac{s}{\sqrt{2}}.$$

Let $(x_1, \dots, x_{n+1})$ be a point in this hyperplane. We have

$$a_i^2 = -\sqrt{2}sx_i + \frac{s^2}{2} + \sum_{i=1}^{n+1} x_i^2. \tag{1}$$

Expanding and summing these equations as $i = 1, \dots, n+1$, we obtain

$$\sum_{i=1}^{n+1} a_i^2 = -\sqrt{2}s\sum_{i=1}^{n+1} x_i + (n+1)\frac{s^2}{2} + (n+1)\sum_{i=1}^{n+1} x_i^2$$

$$= -\sqrt{2}s\left(\frac{s}{\sqrt{2}}\right) + (n+1)\frac{s^2}{2} + (n+1)\sum_{i=1}^{n+1} x_i^2$$

$$= (n-1)\frac{s^2}{2} + (n+1)\sum_{i=1}^{n+1} x_i^2.$$

Therefore

$$\sum_{i=1}^{n+1} x_i^2 = -\frac{n-1}{2(n+1)}s^2 + \frac{1}{n+1}\sum_{i=1}^{n+1} a_i^2. \tag{2}$$

Substituting into (1), we have

$$a_i^2 = -\sqrt{2}sx_i + \frac{s^2}{2} - \frac{n-1}{2(n+1)}s^2 + \frac{1}{n+1}\sum_{i=1}^{n+1} a_i^2$$

$$= -\sqrt{2}sx_i + \frac{1}{n+1}s^2 + \frac{1}{n+1}\sum_{i=1}^{n+1} a_i^2.$$

Solving for x_i, we find that

$$x_i = \frac{1}{\sqrt{2}s}\left(-a_i^2 + \frac{1}{n+1}s^2 + \frac{1}{n+1}\sum_{i=1}^{n+1} a_i^2\right).$$

Substituting into (2) and letting $s = a_0$, we have

$$-\frac{n-1}{2(n+1)}a_0^2 + \frac{1}{n+1}\sum_{i=1}^{n+1} a_i^2 = \frac{1}{2a_0^2}\sum_{i=1}^{n+1}\left(-a_i^2 + \frac{1}{n+1}\sum_{i=0}^{n+1} a_i^2\right)^2$$

$$2a_0^2\left(-\frac{1}{2}a_0^2 + \frac{1}{n+1}\sum_{i=0}^{n+1} a_i^2\right) = \sum_{i=1}^{n+1}\left(-a_i^2 + \frac{1}{n+1}\sum_{i=0}^{n+1} a_i^2\right)^2.$$

Letting $T = \sum_{i=0}^{n+1} a_i^2/(n+1)$, we have

$$-a_0^4 + 2a_0^2 T = \sum_{i=1}^{n+1} a_i^4 - 2T\sum_{i=1}^{n+1} a_i^2 + (n+1)T^2$$

$$0 = \sum_{i=0}^{n+1} a_i^4 - 2T\sum_{i=0}^{n+1} a_i^2 + (n+1)T^2$$

$$0 = \sum_{i=0}^{n+1} a_i^4 - 2T(n+1)T + (n+1)T^2$$

$$\sum_{i=0}^{n+1} a_i^4 = (n+1)T^2 = \frac{1}{n+1}\left(\sum_{i=0}^{n+1} a_i^2\right)^2$$

$$(n+1)\sum_{i=0}^{n+1} a_i^4 = \sum_{i=0}^{n+1} a_i^4 + 2\sum_{0\le i<j\le n+1} a_i^2 a_j^2$$

$$n \sum_{i=0}^{n+1} a_i^4 = 2 \sum_{0 \le i < j \le n+1} a_i^2 a_j^2,$$

which is the desired relation.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Albert Stadler (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

Find the normalizer **April 2021**

2120. *Proposed by Gregory Dresden, Jackson Gazin (student), and Kathleen McNeill (student), Washington & Lee University, Lexington, VA.*

Recall that the normalizer of a subgroup H of G is defined as

$$N_G(H) = \left\{ g \in G | ghg^{-1} \in H \text{ for all } h \in H \right\}.$$

Determine $N_G(H)$, when $G = GL_2(\mathbb{R})$, the group of all invertible 2×2 matrices with real entries, and

$$H = SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.
More generally, for any $n \geq 1$, let $G = GL_n(\mathbb{R})$ and $H = SO_n(\mathbb{R})$, the subgroup of $O_n(\mathbb{R})$, the group of orthogonal matrices, consisting of matrices whose determinant is 1. We will show that

$$N_G(H) = \{aU \,|a \in \mathbb{R} - \{0\}, U \in O_n(\mathbb{R})\}.$$

Suppose $A = aU$, where $a \neq 0$ and U is orthogonal. Then for any $M \in SO_n(\mathbb{R})$,

$$AMA^{-1} = aUM \frac{1}{a} U^{-1} = UMU^{-1}.$$

Since

$$\det(UMU^{-1}) = \det(U)\det(M)/\det(U) = 1,$$

and the product of orthogonal matrices is orthogonal, we see that $AMA^{-1} \in SO_n(\mathbb{R})$.

For the converse, we use a polar decomposition. For $A \in N_G(H)$, write $A = PU$, where P is positive-definite and U is orthogonal. For any $M \in SO_n(\mathbb{R})$, let $N = U^{-1}MU$. Then $N \in SO_n(\mathbb{R})$, so $ANA^{-1} \in SO_n(\mathbb{R})$. But

$$ANA^{-1} = P(UNU^{-1})P^{-1} = PMP^{-1},$$

so $P \in N_G(H)$. Therefore, it remains only to determine which positive-definite matrices are in the normalizer. Now every positive-definite matrix can be written as $P = VDV^{-1}$, where $D = \operatorname{diag}(d_1, \ldots, d_n)$ is a diagonal matrix with $d_i > 0$ and $V \in O_n(\mathbb{R})$. For any $M \in SO_n(\mathbb{R})$, let $N = VMV^{-1}$. Then $B = PNP^{-1} \in SO_n(\mathbb{R})$ and

$$B = VDMD^{-1}V^{-1} \in SO_n(\mathbb{R}), \quad \text{so} \quad DMD^{-1} = V^{-1}BV \in SO_n(\mathbb{R}).$$

Therefore, $D \in N_G(H)$.

For $k > 1$, let $M_k = [m_{ij}]$, where

$$m_{11} = 0, m_{1k} = -1, m_{k1} = 1, m_{kk} = 0, m_{ii} = 1\ (i \neq 1, k), \quad \text{and} \quad m_{i,j} = 0 \ \text{otherwise}.$$

Then $R \in SO_n(\mathbb{R})$ and the first column of DRD^{-1} consists of zeros except the kth entry, which is d_k/d_1. Since DRD^{-1} is orthogonal, this column must have length 1, which means that $d_k = d_1$ for all $k > 1$. Therefore D is a positive multiple of the identity, and so A is a multiple of an orthogonal matrix.

Note: The same proof works for the complex version. In that case, $G = GL_n(\mathbb{C})$ and $H = SU_n(\mathbb{C})$, where the latter is the group of $n \times n$ unitary matrices whose determinant equals 1. Then $N_G(H)$ is the group of all nonzero complex multiples of $n \times n$ unitary matrices.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Eagle Problem Solvers (Georgia Southern University), John Fitch, Dmitry Fleischman, Mark Kaplan & Michael Goldenberg, Koopa Tak Lun Koo (Hong Kong), Didier Pinchon (France), Albert Stadler (Switzerland) and the proposers. There were two incomplete or incorrect solutions.

Solutions

A series involving central binomial coefficients **December 2020**

2111. *Proposed by Enrique Treviño, Lake Forest College, Lake Forest, IL.*

Evaluate

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)(2n+2)}.$$

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.
The value of the series is $\frac{4}{3}(\sqrt{2}-1)$. To this end, recall the generating function for the central binomial coefficients

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}, \quad \text{for } |x| < \tfrac{1}{4}.$$

Replacing x by $-x$ gives

$$\sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} x^n = \frac{1}{\sqrt{1+4x}}, \quad \text{for } |x| < \tfrac{1}{4}.$$

Adding these series gives

$$\sum_{n=0}^{\infty} \binom{4n}{2n} x^{2n} = \frac{1}{2}\left(\frac{1}{\sqrt{1-4x}} + \frac{1}{\sqrt{1+4x}}\right).$$

Replacing x by $x/4$ yields

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}} x^{2n} = \frac{1}{2}\left(\frac{1}{\sqrt{1-x}} + \frac{1}{\sqrt{1+x}}\right), \quad \text{for } |x| < 1.$$

Integrating this series on $[0, x]$ with $0 < x < 1$, we find

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)} x^{2n+1} = \sqrt{1+x} - \sqrt{1-x}.$$

Integrating this series on $[0, x]$ with $0 < x < 1$ again, we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)(2n+2)} x^{2n+2} &= \int_0^x (\sqrt{1+t} - \sqrt{1-t})\mathrm{dt} \\ &= \frac{2}{3}\left((1+x)^{3/2} + (1-x)^{3/2}\right) - \frac{4}{3}.\end{aligned}$$

Applying Abel's convergence theorem and letting $x \to 1$, we conclude

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)(2n+2)} = \frac{4}{3}(\sqrt{2}-1)$$

as claimed.

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Farrukh Ataev (Uzbekistan), Michel Bataille (France), Khristo Boyadzhiev, Paul Bracken, Brian Bradie, Cal Poly Pomona Problem Solving Group, Robert Doucette, Gerald Edgar, Dmitry Fleischman, Mohit Hulse (India), Dixon Jones & Marty Getz, Mark Kaplan & Michael Goldenberg, GWstat Problem Solving Group, Omran Kouba (Syria), Sushanth Sathish Kumar, Elias Lampakis (Greece), Kee-Wai Lau (China), James Magliano, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Shing Hin Jimmy Pa (Canada), Angel Plaza (Spain), Rob Pratt, Volkhard Schindler (Germany), Edward Schmeichel, Randy Schwartz, Albert Stadler (Switzerland), Seán M. Stewart (Australia), Ibrahim Suleiman (United Arab Emirates), Michael Vowe (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

A problem from commutative algebra **December 2020**

2112. *Proposed by Souvik Dey, (graduate student), University of Kansas, Lawrence, KS.*

Let R be an integral domain and I and J be two ideals of R such that IJ is a non-zero principal ideal. Prove that I and J are finitely-generated ideals.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.
Let $IJ = \langle x \rangle$, where x is a nonzero element of R. Since $x \in IJ$, there exist

$$i_1, \dots, i_n \in I \text{ and } j_1, \dots, j_n \in J$$

such that

$$x = i_1 j_1 + \cdots i_n j_n.$$

We claim that

$$I = \langle i_1, \dots, i_n \rangle \text{ and } J = \langle j_1, \dots, j_n \rangle.$$

In each of these two equations, it suffices to prove that the left side is contained in the right. For any $i \in I$, there exist $r_1, \dots, r_n \in R$ such that

$$i j_k = r_k x, k = 1, \dots, n.$$

Multiply the kth equation by i_k and add the the resulting equations to obtain

$$ix = \sum_{k=1}^{n} r_k i_k x$$

Since R is an integral domain,

$$i = \sum_{k=1}^{n} r_k i_k,$$

and so $I = \langle i_1, \dots, i_n \rangle$. Similarly, $J = \langle j_1, \dots, j_n \rangle$.

Also solved by Paul Budney, Noah Garson (Canada), Elias Lampakis (Greece), and the proposer.

A condition for the nilpotency of a matrix **December 2020**

2113. *Proposed by George Stoica, Saint John, NB, Canada.*

Let A be an $n \times n$ complex matrix such that $\det(A^k + I_n) = 1$ for $k = 1, 2, \dots, 2^n - 1$.

(a) Prove that $A^n = O_n$.

(b) Show that the result does not hold if $2^n - 1$ is replaced by any smaller positive integer.

Solution by Michael Reid, University of Central Florida, Orlando, FL.

(a) First we have a lemma.

Lemma. *Suppose $z_1, \ldots, z_m \in \mathbb{C}$ are such that the power sums $S_k = z_1^k + \cdots + z_m^k$ vanish for $k = 1, 2, \ldots, m$. Then each $z_j = 0$.*

Proof. For $k = 1, 2, \ldots, m$, let σ_k denote the kth elementary symmetric function of $z_1, \ldots, z_m$. By Newton's identities,

$$k\sigma_k = (-1)^{k-1} S_k + \sum_{i=1}^{k-1} (-1)^{i-1} \sigma_{k-i} S_i = 0,$$

for $k = 1, 2, \ldots, m$. Hence each $\sigma_k = 0$. Therefore,

$$(T + z_1)(T + z_2) \cdots (T + z_m) = T^m + \sigma_1 T^{m-1} + \cdots + \sigma_{m-1} T + \sigma_m = T^m.$$

By unique factorization of polynomials, each factor, $T + z_j$, on the left is a constant multiple of T, so each $z_j = 0$. $\square$

The matrix A is similar to an upper triangular matrix M (for example, take M to be a Jordan canonical form of A). Let $d_1, \ldots, d_n$ be the diagonal entries of M. Then $A^k + I_n$ is similar to $M^k + I_n$, which is an upper triangular matrix with diagonal entries $d_1^k + 1, \ldots, d_n^k + 1$, so

$$\det(A^k + I_n) = (d_1^k + 1) \cdots (d_n^k + 1).$$

For each subset $\mathcal{S} \subseteq \{1, 2, \ldots, n\}$, let $b_{\mathcal{S}} = \prod_{j \in \mathcal{S}} d_j$. Thus

$$1 = \det(A^k + I_n) = \prod_{j=1}^{n} (d_j^k + 1) = \sum_{\mathcal{S} \subseteq \{1,2,\ldots,n\}} b_{\mathcal{S}}^k.$$

Let $m = 2^n - 1$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_m$ be the non-empty subsets of $\{1, 2, \ldots, n\}$, and put $z_j = b_{\mathcal{S}_j}$. Then, the equation above becomes $z_1^k + \cdots + z_m^k = 0$, which holds for $k = 1, 2, \ldots, m$. From the lemma, each $z_j = 0$. In particular, for a singleton subset $\{i\}$, we have $d_i = b_{\{i\}} = 0$. Hence M is upper triangular, with all zeros on its diagonal, so its characteristic polynomial is T^n. Since A is similar to M, it has the same characteristic polynomial, so by the Cayley–Hamilton theorem, $A^n = 0_n$.

(b) Let $m = 2^n - 1$, and let $\zeta \in \mathbb{C}$ be a primitive mth root of 1. Let A be the diagonal matrix with diagonal entries $\zeta, \zeta^2, \zeta^4, \ldots, \zeta^{2^{n-1}}$. Then, A is non-singular, so it is not nilpotent. For $k \in \mathbb{N}$, $A^k + I_n$ is the diagonal matrix whose diagonal is

$$\zeta^k + 1, \zeta^{2k} + 1, \zeta^{4k} + 1, \ldots, \zeta^{2^{n-1}k} + 1.$$

Thus,

$$\det(A^k + I_n) = (\zeta^k + 1)(\zeta^{2k} + 1) \cdots (\zeta^{2^{n-1}k} + 1).$$

If k is not divisible by m, then this product telescopes to give

$$\det(A^k + I_n) = \prod_{j=0}^{n-1} \frac{\zeta^{2^{j+1}k} - 1}{\zeta^{2^j k} - 1} = \frac{\zeta^{2^n k} - 1}{\zeta^k - 1} = 1,$$

because $\zeta^{2^n k} = \zeta^{mk+k} = \zeta^k$. Hence,

$$\det(A^k + I_n) = 1$$

for $k = 1, 2, \dots, 2^n - 2$.

Also solved by Lixing Han & Xinjia Tang, Koopa Tak Lan Koo (Hong Kong), Elias Lampakis (Greece), Albert Stadler (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

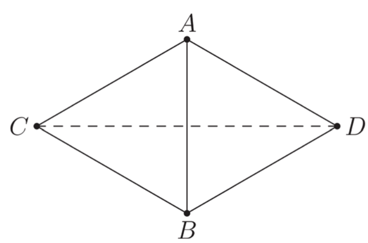
Planar 2-distance sets having four points **December 2020**

2114. *Proposed by Robert Haas, Cleveland Heights, OH.*

Find all configurations of four points in the plane (up to similarity) such that the set of distances between the points consists of exactly two lengths.

Solution by Robert L. Doucette, McNeese State University, Lake Charles, LA.
Suppose A, B, C, D are distinct points in the plane such that the list of six segment distances, AB, AC, AD, BC, BD, and CD, has exactly two real values. For convenience, we may suppose that one of these values is 1. We consider three cases.

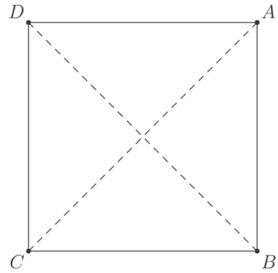
Case 1. Exactly five of the six distances equal 1. Suppose $AB = AC = AD = BC = BD = 1$, $CD \neq 1$. In this case, ABC and ABD must form equilateral triangles. Since $C \neq D$, $ADBC$ must form a rhombus with side length 1 and one pair of opposite angles measuring 60°. This yields a configuration in which the distance not equal to 1 is $CD = \sqrt{3}$.



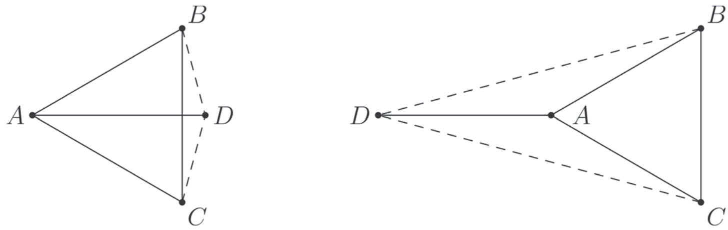
Case 2. Exactly four of the six distances equal 1. There are two subcases to consider.

(i) Suppose first that the two segments with length not equal to 1 do not have an endpoint in common. Say $AC = BD \neq 1$. Since $ABCD$ is a rhombus with congruent diagonals, it must be a square. This yields a configuration in which the distances not equal to 1 are $AC = BD = \sqrt{2}$.

(ii) Suppose next that the two segments with length not equal to 1 do share an endpoint. Say $BD = CD \neq 1$. In this case, ABC forms an equilateral triangle of side length 1. The point D must lie on the perpendicular bisector of segment BC. Either D lies on the same side of $\overleftrightarrow{BC}$ as A or on the opposite side. In the former case, $\triangle BCD$ is a 30°-75°-75° triangle, A is its circumcenter, and $BD = CD = \sqrt{2 + \sqrt{3}}$.

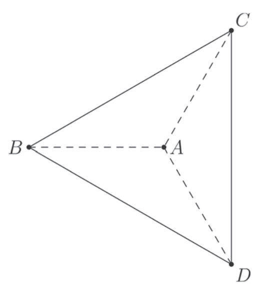


In the latter case, $ABDC$ is a kite with opposite angles of measure 60° and 150°, and $BD = CD = \sqrt{2 - \sqrt{3}}$.

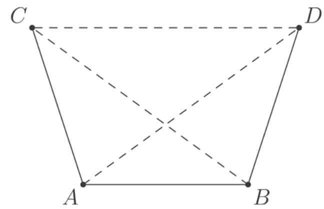


Case 3. Exactly three of the six distances equal 1. Again we consider two subcases.

(i) Suppose that three of the segments of equal length have an endpoint in common. We may assume that $AB = AC = AD = 1$ and $BC = BD = CD \neq 1$. In this case, the points B, C and D lie on the circle with center A and radius 1 and form an equilateral triangle. In other words, BCD forms an equilateral triangle with circumcenter A. In this case, $BC = BD = CD = \sqrt{3}$.



(ii) Next suppose that no three of the segments of equal length share a common endpoint. We may assume that $AB = AC = BD = 1$ and $AD = BC = CD = x > 1$. Since $\triangle ABC \cong \triangle BAD$, $\angle BAC \cong \angle ABD$. If C and D are on opposite sides of $\overleftrightarrow{AB}$, then $ACBD$ is a parallelogram. But by the parallelogram law, $AB^2 + CD^2 = 2AC^2 + 2AD^2$, implying that $1 + x^2 = 2 + 2x^2$, which is impossible. Therefore C and D lie on the same side of $\overleftrightarrow{AB}$ and $ABDC$ is an isosceles trapezoid. Let $m(\angle ADC) = \alpha$. Then $m(\angle BCD) = \alpha$ (since $\triangle ADC \cong \triangle BCD$), $m(\angle ABC) = m(\angle BAD) = \alpha$ (alternating interior angles), $m(\angle ACB) = m(\angle ADB) = \alpha$ (base angles of isosceles triangles), and $m(\angle CAD) = m(\angle CBD) = 2\alpha$ (base angles of isosceles triangles). The sum of the measure of the interior angles of a quadrilateral is 360°, so $10\alpha = 360°$ and $\alpha = 36°$. This means that A, B, C, and D are four of the five vertices of a regular pentagon. In this case, $AD = BC = CD = (1 + \sqrt{5})/2$.



We have shown that there are six configurations of four points satisfying the requirements described in the problem statement: (1) a rhombus with one pair of opposite angles measuring 60°, (2) a square, (3) an isosceles triangle with vertex angle of 30° and its circumcenter, (4) a kite with a pair of opposite angles measuring 60° and 150°, (5) an equilateral triangle and its circumcenter, and (6) four of the five vertices of a regular pentagon.

Also solved by Diya Bhatt & Riley Platz & Tony Luo (students), Viera Cernanova (Slovakia), M. V. Channakeshava (India), Seungheon Lee (Korea), Eagle Problem Solvers, Michael Reid, Celia Schacht, Albert Stadler (Switzerland), Tianyue Ruby Sun (student), Randy K. Schwartz, and the proposer. There were six incomplete or incorrect solutions.

Two compass and straightedge constructions **December 2020**

2115. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let A and B be two distinct points on a circle and let k be a positive rational number.

(a) Give a compass and straightedge construction of a point C on the circle such that $AC/BC = k$.

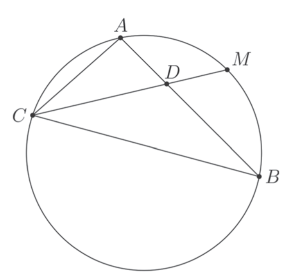
(b) Give a compass and straightedge construction of a point C on the circle such that $AC \cdot BC = k$. As part of your solution, find the restrictions on k in terms of AB and the radius of the circle necessary for such a C to exist.

Solution by Enrique Treviño, Lake Forest College, Lake Forest, IL.

(a) It is well known that we can construct a point D on segment AB such that $AD/BD = k$. Let M be a point of intersection of the perpendicular bisector of AB with the given circle. Then $AM = BM$. Let C be the second point of intersection of $\overleftrightarrow{MD}$ with the circle. Since $AM = BM$, then $\angle ACM = \angle BCM$. Therefore D is on the angle bisector of $\angle ACB$ and by the angle bisector theorem

$$\frac{AC}{BC} = \frac{AD}{BD} = k.$$

An alternative solution is to note that $\{X | AX/BX = k\}$ is a circle or the perpen-



dicular bisector of AB. This curve is readily constructible, and we then find its intersection with the original circle.

(b) To solve this problem in full generality, we need a segment of length 1 to be given. It is well known that given such a segment, $\lambda \in \mathbb{Q}^+$, and a segment of length a, we can use similar triangles to construct a segment of length λ/a.

Denote the center of the circle by O and let $m\angle AOB = 2\alpha$. Then $x = 2r \sin\alpha$. For any point C on the circle, $m\angle ACB = \alpha$ or $m\angle ACB = \pi - \alpha$. In either case $\sin\angle ACB = \sin\alpha$. We will denote the area of $\triangle PQR$ by (PQR). We know

$$(ABC) = \frac{AC \cdot BC \cdot \sin\alpha}{2}.$$

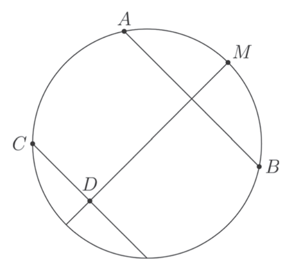
Let h be the height of $\triangle ABC$ with base AB, then

$$(ABC) = \frac{xh}{2}.$$

Therefore

$$AC \cdot BC = 2rh.$$

Let ℓ be the perpendicular bisector of AB. Using the facts stated above, we can construct a point D on ℓ such that the distance from D to AB is $k/(2r)$. Next, we draw a line through D parallel to $\overleftrightarrow{AB}$ and let C be one of the points of intersection of this line with the given circle. This point C satisfies $AC \cdot BC = 2rh = k$.



The largest possible value of $AC \cdot BC$ occurs when C lies on the perpendicular bisector of AB at a maximum distance from AB, namely when

$$AC = BC = r + \sqrt{r^2 - \frac{x^2}{4}}.$$

Therefore

$$\begin{aligned} k &= AC \cdot BC \\ &\le \left(r + \sqrt{r^2 - \frac{x^2}{4}}\right)^2 \\ &= r\left(2r + \sqrt{4r^2 - x^2}\right). \end{aligned}$$

Also solved by Michel Bataille (France), Ivko Dimitrić, Elias Lampakis (Greece), Celia Schacht, Albert Stadler (Switzerland), and the proposer. There were three incomplete or incorrect solutions.

Solutions

Invariance of a ratio of sums of cotangents **October 2020**

2101. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore MD and Mark Kaplan, Towson University, Towson, MD.*

Recall that the Steiner inellipse of a triangle is the unique ellipse that is tangent to each side of the triangle at the midpoints of those sides. Consider the Steiner inellipse E_S of $\triangle ABC$ and another ellipse, E_A, passing through the centroid G of $\triangle ABC$ and tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C. If E_S and E_A meet at M and N, let $\angle MAN = \alpha$. Construct ellipses E_B and E_C, introduce their points of intersection with E_S, and define angles β and γ in an analogous way. Prove that

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{\cot A + \cot B + \cot C} = \frac{11}{3\sqrt{5}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland.
We first consider the equilateral triangle with vertices

$$A = (16, 0),\ B = (-8, 8\sqrt{3}), \quad \text{and} \quad C = (-8, -8\sqrt{3}),$$

whose centroid is the origin. In this case, E_S is the circle whose equation is $x^2 + y^2 = 8^2$ and E_A is the circle whose equation is $(x + 16)^2 + y^2 = 16^2$. Solving this system of equations we find

$$M = (-2, 2\sqrt{15}) \quad \text{and} \quad N = (-2, -2\sqrt{15}).$$

Let $\angle(\overrightarrow{u}, \overrightarrow{v})$ denote the angle between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. Then

$$A = \angle\left((-24, 8\sqrt{3}), (-24, -8\sqrt{3})\right) \quad \text{and} \quad \alpha = \angle\left((-18, 2\sqrt{15}), (-18, -2\sqrt{15})\right).$$

Rotating the vectors above 120° and 240° counter-clockwise gives

$$B = \angle\left((0, -16\sqrt{3}), (24, -8\sqrt{3})\right),$$

$$\beta = \angle\left((9 - 3\sqrt{5}, -9\sqrt{3} - \sqrt{15}), (9 + 3\sqrt{5}, -9\sqrt{3} + \sqrt{15})\right),$$

$$C = \angle\left((24, 8\sqrt{3}), (0, 16\sqrt{3})\right), \quad \text{and}$$

$$\gamma = \angle\left((9 + 3\sqrt{5}, 9\sqrt{3} - \sqrt{15})), (9 - 3\sqrt{5}, 9\sqrt{3} + \sqrt{15})\right).$$

Now let $\triangle A'B'C'$ be any non-degenerate triangle whose centroid is at the origin. There is an invertible linear map $f(x, y) = (ax + by, cx + dy)$ such that $\triangle A'B'C' = f(\triangle ABC)$. This linear mapping preserves the centroid, all midpoints, all tangencies, and it maps lines to lines and circles to ellipses. It remains to analyze how this linear mapping transforms the six numbers $\cot A$, $\cot B$, $\cot C$, $\cot\alpha$, $\cot\beta$, and $\cot\gamma$ to $\cot A'$, $\cot B'$, $\cot C'$, $\cot\alpha'$, $\cot\beta'$, and $\cot\gamma'$.

We will use the fact if $\phi = \angle((u_1, u_2), (v_1, v_2))$, then

$$\cot\phi = \frac{u_1v_1 + u_2v_2}{u_1v_2 - u_2v_1}$$

by the difference formula for cotangent.

Now

$$A' = \angle\left(f(-24, 8\sqrt{3}), f(-24, -8\sqrt{3})\right),$$

$$B' = \angle\left(f(0, -16\sqrt{3}), f(24, -8\sqrt{3})\right), \quad \text{and}$$

$$C' = \angle\left(f(24, 8\sqrt{3}), f(0, 16\sqrt{3})\right).$$

This gives

$$\cot A' = \frac{3a^2 - b^2 + 3c^2 - d^2}{2\sqrt{3}(ad - bc)}$$
$$\cot B' = \frac{b^2 - \sqrt{3}ab + d^2 - \sqrt{3}cd}{\sqrt{3}(ad - bc)}$$
$$\cot C' = \frac{b^2 + \sqrt{3}ab + d^2 + \sqrt{3}cd}{\sqrt{3}(ad - bc)}.$$

Therefore,

$$\cot A' + \cot B' + \cot C' = \frac{\sqrt{3}\left(a^2 + b^2 + c^2 + d^2\right)}{2(ad - bc)}.$$

A similar calculation yields

$$\cot \alpha' + \cot \beta' + \cot \gamma' = \frac{11\left(a^2 + b^2 + c^2 + d^2\right)}{2\sqrt{15}(ad - bc)}.$$

Finally,

$$\frac{\cot \alpha' + \cot \beta' + \cot \gamma'}{\cot A' + \cot B' + \cot C'} = \frac{11}{3\sqrt{5}}$$

as desired.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia) and the proposers. There were two incomplete or incorrect solutions.

Trigonometric identities for the heptagonal triangle **October 2020**

2102. *Proposed by Donald Jay Moore, Wichita, KS.*

Let $\alpha = \pi/7$, $\beta = 2\pi/7$, and $\gamma = 4\pi/7$. Prove the following trigonometric identities.

$$\frac{\cos^2 \alpha}{\cos^2 \beta} + \frac{\cos^2 \beta}{\cos^2 \gamma} + \frac{\cos^2 \gamma}{\cos^2 \alpha} = 10,$$
$$\frac{\sin^2 \alpha}{\sin^2 \beta} + \frac{\sin^2 \beta}{\sin^2 \gamma} + \frac{\sin^2 \gamma}{\sin^2 \alpha} = 6,$$
$$\frac{\tan^2 \alpha}{\tan^2 \beta} + \frac{\tan^2 \beta}{\tan^2 \gamma} + \frac{\tan^2 \gamma}{\tan^2 \alpha} = 83.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.
Denote the trigonometric expressions by $\mathcal{C}, \mathcal{S}, \mathcal{T}$, respectively. The expansion

$$\sin(7t) = \sin t\left(64\cos^6 t - 80\cos^4 t + 24\cos^2 t - 1\right)$$

yields the key polynomial as follows. When $t = \alpha$ or $t = \beta$ or $t = \gamma$, then $\sin(7t) = 0$ but $\sin t \neq 0$. Hence the cubic polynomial

$$p(x) = 64x^3 - 80x^2 + 24x - 1$$

has the three zeros $a = \cos^2\alpha, b = \cos^2\beta, c = \cos^2\gamma$. Since

$$p(x) = 64(x-a)(x-b)(x-c),$$

we have values for the three elementary symmetric polynomials:

$$a+b+c = \frac{5}{4}, \quad ab+bc+ca = \frac{3}{8}, \quad abc = \frac{1}{64}.$$

We use the double angle formula for sine as follows:

$$\frac{\sin^2 t}{\sin^2 2t} = \frac{\sin^2 t}{4\sin^2 t\cos^2 t} = \frac{1}{4\cos^2 t}.$$

Hence, since $\sin^2 2\gamma = \sin^2\alpha$,

$$\mathcal{S} = \frac{\sin^2\alpha}{\sin^2\beta} + \frac{\sin^2\beta}{\sin^2\gamma} + \frac{\sin^2\gamma}{\sin^2\alpha} = \frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} = \frac{bc+ca+ab}{4abc} = \frac{3/8}{4/64} = 6.$$

We use the double angle formula for cosine as follows:

$$\frac{\cos^2 t}{\cos^2 2t} = \frac{\cos^2 t}{(2\cos^2 t - 1)^2}.$$

Hence, since $\cos^2 2\gamma = \cos^2\alpha$,

$$\mathcal{C} = \frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\gamma} + \frac{\cos^2\gamma}{\cos^2\alpha} = \frac{a}{(2a-1)^2} + \frac{b}{(2b-1)^2} + \frac{c}{(2c-1)^2}.$$

Substituting $x = (y+1)/2$ into the polynomial $p(x)$ yields

$$q(y) = 8y^3 + 4y^2 - 4y - 1.$$

Since $y = 2x - 1$, the zeros of $q(y)$ are $a' = 2a-1, b' = 2b-1, c' = 2c-1$ and the elementary symmetric polynomial expressions are

$$a'+b'+c' = -\frac{1}{2}, \quad a'b'+b'c'+c'a' = -\frac{1}{2}, \quad a'b'c' = \frac{1}{8}.$$

Hence,

$$\begin{aligned}\mathcal{C} &= \frac{a'+1}{2a'^2} + \frac{b'+1}{2b'^2} + \frac{c'+1}{2c'^2} = \frac{a'b'^2c'^2 + b'a'^2c'^2 + c'a'^2b'^2 + b'^2c'^2 + a'^1c'^2 + a'^2b'^2}{2(a'b'c')^2} \\ &= \frac{(a'b'c')(a'b'+b'c'+c'a') + (a'b'+b'c'+c'a')^2 - 2(a'b'c')(a'+b'+c')}{2(a'b'c')^2} \\ &= \frac{-1/16 + 1/4 + 1/8}{2/64} = 10.\end{aligned}$$

For the third identity, we use both double angle formulas:

$$\frac{\tan^2 t}{\tan^2 2t} = \frac{\sin^2 t\ \cos^2 2t}{\cos^2 t\ \sin^2 2t} = \frac{(2\cos^2 t - 1)^2}{4\cos^4 t}$$

Thus, since $\tan^2 2\gamma = \tan^2\alpha$,

$$\mathcal{T} = \frac{\tan^2\alpha}{\tan^2\beta} + \frac{\tan^2\beta}{\tan^2\gamma} + \frac{\tan^2\gamma}{\tan^2\alpha} = \left(\frac{2a-1}{2a}\right)^2 + \left(\frac{2b-1}{2b}\right)^2 + \left(\frac{2c-1}{2c}\right)^2.$$

Substituting $x = 1/(2(1-z))$ into the polynomial $p(x)$ and clearing fractions yields

$$r(z) = 8(z^3 + 9z^2 - z - 1).$$

Since $z = (2x-1)/(2x)$, the zeros of $r(z)$ are

$$a' = \frac{2a-1}{2a}, \quad b' = \frac{2b-1}{b}, \quad c' = \frac{2c-1}{c}$$

and the elementary symmetric polynomial expressions are

$$a' + b' + c' = -9, \quad a'b' + b'c' + c'a' = -1, \quad a'b'c' = 1.$$

Hence,

$$\mathcal{T} = a'^2 + b'^2 + c'^2 = (a' + b' + c')^2 - 2(a'b' + b'c' + c'a') = 9^2 - 2(-1) = 83.$$

Also solved by Michel Bataille (France), Anthony J. Bevelacqua, Brian Bradie, Robert Calcaterra, Hongwei Chen, John Christopher, Robert Doucette, Habib Y. Far, J. Chris Fisher, Dmitry Fleischman, Michael Goldenberg & Mark Kaplan, Russell Gordon, Walther Janous (Austria), Kee-Wai Lau (Hong Kong), James Magliano, Ivan Retamoso, Volkhard Schindler (Germany), Randy Schwartz, Allen J.Schwenk, Albert Stadler (Switzerland), Seán M. Stewart (Australia), Enrique Treviño, Michael Vowe (Switzerland), Edward White & Roberta White, Lienhard Wimmer (Germany), and the proposer. There were two incomplete or incorrect solutions.

How many tickets to buy to guarantee three out of four? October 2020

2103. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.*

In a soccer game there are three possible outcomes: a win for the home team (denoted 1), a draw (denoted X), or a win for the visiting team (denoted 2). If there are n games, betting slips are printed for all 3^n possible outcomes. For four games, what is the minimum number of slips you must purchase to guarantee that at least three of the outcomes are correct on at least one of your slips?

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

The answer is nine.

First, we prove that it is impossible to guarantee at least three correct outcomes with fewer than nine slips.

Let T be the set of all possible outcomes, i.e., all 4-tuples of 1, X, and 2. There are $3^4 = 81$ such 4-tuples. In that set, we define the Hamming distance d as the number of places in which two tuples differ. For example, $d(1X21, 2X12) = 3$ because 1X21 and 2X12 differ in three places, namely the first, third and fourth places. The Hamming distance satisfies the usual axioms for a metric, and we can define balls in T in the usual way, i.e., a ball with center $c \in T$ and radius $r \in \mathbb{R}$ is

$$B_r(c) = \{t \in T \mid d(t, c) \leq r\}.$$

Given a tuple $c \in T$, the set of tuples that coincide with c in at least three places consists of those that differ from c in no more than one place. In other words, this set is $B_1(c)$. Note that $B_1(c)$ contains exactly 9 elements: the center c, the two tuples that differ from c exactly in the first element, the two that differ in the second, the two that differ in the third, and the two that differ in the fourth.

In order to ensure that our slips $c_1, c_2, \ldots, c_n$ contain at least three correct entries, the balls $B_1(c_i)$, $i = 1, 2, \ldots, n$ must cover T, i.e.,

$$T = \bigcup_{i=1}^{n} B_1(c_i).$$

Since $|B_1(c)| = 9$ and $|T| = 81$, we will need at least $81/9 = 9$ slips.

Next, we will prove that nine slips suffice. That can be accomplished by exhibiting nine 4-tuples $c_1, \ldots, c_9$ such that $B_i(c_1), \ldots, B_i(c_9)$ cover T, i.e., such that every element in T has a Hamming distance of at most 1 from at least one of the c_i. The following 4-tuples satisfy the condition:

1111 1XXX 1222 X1X2 XX21 X21X 2X12 212X 22X1

One (somewhat tedious) way to check it is to verify that each of the 81 elements in T differ from at least one of these tuples in no more one place.

A slightly easier way to verify the assertion is to observe that these tuples differ from each other in exactly three places, so the Hamming distance between any two of them is 3. Because of the triangle inequality, it is impossible for balls of radius 1 centered on the c_i to overlap. Therefore the total number of elements contained in the union of these balls is $9 \cdot 9 = 81$, so the union must be all of T.

This completes the proof.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Eagle Problem Solvers, Fresno State Problem Solving Group, Dan Hletko, Rob Pratt, Allen J. Schwenk, and the proposer. There were seven incomplete or incorrect solutions.

Vector spaces as unions of proper subspaces **October 2020**

2104. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

It is well known that no vector space can be written as the union of two proper subspaces. For which m does there exist a vector space V that can be written as a union of m proper subspaces with this collection of subspaces being minimal in the sense that no union of a proper subcollection is equal to V?

Solution by Paul Budney, Sunderland, MA.

Such a decomposition exists for any $m > 2$.

Let $V = \mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field with two elements. Let

$$V_i = \{(x_1, \ldots, x_n) \in V \,|\, x_i = 0\}$$

for $1 \leq i \leq n$ and let

$$W = \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\}.$$

Clearly W and the V_i are proper subspaces of V. Since $(1, 1, \ldots, 1)$ is the only vector not in $V_1 \cup V_2 \cup \ldots \cup V_n$,

$$W \cup V_1 \cup V_2 \cup \ldots \cup V_n = V.$$

Deleting W from this union excludes $(1, 1, \ldots, 1)$. Deleting V_i from this union excludes $(1, \ldots, 1, 0, 1, \ldots, 1)$, with 0 for the ith component and 1's elsewhere. Thus, there is no proper subcollection of these subspaces whose union is V. There are

$n + 1$ subspaces, and since $n \geq 2$ is arbitrary, the desired decomposition exists for any $m > 2$.

Also solved by Anthony Bevelacqua, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Doucette, Eugene Herman, and the proposer. There was one incomplete or incorrect solution.

An asymptotic formula for a definite integral **October 2020**

2105. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.*

Let $f : [0, 1] \to \mathbb{R}$ be a function that is k times differentiable on $[0, 1]$, with the kth derivative integrable on $[0, 1]$ and (left) continuous at 1. For integers $i \geq 1$ and $j \geq 0$ let

$$\sigma_j^{(i)} = \sum_{j_1+j_2\cdots+j_i=j} 1^{j_1} 2^{j_2} \cdots i^{j_i},$$

where the sum is extended over all i-tuples $(j_1, \ldots, j_i)$ of nonnegative integers that sum to j. Thus, for example, $\sigma_0^{(i)} = 1$, and $\sigma_1^{(i)} = 1 + 2 + \cdots + i = i(i + 1)/2$ for all $i \geq 1$. Also, for $0 \leq j \leq k$ let

$$a_j = \sigma_j^{(1)} f(1) + \sigma_{j-1}^{(2)} f'(1) + \cdots + \sigma_1^{(j)} f^{(j-1)}(1) + \sigma_0^{(j+1)} f^{(j)}(1).$$

Prove that

$$\int_0^1 x^n f(x)dx = \frac{a_0}{n} - \frac{a_1}{n^2} + \cdots + (-1)^k \frac{a_k}{n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right),$$

for $n \to \infty$. As usual, we denote by $f^{(s)}$ the sth derivative of f (with $f^{(0)} = f$), and by $o(x_n)$ a sequence (y_n) with the property that $\lim_{n\to\infty} y_n/x_n = 0$.

Solution by Michel Bataille, Rouen, France.
For $x \in [0, 1]$, let $f_0(x) = f(x)$ and

$$f_j(x) = \frac{d}{dx}\left(xf_{j-1}(x)\right), 1 \leq j \leq k.$$

An easy induction shows that for $0 \leq j \leq k$, the function f_j is a linear combination of the functions $f(x), xf'(x), \ldots, x^j f^{(j)}(x)$. It follows that $f_0, f_1, \ldots, f_{k-1}$ are differentiable on $[0, 1]$ and that f_k is integrable on $[0, 1]$ and continuous at 1.

Integrating by parts, we obtain the following recursion that holds for $1 \leq j \leq k - 1$:

$$\begin{aligned}\int_0^1 x^n f_{j-1}(x)\,dx &= \left[\frac{x^n}{n} \cdot (xf_{j-1}(x))\right]_0^1 - \frac{1}{n}\int_0^1 x^n f_j(x)\,dx \\ &= \frac{f_{j-1}(1)}{n} - \frac{1}{n}\int_0^1 x^n f_j(x)\,dx.\end{aligned}$$

With the help of this recursion, we are readily led to

$$\int_0^1 x^n f(x)\,dx = \int_0^1 x^n f_0(x)\,dx$$

$$= \sum_{j=0}^{k-1} (-1)^j \frac{f_j(1)}{n^{j+1}} + \frac{(-1)^k}{n^k} \int_0^1 x^n f_k(x)\,dx.$$

Now, if $g : [0,1] \to \mathbb{R}$ is integrable on $[0,1]$ and continuous at 1, then

$$\lim_{n\to\infty} n \cdot \int_0^1 x^n g(x)\,dx = g(1)$$

(Paulo Ney de Souza, Jorge-Nuno Silva, *Berkeley Problems in Mathematics*, Springer, 2004, Problem 1.2.13). With $g = f_k$, this yields

$$\int_0^1 x^n f_k(x)\,dx = \frac{f_k(1)}{n} + o\left(\frac{1}{n}\right)$$

and therefore

$$\int_0^1 x^n f(x)\,dx = \sum_{j=0}^{k-1} (-1)^j \frac{f_j(1)}{n^{j+1}} + \frac{(-1)^k}{n^k}\left(\frac{f_k(1)}{n} + o\left(\frac{1}{n}\right)\right)$$

$$= \sum_{j=0}^{k} (-1)^j \frac{f_j(1)}{n^{j+1}} + o\left(\frac{1}{n^{k+1}}\right).$$

Comparing this with the statement of the problem, it remains to prove that $a_j = f_j(1)$ for $0 \le j \le k$. Clearly, it is sufficient to prove that for $x \in [0,1]$

$$f_j(x) = \sum_{i=0}^{j} \sigma_{j-i}^{(i+1)} x^i f^{(i)}(x). \tag{E_j}$$

We use induction. Since $f_0(x) = f(x) = 1 \cdot x^0 f^{(0)}(x)$, (E_0) holds. Before addressing the induction step, we establish two results about the numbers $\sigma_j^{(i)}$. The first result is

$$\sigma_j^{(i+1)} = \sum_{r=0}^{j} (1+i)^r \sigma_{j-r}^{(i)}. \tag{1}$$

Proof. When $j_1 + \cdots + j_i + j_{i+1} = j$, then j_{i+1} can take the values $0, 1, \ldots, j$. It follows that

$$\sigma_j^{(i+1)} = \sum_{j_1+\cdots+j_{i+1}=j} 1^{j_1} 2^{j_2} \cdots i^{j_i} (i+1)^{j_{i+1}}$$

$$= \sum_{r=0}^{j} (1+i)^r \sum_{j_1+\cdots+j_i=j-r} 1^{j_1} 2^{j_2} \cdots i^{j_i}$$

$$= \sum_{r=0}^{j} (1+i)^r \sigma_{j-r}^{(i)}.$$

The second result is

$$\sigma_{j+1}^{(i+1)} = \sigma_{j+1}^{(i)} + (1+i)\sigma_j^{(i+1)}. \tag{2}$$

Proof. Applying (1),

$$\sigma_{j+1}^{(i+1)} = \sum_{r=0}^{j+1}(1+i)^r\sigma_{j+1-r}^{(i)}$$

$$= \sigma_{j+1}^{(i)} + (1+i)\sum_{r=1}^{j+1}(1+i)^{r-1}\sigma_{j-(r-1)}^{(i)}$$

$$= \sigma_{j+1}^{(i)} + (1+i)\sum_{r=0}^{j}(1+i)^{r}\sigma_{j-r}^{(i)}$$

and applying (1) again we conclude that $\sigma_{j+1}^{(i+1)} = \sigma_{j+1}^{(i)} + (1+i)\sigma_j^{(i+1)}$.

Now, assume that (E_j) holds for some integer j such that $0 \le j \le k-1$. Then, we calculate

$$f_{j+1}(x) = \frac{d}{dx}\left[\sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}x^{i+1}f^{(i)}(x)\right]$$

$$= \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}(i+1)x^i f^{(i)}(x) + \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}x^{i+1}f^{(i+1)}(x)$$

$$= \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}(i+1)x^i f^{(i)}(x) + \sum_{i=1}^{j+1}\sigma_{j-i+1}^{(i)}x^{i}f^{(i)}(x)$$

$$= \sigma_j^{(1)}f(x) + \sum_{i=1}^{j}\Big([\sigma_{j-i+1}^{(i)} + (i+1)\sigma_{j-i}^{(i+1)}]x^i f^{(i)}(x)\Big) + \sigma_0^{(j+1)}x^{j+1}f^{(j+1)}(x).$$

Using (2) and $\sigma_j^{(1)} = \sigma_{j+1}^{(1)} = 1 = \sigma_0^{(j+1)} = \sigma_0^{(j+2)}$, we see that

$$f_{j+1}(x) = \sum_{i=0}^{j+1}\sigma_{j+1-i}^{(i+1)}x^i f^{(i)}(x)$$

so that (E_{j+1}) holds. This completes the induction step and the proof.

Note. The number $\sigma_j^{(i)}$ is the Stirling number of the second kind $S(i+j,i) = \left\{{i+j \atop i}\right\}$ (see L. Comtet, *Advanced Combinatorics*, Reidel, 1974, Theorem D p. 207).

Also solved by Albert Stadler (Switzerland) and the proposer.

Solutions

The number of isosceles triangles in various polytopes **June 2020**

2096. *Proposed by H. A. ShahAli, Tehran, Iran.*

Any three distinct vertices of a polytope P form a triangle. How many of these triangles are isosceles if P is (a) a regular n-gon? (b) one of the Platonic solids? (c) an n-dimensional cube?

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI.
Let m denote the number of vertices of P. For a fixed vertex A of P, let $F(P)$ denote the number of unordered triplets of distinct vertices A, B, and C of P for which $AB = AC$, $G(P)$ is the number of such triplets for which $AB = AC = BC$, and $I(P)$ the number of isosceles triangles that can be formed using the vertices of P. Note that since all of the polytopes under consideration are uniform, $F(P)$ and $G(P)$ do not depend on A. Since each equilateral triangle is counted in $F(P)$ for three different choices of A,

$$I(P) = m\,(F(P) - G(P)) + \frac{m}{3}G(P) = mF(P) - \frac{2}{3}mG(P).$$

(a) If P is a regular n-gon, then $F(P) = \lfloor (n-1)/2 \rfloor$. Moreover, $G(P) = 1$ if n is a multiple of 3 and $G(P) = 0$ if not. Therefore,

$$I(P) = \begin{cases} n\left\lfloor \frac{n-1}{2} \right\rfloor & \text{if } 3 \nmid n \\ n\left\lfloor \frac{n-1}{2} \right\rfloor - \frac{2n}{3} & \text{if } 3 | n \end{cases}$$

(b) Let P be a Platonic solid. If A and B are vertices of P, the minimum number of edges of the solid that must be traversed to get from A to B will be called the span from A to B. For the Platonic solids, the spans for two pairs of vertices are the same if and only if the Euclidean distances are the same.

- If P is a tetrahedron, every triplet of distinct vertices forms an isosceles (in fact, equilateral) triangle. Therefore $I(P) = \binom{4}{3} = 4$.
- If P is a cube, then the numbers of vertices with spans 1, 2, and 3 from the fixed vertex A are 3, 3, and 1, respectively. Therefore, $F(P) = \binom{3}{2} + \binom{3}{2} = 6$. Moreover, 0 pairs of the vertices with span 1 from A have span 1 from each other, and 3 pairs with span 2 from A have span 2 from each other. Thus $G(P) = 3$ and $I(P) = 8 \cdot 6 - \frac{2}{3} \cdot 8 \cdot 3 = 32$. (This also follows from part (c) below).
- If P is an octahedron, every triplet of distinct vertices forms an isosceles triangle. Therefore $I(P) = \binom{6}{3} = 20$.
- If P is an icosahedron, then the numbers of vertices with spans 1, 2, and 3 from the fixed vertex A are 5, 5, and 1, respectively. Therefore, $F(P) = \binom{5}{2} + \binom{5}{2} = 20$. Moreover, 5 pairs of the vertices with span 1 from A have span 1 from each other, and 5 pairs with span 2 from A have span 2 from each other; thus $G(P) = 10$ and $I(P) = 12 \cdot 20 - \frac{2}{3} \cdot 12 \cdot 10 = 160$.
- If P is a dodecahedron, then the numbers of vertices with spans 1, 2, 3, 4, and 5 from A are 3, 6, 6, 3, and 1, respectively. So, $F(P) = \binom{3}{2} + \binom{6}{2} + \binom{6}{2} + \binom{3}{2} = 36$. Moreover, 0 pairs of vertices with span 1 from A have span 1 from each other, 3 pairs with span 2 from A have span 2 from each other, 6 pairs

with span 3 from A have span 3 from each other, and 0 pairs with span 4 from A have span 4 from each other; thus, $G(P) = 9$ and $I(P) = 20 \cdot 36 - \frac{2}{3} \cdot 20 \cdot 9 = 600$.

(c) Let P be a cube in $\mathbb{R}^n$. We may view the vertices of P as binary n-tuples, so that the distance between two vertices is the square root of the number of components at which they differ. The number of vertices of P at distance $\sqrt{k}$ from A is $\binom{n}{k}$ for $k = 0, 1, \ldots, n$. Recall that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ and } \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Therefore,

$$\begin{aligned} F(P) &= \sum_{k=1}^{n-1} \frac{1}{2}\binom{n}{k}\left(\binom{n}{k} - 1\right) = \frac{1}{2}\left(\sum_{k=1}^{n-1} \binom{n}{k}^2 - \sum_{k=1}^{n-1} \binom{n}{k}\right) \\ &= \frac{1}{2}\left(\left(\binom{2n}{n} - 2\right) - (2^n - 2)\right) \\ &= \frac{1}{2}\left(\binom{2n}{n} - 2^n\right) \end{aligned}$$

For the vertices A, B, and C to form an equilateral triangle with sides of length $\sqrt{k}$, three disjoint subsets, say X, Y, and Z, must be chosen from $\{1, 2 \ldots, n\}$ in such a way that the components of A differ from those of B at precisely the positions in $X \cup Y$, the components of A differ from those of C at precisely the positions in $X \cup Z$, and the components of B differ from those of C at precisely the positions in $Y \cup Z$. This forces $|X \cup Y| = |X \cup Z| = |Y \cup Z| = k$, which yields $|X| = |Y| = |Z| = \ell$ and $k = 2\ell$. There will be $n - 3\ell$ positions at which the components of A, B, and C all agree (the positions in the complement of $X \cup Y \cup Z$). Note that each equilateral triangle will be generated twice using this procedure because interchanging Y and Z will reverse the roles of B and C. Therefore (using multinomial coefficients), we have

$$G(P) = \frac{1}{2} \sum_{\ell=1}^{\lfloor n/3 \rfloor} \binom{n}{n - 3\ell, \ell, \ell, \ell} \text{ and}$$

$$I(P) = 2^{n-1}\left(\binom{2n}{n} - 2^n\right) - \frac{2^n}{3} \sum_{\ell=1}^{\lfloor n/3 \rfloor} \binom{n}{n - 3\ell, \ell, \ell, \ell}$$

Also solved by Allen J. Schwenk, Albert Stadler (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

A series involving the floor, ceiling, and round functions June 2020

2097. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

For a real number $x \notin \frac{1}{2} + \mathbb{Z}$, denote the nearest integer to x by $\langle x \rangle$. For any real number x, denote the largest integer smaller than or equal to x and the smallest integer larger than or equal to x by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. For a positive integer n let

$$a_n = \frac{2}{\langle\sqrt{n}\rangle} - \frac{1}{\lfloor\sqrt{n}\rfloor} - \frac{1}{\lceil\sqrt{n}\rceil}.$$

(a) Prove that the series $\sum_{n=1}^{\infty} a_n$ is convergent and find its sum L.

(b) Prove that the set

$$\left\{\sqrt{n}(\sum_{k=1}^{n} a_k - L) : n \geq 1\right\}$$

is dense in [0, 1].

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.

(a) We show that the sum converges to zero. To see this, first, we can easily check the following facts:

$$\begin{aligned} \langle\sqrt{n}\,\rangle &= k, \text{for } n \in [k(k-1)+1, k(k+1)], \\ \lfloor\sqrt{n}\rfloor &= k, \text{for } n \in [k^2, (k+1)^2), \\ \lceil\sqrt{n}\,\rceil &= k+1, \text{for } n \in (k^2, (k+1)^2]. \end{aligned}$$

These imply that $a_{k^2} = 0$ and

$$a_n = \frac{2}{k} - \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}, \text{ for } n \in (k^2, k(k+1)],$$

$$a_n = \frac{2}{k+1} - \frac{1}{k} - \frac{1}{k+1} = -\frac{1}{k(k+1)}, \text{ for } n \in (k(k+1), (k+1)^2).$$

Therefore, for $k^2 \leq n \leq (k+1)^2$, we have $\sum_{m=1}^{k^2} a_m = 0$ and

$$0 \leq \sum_{m=1}^{n} a_m \leq \frac{1}{k(k+1)} \cdot [k(k+1) - k^2] = \frac{1}{k+1}$$

As $n \to \infty$, we have $k \to \infty$ and so

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{m=1}^{n} a_m = 0.$$

(b) Let $x \in [0, 1]$. We show that there exists a subsequence from the set $\{\sqrt{n}\ \sum_{m=1}^{n} a_m\}$, which converges to x. Notice that there exist two integer sequences p_k and q_k with $0 \leq p_k \leq q_k$ such that $p_k/q_k \to x$, as $k \to \infty$. Let $n_k = q_k^2 + p_k$. Then

$$q_k^2 \leq n_k \leq q_k^2 + q_k < \left(q_k + \frac{1}{2}\right)^2.$$

This implies that

$$\langle\sqrt{n_k}\,\rangle = q_k,\ \lfloor\sqrt{n_k}\rfloor = q_k,\ \lceil\sqrt{n_k}\,\rceil = q_k + 1.$$

Therefore, as $k \to \infty$, we have

$$\sqrt{n_k} \sum_{m=1}^{n_k} a_m = \sqrt{n_k} \cdot \frac{n_k - q_k^2}{q_k(q_k+1)} = \frac{p_k}{q_k} \cdot \frac{\sqrt{n_k}}{q_k+1} \to x.$$

This proves that the set $\{\sqrt{n} \sum_{m=1}^{n} a_m\}$ is dense in $[0, 1]$.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Brian Bradie, Robert Calcaterra, Dmitry Fleischman, Maxim Galushka (UK), GWstat Problem Solving Group, Eugene A. Herman, Walter Janous (Austria), Donald E. Knuth, Sushanth Sathish Kumar, Elias Lampakis (Greece), Shing Hin Jimmy Pa (Canada), Allen Schwenk, Albert Stadler (Switzerland), and the proposer. There was one incorrect or incomplete solution.

A zigzag sequence of random variables **June 2020**

2098. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Let $Z_0 = 0$, $Z_1 = 1$, and recursively define random variables $Z_2, Z_3, \ldots,$ taking values in $[0, 1]$ as follows: For each positive integer k, Z_{2k} is chosen uniformly in $[Z_{2k-2}, Z_{2k-1}]$, and Z_{2k+1} is chosen uniformly in $[Z_{2k}, Z_{2k-1}]$.

Prove that, with probability 1, the limit $Z^* = \lim_{n\to\infty} Z_n$ exists and find its distribution.

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

We will prove:

1. The limit Z^* exists.
2. The limit Z^* has probability density $f(x) = 2x$ on $[0, 1]$.

Proof of 1. We have that $[Z_0, Z_1] \supseteq [Z_2, Z_1] \supseteq [Z_2, Z_3] \supseteq [Z_4, Z_3] \supseteq \ldots$ is a sequence of nested closed intervals. By the nested interval theorem, their intersection will be non-empty, and will consist of a unique point precisely if the sequence of lengths of the nested intervals tends to zero. We prove that this happens with probability 1.

Let I_n $(n = 0, 1, 2, , \ldots)$ be the nth interval in the sequence, and $L_n =$ length of I_n, i.e., $L_{2k} = Z_{2k+1} - Z_{2k}$ and $L_{2k+1} = Z_{2k+1} - Z_{2k+2}$. Pick $\delta > 0$. We will prove by induction that the probability of $L_n > \delta$ is $P(L_n > \delta) \leq (1-\delta)^n$. Since $P(L_n > 1) = 0$ the result is trivially true for $\delta \geq 1$, so we may assume $1 > \delta > 0$.

Base case: For $n = 0$ the inequality $P(L_0 > \delta) \leq (1-\delta)^0$ obviously holds because $L_0 = 1$, hence $P(L_0 > \delta) = P(1 > \delta) = 1$ and $(1-\delta)^0 = 1$.

Induction step: Assume $P(L_n > \delta) \leq (1-\delta)^n$. Then

$$P(L_{n+1} > \delta) = P(L_n \leq \delta) \cdot P(L_{n+1} > \delta \mid L_n \leq \delta) + P(L_n > \delta) \cdot P(L_{n+1} > \delta \mid L_n > \delta).$$

Note that the first term is zero because if $L_n \leq \delta$ then $L_{n+1} > \delta$ is impossible. On the other hand, if $L_n > \delta$ then we only have $L_{n+1} > \delta$ if the next endpoint Z_{n+2} is selected at a distance less than $L_n - \delta$ from the right or left (depending on the parity of n) endpoint of I_n. The probability is

$$P(L_{n+1} > \delta \mid L_n > \delta) = \frac{L_n - \delta}{L_n} = 1 - \frac{\delta}{L_n} \leq 1 - \delta.$$

Hence

$$P(L_{n+1} > \delta) \leq (1-\delta)^n(1-\delta) = (1-\delta)^{n+1},$$

and this completes the induction.

From here we get $\lim_{n\to\infty} P(L_{n+1} > \delta) = 0$ for every $\delta > 0$, hence $L_n \to 0$ as $n \to \infty$ with probability 1.

Proof of 2. For each $n \geq 0$ define the new random variable U_n, chosen between Z_{2n} and Z_{2n+1} with probability density

$$f_{U_n|Z_{2n}=z_n,Z_{2n+1}=z_{2n+1}}(x) = \frac{2(x-z_{2n})}{(z_{2n+1}-z_{2n})^2}$$

on $[z_{2n}, z_{2n+1}]$, where "$U_n|Z_{2n} = z_{2n}, Z_{2n+1} = z_{2n+1}$" means the random variable U_n given $Z_{2n} = z_{2n}$ and $Z_{2n+1} = z_{2n+1}$ (we ignore the case $z_{2n+1} = z_{2n}$ because its probability is zero).

Since U_n is between Z_{2n} and Z_{2n+1}, its limit U^* will coincide with Z^*.

Next, we will prove by induction that for every $n \geq 0$, the probability density of U_n is always the same, namely $f_{U_n}(x) = 2x$ on $[0, 1]$.

Base case: For $n = 0$ we have $Z_0 = 0$, $Z_1 = 1$, hence $f_{U_0}(x) = \dfrac{2(x-0)}{(1-0)^2} = 2x$ on $[0, 1]$.

Induction step: Assume $f_{U_n}(x) = 2x$. Next, note that U_{n+1} is defined like U_n but with starting points Z_2 and Z_3 in place of Z_0 and Z_1. So, U_{n+1} given $Z_2 = z_2$ and $Z_3 = z_3$ is just U_n mapped from $[0, 1]$ to $[z_2, z_3]$ with the transformation $(z_3 - z_2)U_n + z_2$. By induction hypothesis we have $f_{U_n}(x) = 2x$, and its transformation to $[z_2, z_3]$ will have probability density

$$f_{U_{n+1}|Z_2=z_2,Z_3=z_3}(x) = \frac{2(x-z_2)}{(z_3-z_2)^2}$$

on $[z_2, z_3]$.

The cumulative distribution function of U_{n+1} is $F_{U_{n+1}}(x) = P(U_{n+1} \leq x)$. By definition U_{n+1} must be in the interval $[Z_2, Z_3]$, while x may be in any of two different intervals, namely $[U_{n+1}, Z_3)$ or $[Z_3, 1]$. So, the event $U_{n+1} \leq x$ can be expressed as the union of $Z_2 \leq Z_3 \leq x$ and $Z_2 \leq U_{n+1} \leq x < Z_3$. Since they are disjoint we have

$$P(U_{n+1} \leq x) = P(Z_2 \leq Z_3 \leq x) + P(Z_2 \leq U_{n+1} \leq x < Z_3)\,.$$

We have that X_2 is random uniform on $[0, 1]$, and X_3 is random uniform on $[Z_2, 1]$, so

$$f_{Z_3|Z_2=z_2}(x) = \frac{1}{1-z_2},$$

hence

$$P(Z_2 \leq Z_3 \leq x) = \int_0^x \frac{x-z_2}{1-z_2}\,dz_2 = x + (1-x)\log(1-x)\,.$$

The second term can be computed as follows:

$$\begin{aligned} P(Z_2 \leq U_{n+1} \leq x < Z_3) &= \int_0^x \int_x^1 \int_{z_2}^x f_{U_{n+1}|Z_2=z_2,Z_3=z_3}(t) f_{Z_3|Z_2=z_2}(x)\,dt\,dz_3\,dz_2 \\ &= \int_0^x \int_x^1 \int_{z_2}^x \frac{2(t-z_2)}{(z_3-z_2)^2}\,\frac{1}{1-z_2}\,dt\,dz_3\,dz_2 \\ &= (x-1)(x+\log(1-x))\,, \end{aligned}$$

hence

$$F_{U_{2n+1}}(x) = x + (1-x)\log(1-x) + (x-1)(x + \log(1-x)) = x^2 .$$

Differentiating we get $f_{U_{2n+1}}(x) = 2x$ on $[0, 1]$, and this completes the induction.

Since the distribution of U_n is the same for every n we have that the limit U^* will have the same distribution too. And since $U^* = Z^*$, the same will hold for Z^*, hence $f_{Z^*}(x) = 2x$.

Also solved by Robert A. Agnew, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Shuyang Gao, John C. Kieffer, Omran Kouba (Syria), Kenneth Schilling, and the proposer.

An almost linear functional equation **June 2020**

2099. *Proposed by Russ Gordon, Whitman College, Walla Walla, WA and George Stoica, Saint John, NB, Canada.*

Let r and s be distinct nonzero rational numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f\left(\frac{x+y}{r}\right) = \frac{f(x) + f(y)}{s}$$

for all real numbers x and y.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.

Clearly the zero function is always a solution and, when $s = 2$, all constant functions are solutions. We show that there are no others. First assume $s \neq 2$. Substituting 0 for both x and y yields $f(0) = 0$. Substituting $y = 0$ and $y = -x$ yield these two identities:

$$f\left(\frac{x}{r}\right) = \frac{f(x)}{s}, \quad f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}.$$

Given any $x \in \mathbb{R}$, we use induction to show that $f(nx) = nf(x)$ for all $n \in \mathbb{N}$. The base case is a tautology. If $f(nx) = nf(x)$ for some $n \in \mathbb{N}$, then

$$\frac{f((n+1)x)}{s} = f\left(\frac{(n+1)x}{r}\right) = f\left(\frac{nx+x}{r}\right) = \frac{f(nx) + f(x)}{s} = \frac{(n+1)f(x)}{s}$$

and so $f((n+1)x) = (n+1)f(x)$. It follows that $f(x/n) = f(x)/n$ for all $n \in \mathbb{N}$ and hence that $f((m/n)x) = (m/n)f(x)$ for all $m, n \in \mathbb{N}$. Since $f(-x) = -f(x)$, this last statement is also true for m negative. Choose m, n so that $r = n/m$. Therefore

$$\frac{f(x)}{s} = f\left(\frac{x}{r}\right) = \frac{f(x)}{r}$$

and so $f(x) = 0$.

Now assume $s = 2$, and let $t = 2/r$. Thus $t \neq 1$ and

$$f\left(\frac{t}{2}(x+y)\right) = \frac{f(x) + f(y)}{2}, \quad \text{for all } x, y \in \mathbb{R}.$$

Substituting $y = x$ and $y = -x$ yield

$$f(tx) = f(x), \quad \frac{f(x) + f(-x)}{2} = f(0) \quad \text{for all } x \in \mathbb{R}.$$

Thus $f(-x/t) = f(-x)$, and so

$$f\left(\frac{t-1}{2}x\right) = f\left(\frac{t}{2}(x - x/t)\right) = \frac{f(x) + f(-x/t)}{2} = \frac{f(x) + f(-x)}{2} = f(0).$$

Therefore f is a constant function.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Paul Budney, Robert Calcaterra, Walther Janous (Austria), Sushanth Sathish Kumar, Omran Kouba (Syria), Elias Lampakis (Greece), Albert Natian, Kangrae Park (South Korea), Kenneth Schilling, Jacob Siehler, Albert Stadler (Switzerland), Michael Vowe (Switzerland), and the proposers.

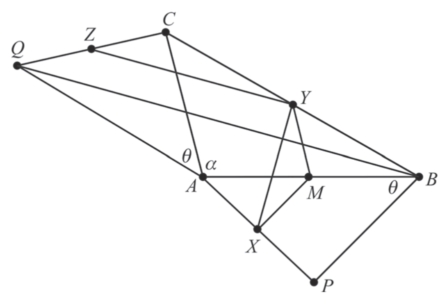
Two congruent triangles on the sides of an arbitrary triangle June 2020

2100. *Proposed by Yevgenya Movshovich and John E. Wetzel, University of Illinois, Urbana, IL.*

Given $\triangle ABC$ and an angle θ, two congruent triangles $\triangle ABP$ and $\triangle QAC$ are constructed as follows: $AQ = AB$, $BP = AC$, $m\angle ABP = m\angle CAQ = \theta$, B and Q are on opposite sides of $\overleftrightarrow{AC}$, and C and P are on opposite sides of $\overleftrightarrow{AB}$, as shown in the figure. Let X, Y, and Z be the midpoints of segments AP, BC, and CQ, respectively.

Show that $\angle XYZ$ is a right angle.

Solution by Sushanth Sathish Kumar (student), Portola High School, Irvine, CA.



Let M be the midpoint of segment AB. Note that $\overline{YZ}$ is a midline of triangle CBQ, and so $\overleftrightarrow{BQ}$ is parallel to $\overleftrightarrow{YZ}$. Thus, it suffices to show that $\overleftrightarrow{XY}$ is perpendicular to $\overleftrightarrow{BQ}$.

Since $\overline{MX}$ and $\overline{MY}$ are midlines of triangles APB and ABC, we have that $MX = BP/2 = AC/2 = MY$. Hence, triangle MXY is isosceles. Moreover, since $\overleftrightarrow{MX} || \overleftrightarrow{BP}$ and $\overleftrightarrow{MY} || \overleftrightarrow{AC}$, we have

$$m\angle XMY = m\angle XMA + m\angle AMY = \theta + 180^\circ - \alpha,$$

where we set $\alpha = m\angle BAC$. It follows that $m\angle MXY = m\angle XYM = (\alpha - \theta)/2$.

We wish to calculate $m\angle(\overleftrightarrow{XM}, \overleftrightarrow{BQ})$, where $m\angle(\ell_1, \ell_2)$ denotes the measure of the non-obtuse angle between ℓ_1 and ℓ_2. Note that

$$m\angle(\overleftrightarrow{XM}, \overleftrightarrow{BQ}) = m\angle PBQ = m\angle PBA + m\angle ABQ.$$

Since $AB = AQ$ and $m\angle BAQ = \alpha + \theta$, we find that $m\angle ABQ = 90^\circ - (\alpha + \theta)/2$. Thus, $m\angle(\overleftrightarrow{XM}, \overleftrightarrow{BQ}) = 90^\circ - (\alpha - \theta)/2$. But since $m\angle(\overleftrightarrow{MX}, \overleftrightarrow{XY}) = (\alpha - \theta)/2$, we find that $m\angle(\overleftrightarrow{BQ}, \overleftrightarrow{XY}) = 90^\circ$, and we are done.

Solutions

A geometric inequality **April 2020**

2091. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bârlad, Romania.*

Let ABC be a triangle with sides of lengths a, b, c, altitudes h_a, h_b, h_c, inradius r, and circumradius R. Prove that the following inequality holds:

$$h_a + h_b + h_c \geq 9r + \frac{a^2 + b^2 + c^2 - ab - ac - bc}{4R},$$

with equality if and only if $\triangle ABC$ is equilateral.

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI. Let K denote the area of $\triangle ABC$. We have

$$\begin{aligned} r &= \frac{2K}{a+b+c}, \\ R &= \frac{abc}{4K}, \\ h_a &= \frac{2K}{a}, \\ h_b &= \frac{2K}{b}, \quad \text{and} \\ h_c &= \frac{2K}{c}. \end{aligned}$$

Note that

$$\frac{abc}{K}(h_a + h_b + h_c) = 2(ab + ac + bc),$$

and

$$\begin{aligned} &\frac{abc}{K}\left(9r + \frac{a^2 + b^2 + c^2 - ab - ac - bc}{4R}\right) \\ &= \frac{18abc}{a+b+c} + a^2 + b^2 + c^2 - ab - ac - bc. \end{aligned}$$

Therefore, it will suffice to show that

$$2(ab + ac + bc) \geq \frac{18abc}{a+b+c} + a^2 + b^2 + c^2 - ab - ac - bc,$$

or equivalently,

$$f(a, b, c) = 2a^2b + 2ab^2 + 2a^2c + 2ac^2 + 2b^2c + 2bc^2 - a^3 - b^3 - c^3 - 9abc \geq 0.$$

Without loss of generality, we may assume that $c \geq b \geq a$. Note that

$$f(a, b, c) = (a + b - c)(c - a)(c - b) + (3c - a - b)(b - a)^2.$$

Since a, b, and c are the side lengths of a triangle, $a + b - c > 0$. Also,

$$3c - a - b = c + c - a + c - b > 0$$

as well. Hence $f(a, b, c) > 0$ if $c > b$ or $b > a$, and consequently $f(a, b, c) = 0$ can only occur when $a = b = c$. This concludes the proof.

Also solved by Arkady Alt, Farrukh Rakhimjanovich Ataev (Uzbekistan), Herb Bailey, Michel Bataille (France), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Scott H. Brown, Habib Y. Far, Subhankar Gayen & Vivekananda Mission Mahavidyalaya & Haldia Purba Medinipur (India), Finbarr Holland (Ireland), Walther Janous (Austria), Parviz Khalili, Koopa Tak Lun Koo (Hong Kong), Omran Kouba (Syria), Sushanth Sathish Kumar. Elias Lampakis (Greece), Kee-Wai Lau (China), Antoine Mhanna (Lebanon), Quan Minh Nguyen (Canada), Sang-Hoon Park (Korea), Volkhard Schindler (Germany), Albert Stadler (Switzerland), Daniel Văcaru (Romania), Michael Vowe (Switzerland), John Zacharias, and the proposer.

An integral involving the tail of a Maclaurin series **April 2020**

2092. *Proposed by Seán M. Stewart, Bomaderry, Australia.*

Let n be a non-negative integer. Evaluate

$$\int_0^\infty \frac{1}{x^{2n+3}} \left(\sin x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right) dx.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.
The answer is

$$(-1)^{n+1} \frac{\pi}{2(2n+2)!}.$$

We define

$$F_{2n}(x) = (-1)^n \left(\cos x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right), \quad \text{and}$$

$$F_{2n+1}(x) = (-1)^n \left(\sin x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right)$$

One easily sees that $F_m' = F_{m-1}$. Further,

$$F_m(x) = O(x^m) \ \text{ as } \ x \to \infty, \ \text{ and}$$

$$F_m(x) = O(x^{m+2}) \ \text{ as } \ x \to 0,$$

so the integral

$$I_m = \int_0^\infty \frac{F_m(x)}{x^{m+2}} \, dx$$

is convergent. A straightforward integration by parts shows that

$$\begin{aligned} I_m &= \left. \frac{-F_m(x)}{(m+1)x^{m+1}} \right|_{x=0}^{\infty} + \frac{1}{m+1} \int_0^\infty \frac{F_{m-1}(x)}{x^{m+1}} dx \\ &= \frac{1}{m+1} I_{m-1}. \end{aligned}$$

This implies that

$$I_m = \frac{I_0}{(m+1)!}.$$

Another integration by parts gives

$$\begin{aligned} I_0 &= \int_0^\infty \frac{\cos x - 1}{x^2}\,dx \\ &= \left.\frac{1-\cos x}{x}\right|_{x=0}^{\infty} - \int_0^\infty \frac{\sin x}{x}dx \\ &= -\int_0^\infty \frac{\sin x}{x}dx \\ &= -\frac{\pi}{2}. \end{aligned}$$

Thus,

$$I_m = -\frac{\pi}{2(m+1)!}.$$

In particular,

$$\begin{aligned} \int_0^\infty \frac{1}{x^{2n+3}}\left(\sin x - \sum_{k=0}^{n}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)dx &= (-1)^n I_{2n+1} \\ &= (-1)^{n+1}\frac{\pi}{2(2n+2)!}, \end{aligned}$$

which is the desired conclusion.

Also solved by Michel Bataille (France), Paul Bracken, Brian Bradie, David M. Bradley, Robert Calcaterra, William Chang, Robin Chapman (UK), Hongwei Chen, G.A. Edgar, Russell Gordon, Lixing Han, Eugene A. Herman, Finbarr Holland (Ireland), Sushanth Sathish Kumar, Elias Lampakis (Greece), Kee-Wai Lau (China), Quan Minh Nguyen (Canada), and the proposer. There were three incomplete or incorrect solutions.

A permutation probability **April 2020**

2093. *Proposed by Jacob Siehler, Gustavus Adolphus College, Saint Peter, MN.*

Suppose π is a permutation of $\{1, 2, \ldots, 2m\}$, where m is a positive integer. Consider the (possibly empty) subsequence of $\pi(m+1), \pi(m+2), \ldots, \pi(2m)$ consisting of only those values which exceed $\max\{\pi(1), \ldots, \pi(m)\}$. Let $P(m)$ denote the probability that this subsequence never decreases (note that the empty sequence has this property), when π is a randomly chosen permutation of $\{1, \ldots, 2m\}$. Evaluate $\lim\limits_{m\to\infty} P(m)$.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
The limit is $\sqrt{e}/2$. Let

$$k = \max\{\pi(1), \ldots, \pi(m)\}.$$

Clearly $m \le k \le 2m$. A permutation π with a given k satisfies the condition if and only if $k+1, k+2, \ldots, 2m$ is a (possibly empty, if $k = 2m$) subsequence of $\pi(m+1)$, $\pi(m+2), \ldots, \pi(2m)$. In the sequence $\pi(1), \ldots, \pi(2m)$ the number k may occupy any of the first m positions. The numbers $k+1, k+2, \ldots, 2m$ may occupy any $2m-k$ places among the last m places (i.e., $\binom{m}{m-k}$ possibilities), and the $2m-1-(m-k) = m+k-1$ remaining elements may be distributed in $(m+k-1)!$ ways. Therefore

$$P(m) = \frac{1}{(2m)!}\sum_{k=m}^{2m} m\binom{m}{2m-k}(m+k-1)!.$$

Putting $j = k - m$ we have

$$P(m) = \frac{1}{(2m)!} \sum_{j=0}^{m} m \binom{m}{j} (2m - j - 1)!.$$

Now

$$\begin{aligned} a_{j,m} &= \frac{1}{(2m)!} m \binom{m}{j} (2m - j - 1)! \\ &= \frac{m(m-1)(m-2)\cdots(m-j+1)}{2j!(2m-1)\cdots(2m-j)}. \end{aligned}$$

For fixed j, we have

$$\begin{aligned} \lim_{m\to\infty} a_{j,m} &= \lim_{m\to\infty} \frac{(1-\frac{1}{m})(1-\frac{2}{m})\cdots(1-\frac{j-1}{m})}{2j!(2-\frac{1}{m})\cdots(2-\frac{j}{m})} \\ &= \frac{1}{j!\,2^{j+1}}. \end{aligned}$$

Also

$$a_{j,m} < \frac{m^j}{2j!(2m-m)^j} = \frac{1}{2j!}$$

and

$$\sum_{j=0}^{\infty} \frac{1}{2j!} = e/2.$$

Hence by the dominated convergence theorem we have

$$\begin{aligned} \lim_{m\to\infty} P(m) &= \lim_{m\to\infty} \sum_{j=0}^{m} a_{j,m} \\ &= \sum_{j=0}^{\infty} \lim_{m\to\infty} a_{j,m} \\ &= \sum_{j=0}^{\infty} \frac{1}{j!\,2^{j+1}} \\ &= \frac{\sqrt{e}}{2}, \end{aligned}$$

as claimed.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Robin Chapman (UK), Kenneth Schilling, Edward Schmeichel, Albert Stadler (Switzerland), and the proposer. There was one incomplete or incorrect solution.

An upper bound for a vector sum **April 2020**

2094. *Proposed by George Stoica, Saint John, NB, Canada.*

Find the smallest number $f(n)$ such that for any set of unit vectors $x_1, \ldots, x_n$ in $\mathbb{R}^n$, there is a choice of $a_i \in \{-1, 1\}$ such that $|a_1x_1 + \cdots + a_nx_n| \le f(n)$.

Solution by Sushanth Sathish Kumar, student, Portola High School, Irvine, CA.
We claim that $f(n) = \sqrt{n}$. To see that this is minimal, consider the unit vectors $x_i = (0, \ldots, 1, \ldots, 0)$, where the ith term is 1 and the rest are 0. Then,

$$a_1x_1 + \cdots + a_nx_n = (\pm 1, \ldots, \pm 1)$$

has magnitude $\sqrt{n}$ regardless of choice of the a_i's.

We now show that $f(n) = \sqrt{n}$ does indeed work. Randomly and independently choose each a_i to be 1 or -1, both with probability $1/2$. We will prove that

$$\mathbb{E}\left[|a_1x_1 + \cdots + a_nx_n|^2\right] = n.$$

To see this, note that

$$\begin{aligned}\mathbb{E}[|a_1x_1 + \cdots + a_nx_n|^2] &= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_ix_i \cdot a_jx_j\right] \\ &= \sum_{i=1}^{n} \mathbb{E}\left[a_i^2|x_i|^2\right] + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathbb{E}[a_ix_i \cdot a_jx_j],\end{aligned}$$

by the dot product and linearity of expectation. Since $a_i^2 = 1$, and x_i is a unit vector, the first sum is just n. To compute the second sum, we note that

$$\begin{aligned}\mathbb{E}[a_ix_i \cdot a_jx_j] &= \mathbb{E}[a_ia_j|x_i||x_j|\cos\theta_{ij}] \\ &= \mathbb{E}[a_ia_j\cos\theta_{ij}] \\ &= 0,\end{aligned}$$

where θ_{ij} is the angle between vectors x_i and x_j. It follows that

$$\mathbb{E}\left[|a_1x_1 + \cdots + a_nx_n|^2\right] = n,$$

as claimed. Hence, there is a choice of $a_1, \ldots, a_n$ for which

$$|a_1x_1 + \cdots + a_nx_n|^2 \le n,$$

and we are done.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, William Chang, Lixing Han, Eugene Herman, Omran Kouba (Syria), Miguel A. Lerma, José Nieto (Venezuela), Celia Schacht, Albert Stadler (Switzerland), Edward Schmeichel, and the proposer. There was one incomplete or incorrect solution.

A floor function sum **April 2020**

2095. *Proposed by Mircea Merca, University of Craiova, Romania.*

Show that

$$\sum_{k=1}^{n} k\left\lfloor\frac{n+1-k}{d}\right\rfloor = \begin{cases}\lceil (n+1)(n-1)(2n+3)/24 \rceil & \text{if } d = 2\\ \left\lceil (n+1)^2(n-2)/18 \right\rceil & \text{if } d = 3\\ \lceil (n+1)(2n+1)(n-3)/48 \rceil & \text{if } d = 4\\ \lceil (n+1)n(n-4)/30 \rceil & \text{if } d = 5\end{cases}.$$

Solution by Russell Gordon, Whitman College, Walla Walla, WA.
We first observe that these four formulas can be combined into one formula by noting that

$$\sum_{k=1}^{n} k \left\lfloor \frac{n+1-k}{d} \right\rfloor = \left\lceil \frac{(n+1)(n+1-d)(2n+5-d)}{12d} \right\rceil$$

is equivalent to the equation above for $d = 2, 3, 4, 5$. We will also show that the analogous formula holds when $d = 1$. It is easy to verify that the formulas are valid for $n = 1, 2, \ldots, d$ for each of these values of d; we omit the simple arithmetic computations that generate 0's and 1's for these values of n and d. Hence, by induction, it is sufficient to show that the equation for a given d is valid for $n + d$ when it is valid for n. To verify this, we will use the fact that

$$\lfloor m + x \rfloor = m + \lfloor x \rfloor \text{ and } \lceil m + x \rceil = m + \lceil x \rceil$$

for any positive integer m and positive number x. We then have

$$\begin{aligned}
&\sum_{k=1}^{n+d} k \left\lfloor \frac{n+d+1-k}{d} \right\rfloor \\
&= \sum_{k=1}^{n} k \left(1 + \left\lfloor \frac{n+1-k}{d} \right\rfloor\right) + (n+1) \\
&= \sum_{k=1}^{n+1} k + \sum_{k=1}^{n} k \left\lfloor \frac{n+1-k}{d} \right\rfloor \\
&= \frac{(n+1)(n+2)}{2} + \left\lceil \frac{(n+1)(n+1-d)(2n+5-d)}{12d} \right\rceil \\
&= \left\lceil \frac{(n+1)(n+2)}{2} + \frac{(n+1)(n+1-d)(2n+5-d)}{12d} \right\rceil \\
&= \left\lceil \frac{(n+1)\left(6dn + 12d + 2n^2 + (7-3d)n + (1-d)(5-d)\right)}{12d} \right\rceil \\
&= \left\lceil \frac{(n+1)\left(2n^2 + (7+3d)n + (1+d)(5+d)\right)}{12d} \right\rceil \\
&= \left\lceil \frac{(n+1)(n+1+d)(2n+5+d)}{12d} \right\rceil,
\end{aligned}$$

as desired.

Remark. The analogous formulas do not hold for $d \geq 6$. For example, when $d = 6$ the two sides agree for all n, except when $n \equiv 0 \pmod 6$. In that case, we must subtract 1 from the right-hand side to maintain equality.

Also solved by Robert Calcaterra, William Chang, Dmitry Fleischman, Walther Janous (Austria), Elias Lampakis (Greece), Jacob Petry, Albert Stadler (Switzerland), and the proposer.

SOLUTIONS

Note that this section includes the solutions to Problems 1241–1244, which would normally have appeared in the January 2024 issue. The solution to Problem 1245 will appear in a later issue.

An inequality for a nonincreasing sequence on $(0, 1]$

1241. *Proposed by Reza Farhadian, Razi University, Kermanshah, Iran.*

Consider a finite sequence $1 = a_0 \geq a_1 \geq \cdots \geq a_{n+1} > 0$ of real numbers. Prove the following inequality:

$$\sqrt[n+1]{a_0 + a_1 + \cdots + a_{n+1}} < \sqrt[n]{a_0 + a_1 + \cdots + a_n}.$$

Solution by Shing Hin Jimmy Pa, China.

We introduce $x = a_0 + a_1 + \cdots + a_n$, and apply the AM–GM inequality:

$$\sqrt[n+1]{1 \cdot (x + a_{n+1})^n} < \frac{1 + n(x + a_{n+1})}{n+1}.$$

We also observe that $na_{n+1} + 1 \leq x$. Add nx to both sides and divide by $n + 1$:

$$\frac{1 + n(x + a_{n+1})}{n+1} \leq x.$$

Thus,

$$\sqrt[n+1]{(x + a_{n+1})^n} < x,$$

which is equivalent to $\sqrt[n+1]{x + a_{n+1}} < \sqrt[n]{x}$, as desired.

Also solved by Michel Bataille, Rouen, France; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Lau Kee-Wai, Hong Kong, China; JHSLPN Group, North Carolina School of Science and Mathematics, Durham, NC; Naïm Mégarbané, Lycée Stanislas High School, Paris, France; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Solution to a transcendental equation

1242. *Proposed by Adam Glesser, California State University, Fullerton, CA.*

Find all solutions to the following transcendantal equation (via proof, not by computer calculation):

$$\frac{6}{\pi e^6} \arcsin\left(\frac{x^4+48}{64x}\right) \exp\left(1+x+\frac{3}{\sqrt[3]{x-1}}\right) = 1.$$

Solution by Robert Doucette, McNeese State University, Lake Charles, LA.

Let $\alpha(x) = (x^4+48)/(64x)$, $\beta(x) = 1+x+3/\sqrt[3]{x-1}$, and $\phi(x) = \arcsin(\alpha(x))\exp(\beta(x))$. Since $\phi(2) = (\pi/6)e^6$, the given equation can be rewritten as $\phi(x) = \phi(2)$. Since $\phi(x) > 0$ only if $x > 0$, we need only consider positive numbers as possible solutions. Since

$$\alpha'(x) = \frac{3}{4}\left(\frac{x^2}{16} - \frac{1}{x^2}\right) \text{ and } \beta'(x) = 1 - \frac{1}{(x-1)^{4/3}},$$

the function α is strictly decreasing on $(0, 2)$ and strictly increasing on $(2, \infty)$, while the function β is strictly decreasing on $[0, 1)$ and on $(1, 2)$ (note the pole at $x = 1$!) and strictly increasing on $(2, \infty)$. Since $\alpha(x) \to \infty$ as $x \to 0^+$ and $\alpha(1) < 1$, there exists a unique $x_0 \in (0, 1)$ such that $\alpha(x_0) = 1$. Also since $\alpha(2) = 1/2$ and $\alpha(x) \to \infty$ as $x \to \infty$, there exists a unique $x_1 \in (2, \infty)$ such that $\alpha(x_1) = 1$. The set $[x_0, 1) \cup (1, x_1]$ are the positive numbers for which ϕ is defined.

For $x \in [x_0, 1)$, $\beta(x) < \beta(0) = -2$. This implies that $\phi(x) < (\pi/2)e^{-2} < \phi(2)$, so there are no solutions to the given equation in the interval $[x_0, 1)$.

Since both α and β are positive and strictly decreasing on the interval $(1, 2)$, the functions $\arcsin(\alpha)$ and $\exp(\beta)$ are both positive and strictly decreasing on $(1, 2)$. It follows that ϕ is strictly decreasing on $(1, 2)$ and that $\phi(x) > \phi(2)$ for $x \in (1, 2)$.

In a similar way we may show that ϕ is strictly increasing on $(2, x_1]$, so that $\phi(x) > \phi(2)$ for $x \in (2, x_1]$.

It follows that 2 is the unique solution to the given equation.

Also solved by Naïm Mégarbané, Lycée Stanislas High School, Paris, France; Albert Stadler, Switzerland; and the proposer. Received one incomplete solution.

An integral inequality

1243. *Proposed by Cezar Lupu, Yanqi Lak Bimsa and Tsinghua University, Beijing, China.*

Let $f : [0, 1] \to \mathbb{R}$ be an integrable function such that $\int_0^1 f(x)\,dx = 1$ and $\int_0^1 x^2 f(x)\,dx = 1$. Prove that $\int_0^1 f^2(x)\,dx \geq 6$.

Solution by Mark Sand, College of St. Mary, Omaha, NE.
Given such a function $f(x)$, we know that for any real number r,

$$\begin{aligned}1 + r &= \int_0^1 (1 + r \cdot x^2) f(x)\, dx \\ &\le \int_0^1 \left|(1 + r \cdot x^2) f(x)\right| dx \\ &\le \left(\int_0^1 (1 + r \cdot x^2)^2\, dx\right)^{1/2} \left(\int_0^1 f^2(x)\, dx\right)^{1/2},\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the last step. The integral that includes the number r has the value $\sqrt{1 + \frac{2}{3}r + \frac{1}{5}r^2}$. Dividing by this and then squaring and simplifying, we see that

$$\int_0^1 f^2(x)\, dx \ge \frac{15(r+1)^2}{3r^2 + 10r + 15}.$$

Since the left side is fixed once $f(x)$ is given, this inequality must be true for all values of the fraction on the right, including the maximum value of the fraction. We note here that the denominator is never zero, which can be seen by completing the square.
Letting $G(r) = \frac{15(r+1)^2}{3r^2+10r+15}$, we find $G'(r) = \frac{60(r^2+6r+5)}{(3r^2+10r+15)^2}$, so the derivative is zero when r is -1 or -5. The only term in the derivative that changes sign is $(r^2 + 6r + 5)$, and we can easily see that $G'(r)$ is positive on $(-\infty, -5) \cup (-1, \infty)$ and negative on $(-5, -1)$. This, along with $\lim_{r\to\infty} G(r) = 5$, tells us that $G(-5) = 6$ is the maximum value of the fraction we have been investigating.
Thus, $\int_0^1 f^2(x)\, dx \ge 6$, as desired.

Also solved by MICHEL BATAILLE, Rouen, France; RUSSELL GORDON, Whitman University; TOM JAGER, Calvin University; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; MICHAEL LAVIGNE, North Carolina School of Math and Science; KELLY MCLENITHAN, Los Alamos, NM; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Distribution of fractional parts of uniform variables

1244. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*
Suppose $(X_k)_1^n$ is a sequence of n independent random variables uniformly distributed over the interval [0, 1]. Prove that the fractional part of the random variable $\sum_{k=1}^n X_i$ is uniformly distributed over [0, 1].

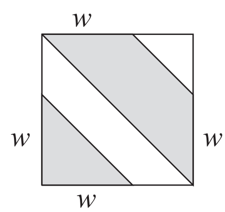
Solution by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA.
If $n = 1$, then X_1 is equal to its fractional part, and the statement is trivially true. We claim that if X and Y are independent random variables uniformly distributed over [0, 1], then the fractional component of the sum $W = X + Y$ is also uniformly distributed on [0, 1]. First, we express the fractional component of the sum as

$$\lfloor W \rfloor = (X + Y)1_{\{X+Y<1\}} + (X + Y - 1)1_{\{X+Y>1\}}.$$

Then for $w \in [0, 1]$,

$$\begin{aligned}
\Pr(\lfloor W \rfloor \le w) &= \Pr(((X + Y \le w) \cap (X + Y < 1)) \cup ((X + Y - 1 \le w) \cap (X + Y > 1))) \\
&= \Pr((X + Y \le w) \cap (X + Y < 1)) + \Pr(0 < X + Y - 1 \le w) \\
&= \frac{1}{2}w^2 + \left(\frac{1}{2} - \frac{1}{2}(1 - w)^2\right) \\
&= \frac{1}{2}w^2 + w - \frac{1}{2}w^2 \\
&= w.
\end{aligned}$$

The probabilities are illustrated geometrically in the following diagram.



Therefore, the fractional part of $W = X + Y$ is uniformly distributed over [0, 1], and the general statement follows by induction with $X = \sum_{k=1}^{n} X_k$ and $Y = X_{n+1}$.

Also solved by Michael P. Cohen, Fairfax, VA; Jan Grzesik, Torrance, CA; Shing Hin Jimmy Pa, China; Albert Stadler, Herrliberg, Switzerland; and the proposer.

SOLUTIONS

To our valued contributors: *CMJ Solutions* is in transition.

Charles N. Curtis, who has served as Solutions editor for nearly 10 years, is retiring from this position. I am thankful for his valuable service on the *CMJ* board over these many years. I'm certain that everyone associated with *CMJ* has been grateful for his leadership and expertise.

I am pleased to announce that Katherine Thompson and Matyas Sustik are joining the *CMJ* editorial board as our new Solutions editors. Both bring significant experience with problem solving competitions, and I am looking forward to working with them.

Dr. Thompson is currently Assistant Professor of Mathematics at the U.S. Naval Academy. She is a regular instructor and grader for the Art of Problem Solving and is the former chair of the Question Writing Committee for MATHCOUNTS.

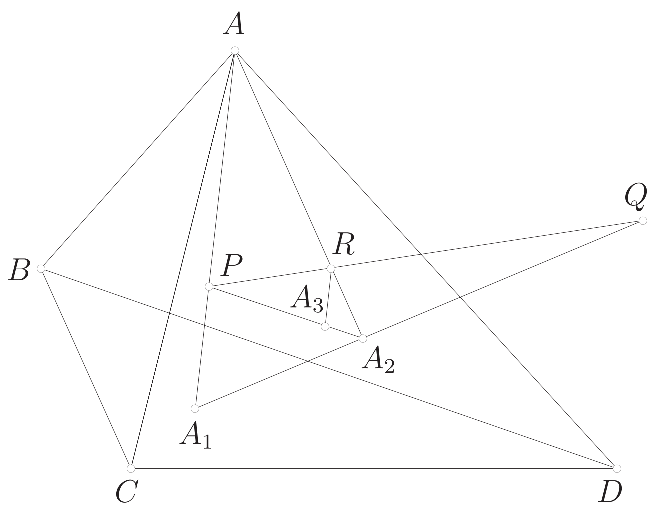
Dr. Sustik works in industry as a mathematician and software engineer. With an active and successful high school math contest participation behind him (that included the IMO) now he gives back by developing, grading, and evaluating mathematical contest problems for AMC, AIME, and BAMO.

I currently expect *CMJ Solutions* to return in the March issue. It will take a couple of issues for us to catch up and resume our typical schedule. I ask for your patience as we complete this transition.

— Tamara Lakins, Editor

The centroid of a tetrahedron

1236. *Proposed by Tran Quang Hung, Vietnam National University, Hanoi, Vietnam.*
Let $ABCD$ be a tetrahedron in 3-space, and let P, Q and R be three collinear points. Assume that lines PA, PB, PC, and PD are not parallel to planes (BCD), (CDA), (DAB), and (ABC), respectively. Line PA meets plane (BCD) at point A_1. In the plane (APR), assume that the two lines AR and A_1Q intersect at A_2. Point A_3 lies on line PA_2 such that RA_3 is parallel to line AA_1. Define similarly the points B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2, and D_3. Prove that R is the centroid of tetrahedron $A_3B_3C_3D_3$ (see figure).



Solution by the proposer.

Proof. Let $\overline{AB}$ and $\overline{BA}$ denote signed lengths of segments. Apply the theorem of Menelaus to $\triangle APR$ with transversal A_1A_2Q to get

$$\frac{\overline{A_1P}}{\overline{A_1A}} \cdot \frac{\overline{A_2A}}{\overline{A_2R}} \cdot \frac{\overline{QR}}{\overline{QP}} = 1. \tag{1}$$

Let the Euclidean vector connecting an initial point X with a terminal point Y be denoted by $\overrightarrow{XY}$. Let x, y, z, t, not all zero, such that

$$x\vec{PA} + y\vec{PB} + z\vec{PC} + t\vec{PD} = \vec{0}. \tag{2}$$

If $x+y+z+t=0$, then equation (2) becomes

$$-(y+z+t)\vec{PA}+y\vec{PB}+z\vec{PC}+t\vec{PD}=\vec{0},$$

which implies

$$y\vec{AB}+z\vec{AC}+t\vec{AD}=\vec{0}.$$

This would mean that $\left\{\vec{AB},\vec{AC},\vec{AD}\right\}$ is linearly dependent, which is impossible since A, B, C, and D are not coplanar. Therefore, we have $x+y+z+t\neq 0$.

Note that applying projections parallel to line AP onto plane (BCD), one has

$$A,P\mapsto A_1,\ B\mapsto B,\ C\mapsto C,\ D\mapsto D. \tag{3}$$

Since parallel projection is an affine transformation, it follows from (2) and (3),

$$y\vec{A_1B}+z\vec{A_1C}+t\vec{A_1D}=\vec{0}. \tag{4}$$

From (4), we may deduce $y\vec{PB}+z\vec{PC}+t\vec{PD}=(y+z+t)\vec{PA_1}$. Combining with (2), we obtain

$$-x\vec{PA}=(y+z+t)\vec{PA_1},$$

or

$$x\vec{A_1A}=(x+y+z+t)\vec{PA_1}.$$

From this,

$$\frac{\overline{A_1P}}{\overline{A_1A}}=\frac{-x}{x+y+z+t}. \tag{5}$$

Let

$$\frac{\overline{QR}}{\overline{QP}}=k. \tag{6}$$

It follows from (1), (5), and (6),

$$\frac{\overline{A_2R}}{\overline{A_2A}}=\frac{-kx}{x+y+z+t}.$$

Since $RA_3\parallel PA$, applying the theorem of Thales yields

$$\vec{RA_3}=\frac{\overline{RA_3}}{\overline{PA}}\cdot\vec{PA}=\frac{\overline{A_2R}}{\overline{AA_2}}\cdot\vec{PA}=\frac{kx\cdot\vec{PA}}{x+y+z+t}. \tag{7}$$

Similarly, we have

$$\vec{RB_3}=\frac{ky\cdot\vec{PB}}{x+y+z+t},\ \vec{RC_3}=\frac{kz\cdot\vec{PC}}{x+y+z+t},\ \vec{RD_3}=\frac{kt\cdot\vec{PD}}{x+y+z+t}. \tag{8}$$

From (7) and (8), we get

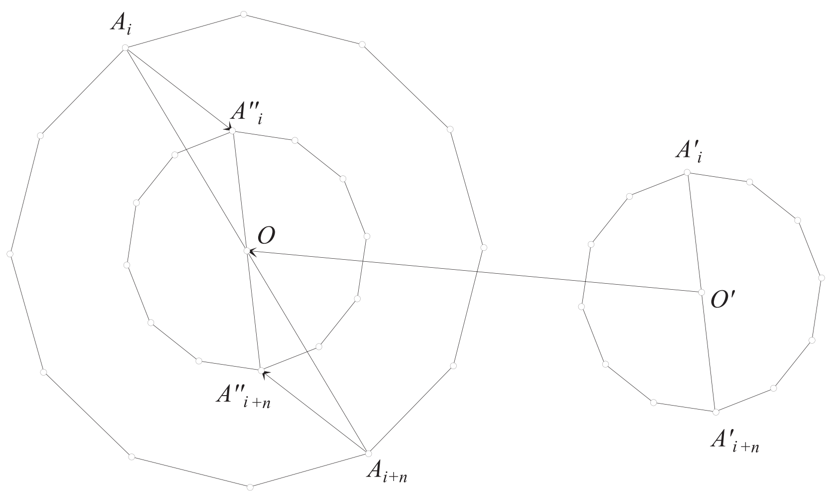
$$\vec{RA_3} + \vec{RB_3} + \vec{RC_3} + \vec{RD_3} = \frac{k\left(x\vec{PA} + y\vec{PB} + z\vec{PC} + t\vec{PD}\right)}{x+y+z+t} = \vec{0},$$

which implies R is the centroid of $A_3B_3C_3D_3$. This completes the proof. ■

No other solutions were received.

Two polygons

1237. *Proposed by Tran Quang Hung, Vietnam National University, Hanoi, Vietnam.*
Let $A_1A_2 \ldots A_{2n}$ and $A'_1A'_2 \ldots A'_{2n}$ ($n \geq 2$) be two directly $2n$-regular polygons. Prove that $\sum_{i=1}^{n} A_{2i}A'^2_{2i} = \sum_{i=0}^{n-1} A_{2i+1}A'^2_{2i+1}$ (see figure).



Solution by Albert Stadler, Herrliberg, Switzerland.
We may assume (without loss of generality) that the vertices of the two polygons are given by

$$A_k = re^{\frac{\pi ik}{n}+i\omega}, \text{ and } A'_k = 1 + r'e^{\frac{\pi ik}{n}+i\omega'}, \text{ for } k = 1, 2, \ldots, 2n.$$

Then

$$\sum_{i=1}^{n}\left(A_{2i}A'_{2i}\right)^2 - \sum_{i=0}^{n-1}\left(A_{2i+1}A'_{2i+1}\right)^2$$
$$= \sum_{k=1}^{n}\left[\left(A_{2k}A'_{2k}\right)^2 - \left(A_{2k-1}A'_{2k-1}\right)^2\right]$$
$$= \sum_{k=1}^{n}\left[\left|r'e^{\frac{\pi i(2k)}{n}+i\omega'} + 1 - re^{\frac{\pi i(2k)}{n}+i\omega}\right|^2\right.$$
$$\left.-\left|r'e^{\frac{\pi i(2k-1)}{n}+i\omega'} + 1 - re^{\frac{\pi i(2k-1)}{n}+i\omega}\right|^2\right]$$

$$= \sum_{k=1}^{n}\Big[-re^{-\frac{2\pi ik}{n}-i\omega}+re^{-\frac{\pi i(2k-1)}{n}-i\omega}-re^{\frac{2\pi ik}{n}+i\omega}$$

$$+re^{\frac{\pi i(2k-1)}{n}+i\omega}+r'e^{-\frac{2\pi ik}{n}-i\omega'}$$

$$-r'e^{-\frac{\pi i(2k-1)}{n}-i\omega'}+r'e^{\frac{2\pi ik}{n}+i\omega'}-r'e^{\frac{\pi i(2k-1)}{n}+i\omega'}\Big]=0,$$

since $\sum_{k=1}^{n} e^{\frac{2\pi ik}{n}} = 0.$

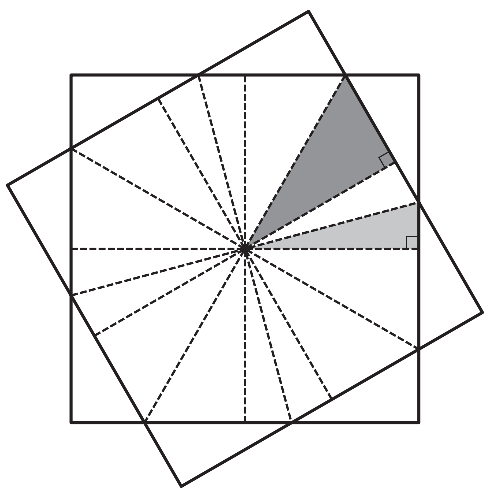
Also solved by Dmitry Fleischman, Santa Monica, CA; Eugene Herman, Grinnell C.; and the proposer.

Rotated squares

1238. *Proposed by Jacob Siehler, Gustavus Adolphus College, St. Peter, MN.*

Consider the intersection of a unit square with a copy of itself rotated through an angle of θ about their mutual center. Note that in general, this region is an octagon. Evaluate the average area of the intersection as θ ranges from 0 to $\frac{\pi}{2}$.

Solution by Kyle Calderhead, Malone University, Canton, Ohio.



By extending lines from the mutual center to the midpoints of each side of each square, as well as to the points of intersection of their sides, we can decompose the octagonal intersection into sixteen right triangles—eight with a leg of length $\frac{1}{2}$ and adjacent angle of $\frac{\theta}{2}$, and eight more with a leg of length $\frac{1}{2}$ and adjacent angle of $\frac{\pi}{4}-\frac{\theta}{2}$. In the figure above, one of each of these types of triangles has been highlighted.

Using right-triangle trigonometry, we see that the length of the other legs of these triangles are $\frac{1}{2}\tan\theta$ and $\frac{1}{2}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$, respectively. Hence the areas of each type of triangle are $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\tan\left(\frac{\theta}{2}\right)$ and $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$, respectively. With eight of each, we have a total area of

$$A = \tan\left(\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right).$$

Note that this formula is consistent with the situation where the squares coincide, corresponding to θ being equal to either 0 or $\frac{\pi}{2}$.

Before taking the average value, note that the integral $\int_0^{\frac{\pi}{2}} \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\, d\theta$ can be shown to be equivalent to the integral $\int_0^{\frac{\pi}{2}} \tan\left(\frac{\theta}{2}\right)\, d\theta$ by means of the substitution $u = \frac{\pi}{2} - \theta$. This simplifies the average value calculation to

$$\frac{1}{\pi/2}\int_0^{\frac{\pi}{2}} 2\tan\left(\frac{\theta}{2}\right)\, d\theta = \frac{4}{\pi}\left[-2\ln\left|\cos\left(\frac{\theta}{2}\right)\right|\right]_0^{\frac{\pi}{2}} = \frac{4\ln 2}{\pi},$$

or approximately 0.8825.

NOTE: Using the same dissection technique, we can show that in the more general case of two overlapping regular n-gons with unit area, the average area of their intersection will be $\frac{n}{\pi}\cot\left(\frac{\pi}{n}\right)\ln\left(\sec^2\left(\frac{\pi}{n}\right)\right)$.

Also solved by RICARDO ALFARO, U. of Michigan - Flint; ANDREW BAUMAN, U. of Arkansas at Little Rock; NATE BELGARD, The Barrie School; BRIAN BRADIE, Christopher Newport U.; ROB DOWNES, Newark Academy; BILL DUNN , Montgomery C.; EAGLE PROBLEM SOLVERS, Georgia Southern U.; HABIB FAR, Lone Star C. - Montgomery; DMITRY FLEISCHMAN, Santa Monica, CA; MICHAEL GOLDENBERG, Reiserstown, MD and MARK KAPLAN, U. of Maryland Global Campus (jointly); AAKASH GURUNG, ASAHI NAGO, and XUAN PHAM (jointly); SPENCER HARRIS, Westmont C. (graduate); EUGENE HERMAN, Grinnell C.; STEPHEN HERSCHKORN, Rutgers U.; LIAM MAUCK and CLAYTON COE, Cal Poly Pomona Problem Solving Group; KELLY MCLENITHAN, Los Alamos, NM; PETER OMAN and HAOHAO WANG, Southeast Missouri St. U.; LEAH RAMOS (STUDENT), Seton Hall U.; VOLKHARD SCHINDLER, Berlin, Germany; SKIDMORE C. PROBLEM GROUP; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

An explicit formula for a sequence from a recursion

1239. *Proposed by Moubinool Omarjee, Lycée Henry IV, Paris, France.*

Let u_0 be a positive real number, and for every $n \in \mathbb{N}$, define $u_{n+1} := \frac{u_n^3 - 3u_n - \sqrt{5}}{3u_n^2 + 3\sqrt{5}u_n + 4}$. Find a closed-form expression for u_n in terms of u_0 and n.

Solution by the Stephen Locke, Florida Atlantic University.

Lemma 1. *Let* $g(x) = \dfrac{x^3}{(x+1)^3 - x^3}$. *Then, the* kth *iterate* $g^{(k)}$ *of* g *is given by* $g^{(k)}(x) = \dfrac{x^{3^k}}{(x+1)^{3^k} - x^{3^k}}$.

Proof. We note that $g^{(1)} = g$ and assume that for some k, $g^{(k)}(x) = \dfrac{x^{3^k}}{(x+1)^{3^k} - x^{3^k}}$. Then,

$$\begin{aligned} g^{(k+1)}(x) &= g\left(\frac{x^{3^k}}{(x+1)^{3^k} - x^{3^k}}\right) \\ &= \left(\frac{x^{3^k}}{(x+1)^{3^k} - x^{3^k}}\right)^3\left(\left(\frac{x^{3^k}}{(x+1)^{3^k} - x^{3^k}} + 1\right)^3 - \left(\frac{x^{3^k}}{(x+1)^{3^k} - x^{3^k}}\right)^3\right)^{-1} \end{aligned}$$

$$= x^{3^{k+1}} \left(\left(x^{3^k} + \left((x+1)^{3^k} - x^{3^k} \right) \right)^3 - \left(x^{3^k} \right)^3 \right)^{-1}$$

$$= x^{3^{k+1}} \left(\left(x+1)^{3^k} \right)^3 - x^{3^{k+1}} \right)^{-1},$$

establishing the inductive proof. ■

Now, let $f(w) = \dfrac{w^3 - 3w - \sqrt{5}}{3w^2 + 3\sqrt{5}w + 4}$, so that $u_{n+1} = f(u_n)$, and let $\tau = \dfrac{1-\sqrt{5}}{2}$. Note that $f(\tau + w) = \tau + \dfrac{w^3}{3w^2 + 3w + 1} = \tau + \dfrac{w^3}{(w+1)^3 - w^3} = \tau + g(w)$. Hence, for $w = u_0 - \tau$,

$$u_n = f^{(n)}(\tau + w) = \tau + g^{(n)}(w) = \tau + \frac{w^{3^n}}{(w+1)^{3^n} - w^{3^n}},$$

providing a closed form for u_n in terms of u_0 and n.

Also solved by Brian Bradie, Christopher Newport U.; Michael Goldenberg, Reistertown, MD and Mark Kaplan, U. of Maryland Globan Campus (jointly); Albert Stadler, Herrliberg, Switzerland; and the proposer.

Fields for which the collection of additive subgroups and the collection of multiplicative subgroups are isomorphic

1240. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let S be a set. Recall that a *partial order* on S is a binary relation $\leq$ which is reflexive, anti-symmetric, and transitive. If S, T are sets and $\leq, \preceq$ are partial orders on S and T, respectively, then we say that the partially ordered set $(S, \leq)$ and $(T, \preceq)$ are *isomorphic* if there is a bijection $f\colon S \to T$ such that for all $s_1, s_2 \in S$: $s_1 \leq s_2$ iff $f(s_1) \preceq f(s_2)$. Now let F be a field, and let $\mathcal{P}^+(F)$ be the collection of additive subgroups of F, partially ordered by set-theoretic inclusion, and let $\mathcal{P}^\times(F)$ be the collection of multiplicative subgroups of $F^\times := F\backslash\{0\}$, partially ordered by inclusion. Find all fields F for which $\mathcal{P}^+(F)$ and $\mathcal{P}^\times(F)$ are isomorphic.

Solution by Anthony Bevelacqua, University of North Dakota, Grand Forks, North Dakota.

Any subgroup H of $F^\times$ corresponds to an additive subgroup A of F in such a way that the subgroup lattices of H and A are isomorphic. Consequently the trivial subgroup $\langle 1 \rangle$ of $F^\times$ must correspond to the trivial subgroup $\langle 0 \rangle$ of F. Since a group is finite if and only if it has finitely many subgroups, finite subgroups of $F^\times$ correspond to finite additive subgroups of F. Since a field of characteristic zero has a nontrivial finite multiplicative subgroup (namely $\{1, -1\}$) and every nontrivial additive subgroup of a field of characteristic zero is infinite, F must have characteristic $p > 0$. Thus $\mathbb{Z}_p$, the field with p elements, is a subfield of F. We note that the additive subgroups of F are precisely the $\mathbb{Z}_p$-subspaces of F.

Assume $\dim_{\mathbb{Z}_p} F > 1$. Then F contains a subspace A of dimension two. A contains exactly $p+1$ proper, nontrivial subgroups, no one of which is contained in another. Now A corresponds to a finite subgroup H of $F^\times$ with exactly $p+1$ proper, nontrivial subgroups, no one of which is contained in another. Recall that $J \mapsto |J|$ gives an isomorphism between the lattice of subgroups of a cyclic group of order n and the lattice of positive divisors of n ordered by divisibility. Since H has $p+1 \geq 3$ proper, nontrivial subgroups, no one of which is contained in another, $|H|$ must be divisible by (at least) three distinct primes q, r, and s. Now q is a proper divisor of qr and qr is a proper divisor qrs, so H contains a pair of nested proper, nontrivial subgroups, a contradiction.

Thus $F = \mathbb{Z}_p$. Since the additive group $\mathbb{Z}_p$ has exactly two subgroups, $\mathbb{Z}_p^\times$ has exactly two subgroups. Therefore $p - 1 = |\mathbb{Z}_p^\times|$ is a prime, and so $p = 3$. Hence $\mathbb{Z}_3$ is the only field F for which $\mathcal{P}^+(F)$ and $\mathcal{P}^\times(F)$ are isomorphic.

Also solved by the proposer.

An inequality for the angles of a triangle

1231. *Proposed by George Apostolopoulos, Messolongi, Greece.*

Let ABC be a triangle. Show that $\sum_{\alpha=A,B,C} \sin^3(\alpha)\cos(\alpha) \leq \frac{9\sqrt{3}}{16}$.
Solution by John Christopher, California State University, Sacramento.

Note that each angle of triangle ABC lies in the interval $(0, \pi)$. Using first semester calculus, it is easily shown that in the interval $(0, \pi)$, the function $f(x) = \sin^3 x \cos x$ attains its maximum value when $x = \pi/3$. Since $f(\pi/3) = \left(\frac{\sqrt{3}}{2}\right)^3 \cdot \frac{1}{2} = \frac{3\sqrt{3}}{16}$, we have $f(\angle A) + f(\angle B) + f(\angle C) \leq \frac{3\sqrt{3}}{16} + \frac{3\sqrt{3}}{16} + \frac{3\sqrt{3}}{16} = \frac{9\sqrt{3}}{16}$. Equality is attained when the triangle is equilateral and each angle is $\pi/3$.

Also solved by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Germany; Michel Bataille, Rouen, France; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Charles Burnette, Xavier U. of Louisiana; M. V. Channakeshava, Bengaluru, India; Ritabrato Chaterjee (student), Western Michigan U.; Danko Dmitry (student), RUDN U., Moscow, Russia; Eagle Problem Solvers, Georgia Southern U.; The Episcopal Academy Problem Solvers; Habib Far, Lone Star C. - Montgomery; Meagan Fisher, Anna Phillips, Juan Martinez, and William French (students), U. of Arkansas at Little Rock; Fresno State Journal Problem Solving Group; Shubham Goel, GGSIPU, Uttar Pradesh, India; Michael Goldenberg, Reierstown, MD and Mark Kaplan, U. of Maryland Global Campus (jointly); Russ Gordon, Whitman C.; Jacob Guerra, Lowell, MA; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; A. Bathi Kasturiararchi, Kent St. U. at Stark; Hidefumi Katsuura, San Jose St. U.; Parviz Khalili, Newport News, VA; Joseph Klaips (student), North Central C.; Panagiotis Krasopoulos, Athens, Greece; Wei-Kai Lai, U. of South Carolina Salkehatchie; Kee-Wai Lau, Hong Kong, China; Shing Hin Jimmy Pa; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma; Chrysostom Petalas, Ioannina, Greece; Volkhard Schindler, Berlin, Germany; Joel Schlosberg, Bayside, NY; Digby Smith, Waterton Lakes Mathematics Guild; Southeast Missouri State U. Math Club, ; Albert Stadler, Herrliberg, Switzerland; Michael Vowe, Therwil, Switzerland; and the proposer. One incomplete solution was received.

The Catalan numbers

1232. *Proposed by Jacob Guerra, Salem State University, Salem, MA.*

Define, for every nonnegative integer n, the nth Catalan number by $C_n := \frac{1}{n+1}\binom{2n}{n}$. Consider the sequence of complex polynomials in z defined by $z_k := z_{k-1}^2 + z$ for every nonnegative integer k, where $z_0 := z$. It is clear that z_k has degree 2^k and thus has the representation $z_k = \sum_{n=1}^{2^k} M_{n,k} z^n$, where each $M_{n,k}$ is a positive integer. Prove that $M_{n,k} = C_{n-1}$ for $1 \le n \le k+1$.

Solution by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

We proceed by induction on k, noting that for the base case $k = 0$, we have $M_{1,0} = 1 = C_0$. For the induction step, suppose that $M_{n,r} = C_{n-1}$ for $1 \le n \le r+1$, where r is a nonnegative integer. Observe that

$$z_{r+1} = \left(\sum_{n=1}^{2^r} M_{n,r} z^n\right)\left(\sum_{n=1}^{2^r} M_{n,r} z^n\right) + z = \sum_{n=2}^{2^{r+1}} \left(\sum_{m=1}^{n-1} M_{m,r} M_{n-m,r}\right) z^n + z,$$

so that then $M_{1,r+1} = 1 = C_0$. Furthermore, because the Catalan numbers satisfy the recurrence $C_{n+1} = \sum_{m=0}^{n} C_m C_{n-m}$, we find that

$$M_{n,r+1} = \sum_{m=1}^{n-1} M_{m,r} M_{n-m,r} = \sum_{m=1}^{n-1} C_{m-1} C_{n-m-1} = \sum_{m=0}^{n-2} C_m C_{n-m-2} = C_{n-1}$$

for $1 \le n \le r+2$. *Also solved by* Ulrich Abel, Technische Hochschule Mittelhessen, Germany; Cal Poly Pomona Problem Solving Group; Hongwei Chen, Christopher Newport U.; Eagle Problem Solvers, Georgia Southern U.; Michael Goldenberg, Reisterstown, MD and Mark Kaplan, U. of Maryland Global Campus (jointly); Eugene Herman, Grinnell C.; Walther Janous, Innsbruck, Austria; Panagiotis Krasopoulos, Athens, Greece; Shing Hin Jimmy Pa; John Quintanilla, U. of North Texas; Ajay Srinivasan, U. of Southern California; Albert Stadler, Herrliberg, Switzerland; Dan Swenson, Black Hills St. U.; and the proposer. One incomplete solution was received.

Uniform random variables

1233. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Suppose that X and Y are independent, uniform random variables over [0, 1]. Define U_X, V_X, and B_X as follows: U_X is uniform over $[0, X]$, V_X is uniform over $[X, 1]$, and $B_X \in \{0, 1\}$, with $P(B_X = 1) = X$, and $P(B_X) = 0 = 1 - X$. Now define random variables Z and W_X as follows:

$$Z = Y - X\mathbf{1}\{Y \ge X\} + (1 - X + Y)\mathbf{1}\{Y < X\}, \text{ and}$$

$$W_X = B_X \cdot U_X + (1 - B_X)V_X.$$

Prove that both Z and W_X are uniform over [0, 1]. Here, $\mathbf{1}[S]$ is the indicator function that is equal to 1 if S is true and 0 otherwise. *Solution by John Quintanilla, University of North Texas, Denton, Texas.*

We proceed by induction on k. The statement clearly holds for $k = 1$:

$$z_1 = z_0^2 + z = z + z^2 = C_0 z + C_1 z^2.$$

We now assume that, for some $k \geq 1$, $M_{n,k} = C_{n-1}$ for all $1 \leq n \leq k+1$, and we define

$$z_{k+1} = z + \left(M_{1,k}z + M_{2,k}z^2 + M_{3,k}z^3 + \cdots + M_{2^k,k}z^{2^k}\right)^2$$

Our goal is to show that $M_{n,k+1} = C_{n-1}$ for $n = 1, 2, \ldots, k+2$.

For $n = 1$, the coefficient $M_{1,k+1}$ of z in z_{k+1} is clearly 1, or C_0. For $2 \leq n \leq k+2$, the coefficient $M_{n,k+1}$ of z^n in z_{k+1} can be found by expanding the above square; every product of the form $M_{j,k}z^j \cdot M_{n-j,k}z^{n-j}$ will contribute to the term $M_{n,k+1}z^n$. Since $n \leq k+2 \leq 2^k + 1$ (since $k \geq 1$), the values of j that will contribute to this term will be $j = 1, 2, \ldots, n-1$. (Ordinarily, the z^0 and z^n terms would also contribute; however, there is no z^0 term in the expression being squared). Therefore,

$$\begin{aligned} M_{n,k+1} &= \sum_{j=1}^{n-1} M_{j,k}M_{n-j,k} \\ &= \sum_{j=1}^{n-1} C_{j-1}C_{n-j-1} && \text{by induction hypothesis} \\ &= \sum_{j=0}^{n-2} C_jC_{n-2-j} && \text{after reindexing} \\ &= C_{n-1}, \end{aligned}$$

where we used a well-known recursive relationship for the Catalan numbers in the last step. ■

Also solved by ROBERT AGNEW, Palm Coast, FL; CHARLES BURNETTE, Xavier U. of Louisiana; DMITRY FLEISCHMAN, Santa Monica, CA; MISSOURI ST. U. PROBLEM SOLVING GROUP; NORTHWESTERN U. MATH PROBLEM SOLVING GROUP; ROB PRATT, Apex, NC; AJAY SRINIVASAN, U. of Southern California; DAN SWENSON, Black Hills St. U.; and the proposer.

The limit of a quotient of sequences defined by sums

1234. *Proposed by Moubinool Omarjee, Lycée Henry IV, Paris, France.*

For every positive integer n, set $a_n := \sum_{k=1}^{n} \frac{1}{k^4}$ and $b_n := \sum_{k=1}^{n} \frac{1}{(2k-1)^4}$. Compute $\lim_{n\to\infty}(\frac{b_n}{a_n} - \frac{15}{16})$.

Solution by Russelle Guadalupe (student), University of the Philippines, Diliman, Quezon City, Philippines.

We note that for integers $n \geq 1$,

$$\sum_{k=1}^{2n} \frac{1}{k^4} = \sum_{k=1}^{n} \frac{1}{(2k)^4} + \sum_{k=1}^{n} \frac{1}{(2k-1)^4} = \frac{a_n}{16} + b_n, \text{ and}$$

$$\sum_{k=1}^{2n} \frac{1}{k^4} = \sum_{k=1}^{n} \frac{1}{k^4} + \sum_{k=n+1}^{2n} \frac{1}{k^4} = a_n + \sum_{k=1}^{n} \frac{1}{(n+k)^4}.$$

Thus, we have

$$\sum_{k=1}^{n} \frac{1}{(n+k)^4} = b_n + \frac{a_n}{16} - a_n = b_n - \frac{15}{16}a_n$$

and

$$\lim_{n\to\infty} n^3 \left(\frac{b_n}{a_n} - \frac{15}{16}\right) = \lim_{n\to\infty} \frac{n^3}{a_n} \sum_{k=1}^{n} \frac{1}{(n+k)^4} = \lim_{n\to\infty} \frac{1}{a_n} \cdot \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1+k/n)^4}.$$

Since it is well-known that a_n approaches $\frac{\pi^4}{90}$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1+k/n)^4}$$

is the limit of a Riemann sum, which is given by the definite integral $\int_1^2 x^{-4}\,dx$, we obtain

$$\lim_{n\to\infty} n^3 \left(\frac{b_n}{a_n} - \frac{15}{16}\right) = \frac{90}{\pi^4} \int_1^2 \frac{dx}{x^4} = \frac{30}{\pi^4}\left(1 - \frac{1}{8}\right) = \frac{105}{4\pi^4}.$$

Also solved by Robert Agnew, Palm Coast, FL; Michel Bataille, Rouen, France; Paul Bracken, U. of Texas, Edinburg (2 solutions); Brian Bradie, Christopher Newport U. (2 solutions); Ritabrato Chaterjee, Western Michigan U. (2 solutions); Hongwei Chen, Christopher Newport U. ; Giuseppe Fera, Vicenza, Italy; Dmitry Fleischman, Santa Monica, CA; Michael Goldenberg, Reistertown, MD and Mark Kaplan, U. of Maryland Globan Campus (jointly); Russ Gordon, Whitman C.; Eugene Herman, Grinnell C.; Eugen Ionaşcu, ; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Stephen Kaczkowski, South Carolina Governor's S. for Science and Mathematics; A. Bathi Kasturiararchi, Kent St. U.; Yoodam Kim, Seoul National U. of Science and Technology; Kee-Wai Lau, Hong Kong, China; Missouri St. U. Problem Solving Group; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Mark Sand, C. of Saint Mary; Kenneth Schilling, U. of Michigan - Flint; Volkhard Schindler, Berlin, Germany; Ajay Srinivasan, U. of Southern California; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, King Abdullah U. of Science and Technology, Saudi Arabia; Southeast Missouri St. U. Math Club; Michael Vowe, Therwil, Switzerland; and the proposer. Three incorrect solutions were received.

Non-finitely generated sets whose proper subsets closed under a given function are all finitely generated

1235. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let S be a set, and let $f: S \to S$ be a function. For $s \in S$, the *orbit* of s is defined by $\mathcal{O}(s) := \{f^n(s): n \geq 0\}$, where $f_0: S \to S$ is the identity map and f^n is the n-fold composition of f with itself for $n > 0$. A subset $X \subseteq S$ is *closed under* f provided that for all $x \in X$, also $f(x) \in X$. Finally, if X is closed under F, we say that X is *finitely generated* if there is a finite $F \subseteq X$ such that $X = \bigcup_{x\in F} \mathcal{O}(x)$. Find all structures (S, f) up to isomorphism where S is not finitely generated, but every proper subset of S closed under f is finitely generated. Note that (S, f) and (T, g) are *isomorphic* if there is a bijection $\varphi: S \to T$ such that $\varphi(f(s)) = g(\varphi(s))$ for all $s \in S$. *Solution by Kenneth Schilling, University of Michigan-Flint.*

Let $f : S \to S$ be as described in the proposal.

First note that $f : S \to S$ is surjective, for if $t \in S \setminus f(S)$, then $S \setminus \{t\}$ is invariant, and if $S \setminus \{t\}$ were finitely generated, then so would be S.

Second, say $s \in S$ is of *finite order* there exists $k > 0$ with $f^k(s) = s$, and of *infinite order* if no such k exists. We claim that there exists $s \in S$ of infinite order. Suppose to the contrary that every $s \in S$ is of finite order. Then it is clear that every orbit is finite. Furthermore, the orbits are disjoint, for if $\mathcal{O}(s) \cap \mathcal{O}(t) \neq \emptyset$, then there exist i, j, k such that $f^i(s) = f^j(t)$ and $f^k(s) = s$. Then $s = f^{ik}(s) = f^{i(k-1)} \circ f^i(s) = f^{i(k-1)} \circ f^j(t)$, so $s \in \mathcal{O}(t)$, and by symmetry $t \in \mathcal{O}(s)$, so $\mathcal{O}(s) = \mathcal{O}(t)$. Now S is a disjoint union of finite orbits, so any union of infinitely many but not all orbits is a proper closed but not finitely generated subset of S, contrary to hypothesis.

Let $s_0 \in S$ be of infinite order. For $n > 0$, let $s_n = f^n(s)$. Choose s_{-1} so that $f(s_{-1}) = s_0$, then choose s_{-2} so that $f(s_{-2}) = s_{-1}$, then choose s_{-3} so that $f(s_{-3}) = s_{-2}$, and so on. The doubly infinite sequence $\underline{s} = \{s_n : n \in \mathbf{Z}\}$ is closed under f. For $n < 0$, s_n is of infinite order, for if $f^k(s_n) = s_n$, then $s_0 = f^{-n}(s_n) = f^{k-n}(s_0)$, contrary to the fact that s_0 is of infinite order. It follows that $\underline{s}$ is not finitely generated; for $n < 0$, s_n is not in the orbit of s_m for $m > n$. Therefore $\underline{s} = S$.

The structure (S, f) is isomorphic to one of the following:

$S = \mathbf{Z}$ and $f(z) = z + 1$, or for some positive integer m, $S = \{z \in \mathbf{Z} : z \leq m\}$ and $f(z) = \begin{cases} z+1 & \text{for } z < m \\ 1 & \text{for } z = m \end{cases}.$

Also solved by Eugen Ionaşcu, Columbus St. U.; Dan Swenson, Black Hills St. U.; and the proposer.

SOLUTIONS

An easy logarithmic inequality

1226. *Proposed by George Apostolopoulos, Messolongi, Greece.*

Let a, b, and c be positive real numbers. Prove that $\ln \frac{27abc}{(a+b+c)^3} \leq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3}$.

Solution by Shing Hin Jimmy Pa.

$$\begin{aligned}\ln\left[\frac{27abc}{(a+b+c)^3}\right] &= \ln\left[\frac{abc}{\left(\frac{a+b+c}{3}\right)^3}\right]\\ &\leq \ln\left[\frac{abc}{(abc)^{3/3}}\right] && \text{(AM-GM Inequality)}\\ &= 0\\ &\leq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3}.\end{aligned}$$

Also solved by F. R. Ataev, Westminster International U. in Tashkent; Michel Bataille, Rouen, France; Soham Bhadra (student), Patha Bhavan, India; Connor Chambers, Rohan Dalal, Jonathan Hong, Kassidy Kryukov, Dylan Lorello (students), Tommy Goebeler, and Molly Konopka, The Episcopal Academy; Carson Dorough, Cuesta C.; Habib Far, Lone Star C. - Montgomery; Dmitry Fleischman, Santa Monica, CA; Philip Wagala Gwanyama, Northeastern Illinois U.; Eugene Herman, Grinnell C.; Donald Hooley, Bluffton, OH; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; A. Bathi Kasturiarachi, Kent

St. U. at Stark; Hidefumi Katsuura, San Jose St. U.; Panagiotis Krasopoulos, Athens, Greece; Wei-Kai Lai, U. of S. Carolina Salkehatchie and JOhn Risher (graduate student), C. of Charleston; Mihat Mammadli; Kelly McLenithan, Los Alamos, NM; Antoine Mhanna, Lebanon; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma; Benjamin Phillabaum; Henry Ricardo, Westchester Area Math Circle; Digby Smith, Waterton Lakes Mathematics Guild; Southeast Missouri St. U. Math Club; Albert Stadler, Herrliberg, Switzerland; Kwame Yeboah and Fatema Ruhi, Southeast Missouri St. U.; and the proposer.

Nonexistence of a pair of functions with intertwined inequalities

1227. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Do there exist functions $f : (0, 1) \to \mathbb{R}$ and $g : (0, 1) \to \mathbb{R}$ such that for all $x \in (0, 1)$, the following two conditions are satisfied:

1. $f(x) < g(x)$, and
2. if $x < y$, then $g(x) < f(y)$?

Either find examples of such f and g or prove that no such f and g exist.

Solution by Bruce Burdick, retired, Providence, RI.

Suppose functions f and g satisfy the given properties. Since $x < y$ implies $f(x) < g(x) < f(y)$, we see that f is strictly increasing. Therefore, f can only have countably many points of discontinuity in $(0, 1)$. We choose $x \in (0, 1)$ with $f(x) = \lim_{y\to x} f(y)$. By property 2, we must have

$$g(x) \le \lim_{y\to x^+} f(y) = f(x).$$

But that contradicts property 1. So, no such pair of functions can exist.

Also solved by Jesús Sistos Barron (student) and Eagle Problem Solvers, Georgia Southern U.; Bobby Benim, U. of Colorado - Boulder; Soham Bhadra (student), Patha Bhavan, India; Michael Ecker (retired), Penn. St. U.; Kaitlyn Gibson and Arthur Rosenthal, Salem St. U.; Lixing Han, U. of Michigan - Flint; Eugene Herman, Grinnell C.; Eugen Ionaşcu, Columbus St. U.; Juniata C. Problem Solving Group Ioana Mihaila and Ivan Ventura, Cal Poly Pomona; Charlie Mumma, Seattle, WA; Katherine Nogin, Clovis North High School; Northwestern U. Math Problem Solving Group; Paolo Perfetti, Universitá degli studi di Tor Vergata Roma; Lawrence Peterson, U. of N. Dakota; Mark Sand, C. of St. Mary; Stephen Scheinberg, Corona del Mar; Joel Schlosberg, Bayside, NY; Omar Sonebi; Nora Thornber; and the proposer.

Rings with few multiplicative maps are rare.

1228. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a ring, and let $f : R \to R$ be a function. Say that f is *multiplicative* if $f(xy) = f(x)f(y)$, $f(0) = 0$, and (if R has an identity) $f(1) = 1$. Find all commutative rings R (not assumed to have an identity) with the following two properties:

1. There exists an element $a \in R$ which is not nilpotent, and
2. every multiplicative map $f : R \to R$ is either the identity map or the zero map.

Solution by Kevin Byrnes.

Claim: The only commutative ring R satisfying conditions 1 and 2 is $R = \mathbb{F}_2$.

Proof. We will prove the claim by showing it is true when $|R| = 2$, and that no commutative ring R with $|R| \geq 3$ satisfies the conditions (observe, no ring of size 1 has a non-nilpotent element). If $|R| = 2$ then the distinguished non-nilpotent element $a \in R$ serves as the identity element 1, and applying Cauchy's Theorem to the additive group of R forces $R = \mathbb{F}_2$. Trivially, the only multiplicative map $f : \mathbb{F}_2 \to \mathbb{F}_2$ with $f(0) = 0$ and $f(1) = 1$ is $f = id$.

Now suppose $|R| \geq 3$ and R has a distinguished non-nilpotent element a, we'll demonstrate the existence of a multiplicative function $f : R \to R$ that is neither 0 nor id.

Case 1: $1 \in R$
Recall that if $1 \in R$ then R contains at least one maximal ideal M and M is also a prime ideal (see Dummit and Foote Chapter 7, Prop. 11–13). Now define $f : R \to R$ by $f(x) = \begin{cases} 0, & \text{if } x \in M \\ 1, & \text{otherwise} \end{cases}$. Observe that $f(0) = 0$, $f(1) = 1$ (as M cannot contain 1). Furthermore, f is multiplicative since for any $x, y \in R$: if x or $y \in M$ then $xy \in M$ so $f(xy) = 0 = f(x)f(y)$; if neither x nor $y \in M$ then $xy \notin M$ as M is prime, so $f(xy) = 1 = f(x)f(y)$. Finally, f must map some $x \in R - \{0, 1\}$ to 0 or 1, so $f \neq id$. Thus we have demonstrated the desired function f.

Case 2: $1 \notin R$
We will show that one of the two functions: $g(x) = x^2$ or $h(x) = ax$ is multiplicative, maps 0 to 0, and is not 0 or id. Clearly g is multiplicative and $g(0) = 0$; if $g \neq id$ we are done, so suppose that $g = id$, hence $x^2 = x \; \forall x \in R$. In particular, $a^2 = a$ and thus $a^2x = ax \; \forall x \in R$, implying $a(ax - x) = 0 \; \forall x \in R$. Since $1 \notin R$ we have $a\tilde{x} \neq \tilde{x}$ for some $\tilde{x} \in R$ and thus $\exists b \in R - \{0\}$ (specifically $b = a\tilde{x} - \tilde{x}$) such that $ab = 0$. In this case, for any $x, y \in R$: $h(xy) = axy = a^2xy = axay = h(x)h(y)$ since $a^2 = a$ and R is commutative, so h is multiplicative. Clearly $h(0) = 0$, and $h(b) = ab = 0 \neq b$, so $h \neq id$. ■

But wait, there's more! Even if condition 1 is dropped it is still possible to find multiplicative functions $\neq 0$ or id for commutative rings of size ≥ 3. Consider the subring $S = \{0, 3, 6\}$ of $\mathbb{Z}_9$. There we have $xy = 0 \; \forall x, y \in S$, hence any function $f : S \to S$ with $f(0) = 0$ is multiplicative. In particular, $f(0) = 0$, $f(3) = 6$, $f(6) = 3$ is multiplicative (even stronger, it is a ring homomorphism).

Also solved by Ioana Mihaila and Ivan Ventura, Cal Poly Pomona; and the proposer.

A bound on the spectral radius of a matrix

1229. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ii} = 0$ and $a_{ij} = b_i c_j$ for $i \neq j$, where $b_i > 0$ and $c_j \geq 0$ for $1 \leq i, j \leq n$. Prove that the spectral radius of A is strictly less than 1 if and only if $\sum_{i=1}^{n} \frac{b_i c_i}{b_i c_i + 1} < 1$.

Solution by Lixing Han, University of Michigan - Flint.

Denote the spectral radius of matrix A by $\rho(A)$. We will use the following well-known result about nonnegative matrices.

If A is an $n \times n$ nonnegative matrix and $x \in \mathbb{R}^n$ is a (entry-wise) positive vector, then

$$\min_{1\le i\le n} \frac{(Ax)_i}{x_i} \le \rho(A) \le \max_{1\le i\le n} \frac{(Ax)_i}{x_i}.$$

For the matrix A in the problem, consider $A - I$. where I is the $n \times n$ identity matrix. Then

$$A - I = bc^T - \text{diag}\left([b_1c_1 + 1, \ldots, b_nc_n + 1]\right),$$

where $b = [b_1, \ldots, b_n]^T$ and $c = [c_1, \ldots, c_n]^T$ are the column vectors and diag $([b_1c_1 + 1, \ldots, b_nc_n + 1])$ is the diagonal matrix whose diagonal entries are $b_1c_1 + 1, \ldots, b_nc_n + 1$. Choose the positive vector

$$y = \left[\frac{b_1}{b_1c_1 + 1}, \ldots, \frac{b_n}{b_nc_n + 1}\right]^T.$$

Then we have

$$(A - I)y = \left(\sum_{i=1}^{n} \frac{b_ic_i}{b_ic_i + 1} - 1\right) b.$$

If $\sum_{i=1}^{n} \frac{b_ic_i}{b_ic_i+1} < 1$, then from (2) we have $(A - I)y$. Thus $Ay < y$. This implies

$$\max_{1\le i\le n} \frac{(Ay)_i}{y_i} < 1.$$

Therefore by (1) we must have $\rho(A) < 1$.

On the other hand, if $\sum_{i=1}^{n} \frac{b_ic_i}{b_ic_i+1} \ge 1$, then from (2) we have $(A - I)y \ge 0$. Thus $Ay \ge y$, which implies

$$\min_{1\le i\le n} \frac{(Ay)_i}{y_i} \ge 1.$$

By (1) we obtain $\rho(A) \ge 1$.

We thus conclude that $\rho(A) < 1$ if and only if $\sum_{i=1}^{n} \frac{b_ic_i}{b_ic_i+1} < 1$.

Also solved by Michel Bataille, Rouen, France; Soham Bhadra (student), Patha Bhavan, India; and the proposer.

Primitive Heronian triangles with equivalent rectangles

1230. *Proposed by Jason Zimba, Amplify, New York, NY.*

A Heronian triangle is a triangle with positive integer side lengths and positive integer area. Denoting the side lengths of a Heronian triangle by a, b, and c, the triangle is called *primitive* if $\gcd(a, b, c) = 1$. We shall say that a primitive Heronian triangle *has an equivalent rectangle* if there exists a rectangle with integer length and width that shares the same perimeter and area as the triangle. Show that infinitely many primitive Heronian triangles have equivalent rectangles.

Solution by Kyle Calderhead, Malone University, Canton, OH.

We provide a constructive solution by parameterizing an infinite family of such triangles.

Consider the triangles with sides

$$a = n^3 + 2n^2 + 2n + 1,$$
$$b = n^3 + 2n^2 + 2n, \text{ and}$$
$$c = 2n^2 + 2n + 1,$$

where n is a positive integer. This must be primitive, since $a = b + 1$.

This gives us a perimeter of $P = 2n^3 + 6n^2 + 6n + 2$. Calculating the area (using Heron's formula, of course), it is straightforward to verify that it simplifies to $A = n(n+1)^2(n^2+n+1)$. The equivalent rectangle has dimensions $n(n+1) \times (n+1)(n^2+n+1)$. We can immediately see that the area is the same, and another straightforward calculation shows that the perimeter is the same as well.

We should note, however, that this parameterization does not cover all such triangles—for example, those with sides (a, b, c) equal to (56, 53, 53) or (95, 87, 68).

Also solved by John Christopher, California St. U., Sacramento; Rohan Dalal (student) and Tommy Goebeler, The Episcopal Academy; Habib Far, Lone Star C. - Montgomery; Eugen Ionaşcu, Columbus St. U.; Michael Vowe, Therwil, Switzerland; and the proposer.

SOLUTIONS

(Note that this section includes solutions that would normally have appeared in the January issue, together with all solutions slated for the March issue.)

Tiling a square with small squares and narrow rectangles

1216. *Proposed by Oluwatobi Alabi, Government Science Secondary School Pyakasa Abuja, Abuja, Nigeria.*

For an integer $n \geq 3$, find a closed form for the number of ways to tile an $n \times n$ square with 1×1 squares and $(n-1) \times 1$ rectangles (each of which may be placed horizontally or vertically).

Solution by Rob Pratt, Apex, NC.

Each tiling is uniquely determined by its placement of h horizontal and v vertical rectangles. We consider nine cases.

- $h = 0, v = 0$: There is clearly 1 such tiling with no rectangles.
- $h = 1, v = 1$: If the horizontal rectangle H is in row 1 or n, there are 2 ways to place H and $n + 1$ ways to place the vertical rectangle V. If H is in one of the other $n - 2$ rows, there are 2 ways to place H and 2 ways to place V. This case yields $4(n + 1) + 4(n - 2) = 8n - 4$ tilings.
- $h = 2, v = 2$: There are 2 such tilings, with the horizontal rectangles in rows 1 and n and the vertical rectangles in columns 1 and n.
- $h = 0, v > 0$: Each column has 3 choices for a vertical rectangle (upper, lower, or empty), but $v > 0$ implies that not all columns are empty. This case yields $3^n - 1$ tilings.
- $h > 0, v = 0$: Same count as $h = 0, v > 0$.
- $h = 1, v > 1$: The horizontal rectangle H must be in row 1 or n, and for each row there are 2 ways to place H. If the remaining column contains a vertical rectangle V, there are 2 ways to place V and $2^{n-1} - 1$ nonempty placements of vertical rectangles in the $n - 1$ columns shared with H. If the remaining column does not contain a vertical rectangle, there are $2^{n-1} - 1 - (n - 1)$ placements of at least 2 vertical rectangles. This case yields $4[2(2^{n-1} - 1) + (2^{n-1} - n)] = 4(3 \cdot 2^{n-1} - n - 2)$ tilings.
- $h > 1, v = 1$: Same count as $h = 1, v > 1$.
- $h \geq 2, v > 2$: There are 0 such tilings because the horizontal rectangles block at least $n - 2$ columns.
- $h > 2, v \geq 2$: Same count as $h \geq 2, v > 2$.

Hence, the total number of tilings is

$$1 + (8n - 4) + 2 + 2(3^n - 1) + 2[4(3 \cdot 2^{n-1} - n - 2)] = 2 \cdot 3^n + 12 \cdot 2^n - 19.$$

Each tiling is uniquely determined by its placement of h horizontal and v vertical rectangles. We consider nine cases.

- $h = 0, v = 0$: There is clearly 1 such tiling with no rectangles.
- $h = 1, v = 1$: If the horizontal rectangle H is in row 1 or n, there are 2 ways to place H and $n + 1$ ways to place the vertical rectangle V. If H is in one of the other $n - 2$ rows, there are 2 ways to place H and 2 ways to place V. This case yields $4(n + 1) + 4(n - 2) = 8n - 4$ tilings.
- $h = 2, v = 2$: There are 2 such tilings, with the horizontal rectangles in rows 1 and n and the vertical rectangles in columns 1 and n.
- $h = 0, v > 0$: Each column has 3 choices for a vertical rectangle (upper, lower, or empty), but $v > 0$ implies that not all columns are empty. This case yields $3^n - 1$ tilings.
- $h > 0, v = 0$: Same count as $h = 0, v > 0$.
- $h = 1, v > 1$: The horizontal rectangle H must be in row 1 or n, and for each row there are 2 ways to place H. If the remaining column contains a vertical rectangle V, there are 2 ways to place V and $2^{n-1} - 1$ nonempty placements of vertical rectangles in the $n - 1$ columns shared with H. If the remaining column does not contain a vertical rectangle, there are $2^{n-1} - 1 - (n - 1)$ placements of at least 2 vertical rectangles. This case yields $4[2(2^{n-1} - 1) + (2^{n-1} - n)] = 4(3 \cdot 2^{n-1} - n - 2)$ tilings.

- $h > 1$, $v = 1$: Same count as $h = 1$, $v > 1$.
- $h \geq 2$, $v > 2$: There are 0 such tilings because the horizontal rectangles block at least $n - 2$ columns.
- $h > 2$, $v \geq 2$: Same count as $h \geq 2$, $v > 2$.

Hence, the total number of tilings is

$$1 + (8n - 4) + 2 + 2(3^n - 1) + 2[4(3 \cdot 2^{n-1} - n - 2)] = 2 \cdot 3^n + 12 \cdot 2^n - 19.$$

Also solved by Kyle Calderhead, Malone U.; Vincent and Owen Zhang high school students from MathILy summer program; Ethan Curb, Peyton Matheson, Aiden Milligan, Cameron Moening, Virginia Rhett Smith and Ell Torek, high school students at The Citadel; Eagle Problem Solvers, Georgia Southern U.; Dmitri Fleishman, Santa Monica, CA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Lawrence Peterson, U. of N. Dakota; and the proposer. Two incorrect solutions were received.

Fibonacci numbers from the solution to an integral equation

1217. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Prove the following:

1. There exists a unique function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the following equation for every $x \in \mathbb{R}$:

$$f(-x) = 1 + \int_0^x \cos(t) f(x - t) dt.$$

   Moreover, express f explicitly in terms of elementary functions.

2. For every nonnegative integer k, $f^k(0) = (-1)^{\lfloor \frac{k+1}{2} \rfloor} F_k$, where $F_0 = 0$, $F_1 = 1$, $F_{k+2} = F_k + F_{k+1}$, and $\lfloor x \rfloor$ denote the greatest integer less than or equal to a real number x.

Solution by Russ Gordon, Whitman College, Walla Walla, WA.

Using some simple substitutions, it is easy to verify that

$$g(-x) = 1 + \int_0^x g(t) \cos(x - t)\, dt$$

and

$$g(x) = 1 - \int_0^x g(-t) \cos(x - t)\, dt.$$

It then follows that

$$u(x) \equiv g(x) + g(-x) = 2 + \int_0^x v(t) \cos(x - t)\, dt;$$

$$v(x) \equiv g(x) - g(-x) = -\int_0^x u(t) \cos(x - t)\, dt.$$

Taking Laplace transforms (with the obvious notation and noting the convolution operator), we find that

$$U(s) = \frac{2}{s} + \frac{s}{s^2+1} V(s)$$

and

$$V(s) = -\frac{s}{s^2+1} U(s).$$

Letting $\alpha = \phi$ and $\beta = -1/\phi$ (the two solutions to the equation $x^2 = x + 1$), where phi represents the golden mean, we find that

$$\begin{aligned} U(s) &= \frac{2}{s} \cdot \frac{s^4 + 3s^2 + 1 - s^2}{s^4 + 3s^2 + 1} = \frac{2}{s} - \frac{2s}{\left(s^2+\alpha^2\right)\left(s^2+\beta^2\right)} \\ &= \frac{2}{s} + \frac{2}{\sqrt{5}} \left(\frac{s}{s^2+\alpha^2} - \frac{s}{s^2+\beta^2} \right), \end{aligned}$$

where we have used the simple facts $\alpha\beta = -1$, $\alpha + \beta = 1$, and $\alpha - \beta = \sqrt{5}$. Taking the inverse Laplace transform, it follows that

$$u(x) = 2 + \frac{2}{\sqrt{5}} \left(\cos(\alpha x) - \cos(\beta x) \right).$$

The function $V(s)$ satisfies

$$\begin{aligned} V(s) &= -\frac{s}{s^2+1} \cdot \frac{2}{s} \cdot \frac{\left(s^2+1\right)^2}{s^4+3s^2+1} = -2 \cdot \frac{s^2+1}{\left(s^2+\alpha^2\right)\left(s^2+\beta^2\right)} \\ &= -\frac{2}{\sqrt{5}} \left(\frac{\alpha}{s^2+\alpha^2} - \frac{\beta}{s^2+\beta^2} \right), \end{aligned}$$

and thus

$$v(x) = -\frac{2}{\sqrt{5}} \left(\sin(\alpha x) - \sin(\beta x) \right).$$

Combining these results gives

$$g(x) = \frac{u(x)+v(x)}{2} = 1 + \frac{1}{\sqrt{5}} \left(\cos(\alpha x) - \cos(\beta x) - \sin(\alpha x) + \sin(\beta x) \right).$$

Using simple derivative properties of the sine and cosine functions, along with the Binet formula for the Fibonacci numbers, we see that

$$g^{(2k-1)}(0) = (-1)^k \cdot \frac{\alpha^{2k-1} - \beta^{2k-1}}{\sqrt{5}} = (-1)^k f_{2k-1}$$

and

$$g^{(2k)}(0) = (-1)^k \cdot \frac{\alpha^{2k} - \beta^{2k}}{\sqrt{5}} = (-1)^k f_{2k}$$

for each positive integer k. This completes the solution.

Also solved by Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport U.; Bruce Burdick (retired), Providence, RI; Hongwei Chen, Christopher Newport U.; Russ Gordon (additional solution), Whitman C.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Kee-Wai Lau, Hong Kong, China; Albert Natian, Los Angeles Valley C.; and the proposer.

Pell numbers and Pell-Lucas numbers

1218. *Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.*

The *Pell* and *Pell-Lucas numbers*, $\{P_n : n \in \mathbb{N}\}$ and $\{Q_n : n \in \mathbb{N}\}$, respectively, are defined recursively as follows: $P_0 = 0$, $P_1 = 1$, $Q_0 = Q_1 = 2$, and (for each sequence) $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Next, let $n \in \mathbb{N}$, and let $A_n(x)$ and $B_n(x)$ be polynomials of degree n with real coefficients such that for $0 \leq i \leq n$, we have $A_n(i) = P_i$ and $B_n(i) = Q_i$. Find $A_n(n+1)$ and $B_n(n+1)$ in terms of P_{n+1} and Q_{n+1}, respectively.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Solution: For each $i = 0, 1, 2, \ldots, n$, let $x_i = i$ and define

$$L_{n,i}(x) = \prod_{j=0,\, j\neq i}^{n} \frac{x-j}{i-j} = \frac{(-1)^{n-i}}{i!(n-i)!} \prod_{j=0,\, j\neq i}^{n} (x-j).$$

Note $L_{n,i}(x)$ is the Lagrange interpolating polynomial associated with the node $x_i = i$ which satisfies

$$L_{n,i}(j) = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases} \quad \text{and} \quad L_{n,i}(n+1) = (-1)^{n-i}\binom{n+1}{i}.$$

The Lagrange form for the interpolating polynomials $A_n(x)$ and $B_n(x)$ is then

$$A_n(x) = \sum_{i=0}^{n} L_{n,i}(x) P_i \quad \text{and} \quad B_n(x) = \sum_{i=0}^{n} L_{n,i}(x) Q_i;$$

consequently,

$$A_n(n+1) = (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} P_i = P_{n+1} + (-1)^n \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P_i$$

and

$$B_n(n+1) = (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} Q_i = Q_{n+1} + (-1)^n \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} Q_i.$$

Now, the Binet forms for P_i and Q_i are

$$P_i = \frac{(1+\sqrt{2})^i - (1-\sqrt{2})^i}{2\sqrt{2}} \quad \text{and} \quad Q_i = (1+\sqrt{2})^i + (1-\sqrt{2})^i,$$

so, by the binomial theorem,

$$\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}P_i = \frac{(-\sqrt{2})^{n+1}-(\sqrt{2})^{n+1}}{2\sqrt{2}}$$

$$= \frac{(\sqrt{2})^n}{2}((-1)^{n+1}-1) = \begin{cases} 0, & n \text{ odd} \\ -(\sqrt{2})^n, & n \text{ even} \end{cases}$$

and

$$\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}Q_i = (-\sqrt{2})^{n+1}+(\sqrt{2})^{n+1}$$

$$= (\sqrt{2})^{n+1}(1+(-1)^{n+1}) = \begin{cases} 2(\sqrt{2})^{n+1}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

Finally,

$$A_n(n+1) = P_{n+1} - \begin{cases} 0, & n \text{ odd} \\ (\sqrt{2})^n, & n \text{ even} \end{cases}$$

and

$$B_n(n+1) = Q_{n+1} - \begin{cases} 2(\sqrt{2})^{n+1}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

Also solved by Michel Bataille, Rouen, France; Eugene Herman, Grinnell C.; Northwestern U. Math Problem Solving Group; Albert Stadler, Herrliberg, Switzerland; and the proposer. One incorrect solution was received.

A criterion for a commutative ring to be a field

1219. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring with identity $1 \neq 0$. Recall that if I and J are ideals of R, then the *product* of I and J is defined as follows:

$$IJ := \{i_1 j_1 + \cdots + i_n j_n \colon i_k \in I, j_k \in J, n \in \mathbb{Z}^+\}.$$

Prove that R is a field if and only if for every ideal I and J of R, we have $IJ \in \{I, J\}$.

Solution by Missouri State Problem Solving Group.

Sufficiency follows directly since if R is a field, then the only ideals of R are 0 and R. For necessity, let $x, y \in R$. Then the assumption implies that either $(xy) = (x)(y) = (x)$ or $(xy) = (x)(y) = (y)$, where (z) denotes the ideal of R generated by $z \in R$. Now if $xy = 0$, then either $(x) = (0)$ or $(y) = (0)$, that is either $x = 0$ or $y = 0$, so R is an integral domain. Let a be a nonzero element of R. Then we have $(a^2) = (a)^2 = (a)(a) \in \{(a), (a)\}$, that is, $(a^2) = (a)$. Since R is a domain, then $a^2 = ua$ for some unit $u \in R$, and by cancelation we get $a = u$. So all nonzero elements are units and hence R is a field.

Also solved by Anthony Bevelacqua, U. of N. Dakota; Paul Budney, Sunderland, MA; Bill Dunn, Montgomery C.; Eugene Herman, Grinnell C.; Scheilla Raffaelli, Indiana U. East; Diego Vurgait; and the proposer.

Cofactors of cofactors

1220. *Proposed by Jeff Stuart, Pacific Lutheran University, Tacoma, WA.*

Let A be an $n \times n$ real or complex matrix with $n \geq 2$. Let $\text{co}(A)$ denote the matrix of cofactors of A, that is, for each i and j, $(\text{co}(A))_{ij}$ is the product of $(-1)^{i+j}$ and the determinant of the matrix obtained by deleting the ith row and jth column of A. Prove the following:

1. If $n = 2$, then $\text{co}(\text{co}(A)) = A$ for every A.
2. If $n > 2$, show that there is a unique singular A such that $\text{co}(\text{co}(A)) = A$.
3. If $n > 2$, find a condition on $\det(A)$ that is satisfied exactly when A is invertible and $\text{co}(\text{co}(A)) = A$.

Solution by Mark Wildon, Royal Holloway, Egham, UK.

Say that a ring R with unit element $1 \neq 0$ is *small* if no proper nontrivial subring of R has an identity.

The subring of R generated by 1 is $\{m1 : m \in \mathbb{Z}\}$. Clearly it contains the identity of R. Therefore, if R is small, R is generated as an abelian group by 1. Hence R has $\mathbb{Z}$-rank 1 as an abelian group and so either $R = \mathbb{Z}$ or $R = \mathbb{Z}/N\mathbb{Z}$ for some $N \in \mathbb{N}$ with $N \geq 2$. Since $m^2 = m$ for $m \in \mathbb{Z}$ if and only if $m = 0$ or $m = 1$, the only possible identity in a subring of $\mathbb{Z}$ is 1. Hence, $\mathbb{Z}$ is small. If N is composite, with $N = AB$ where $\gcd(A, B) = 1$ then, by the Chinese Remainder Theorem,

$$\frac{\mathbb{Z}}{N\mathbb{Z}} \cong \frac{\mathbb{Z}}{A\mathbb{Z}} \times \frac{\mathbb{Z}}{B\mathbb{Z}}$$

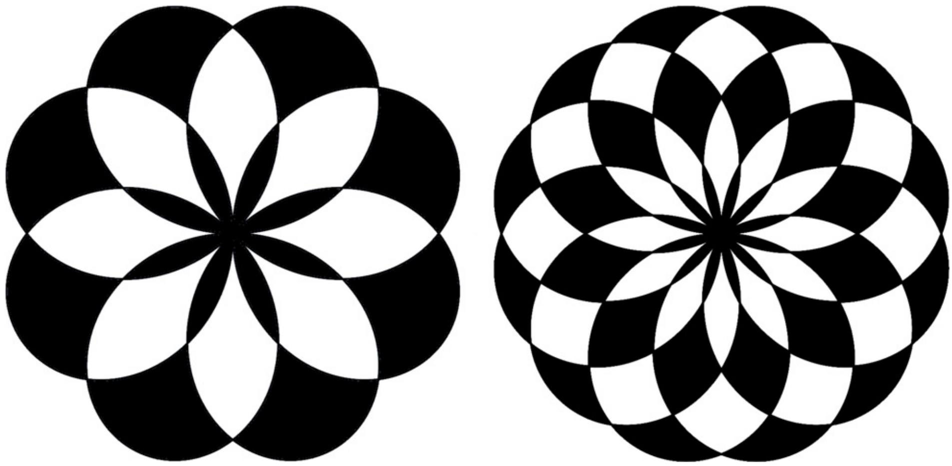
and $\{(x, 1) : x \in \mathbb{Z}/A\mathbb{Z}\}$ is a proper subring with identity of the right-hand side. (In this case the identity is not the identity of $\mathbb{Z}/N\mathbb{Z}$.) Hence, $\mathbb{Z}/N\mathbb{Z}$ is small only if N is a power of a prime. In this case $\mathbb{Z}/N\mathbb{Z}$ *is* small, since $m^2 \equiv m \bmod p^a$ if and only if $m(m-1) \equiv 0 \bmod p^a$, and since m and $m-1$ are coprime integers, either $p^a \mid m$ which implies that $m \equiv 0 \bmod p^a$, or $p^a \mid m-1$, which implies that $m \equiv 1 \bmod p^a$. We conclude that the small rings are precisely $\mathbb{Z}$ and $\mathbb{Z}/p^a\mathbb{Z}$ for p a prime and $a \geq 1$.

Also solved by Michel Bataille, Rouen, France; Missouri State Problem Solving Group, ; Albert Stadler, Herrliberg, Switzerland; and the proposer. One incorrect solution was received.

Area of a polar graph

1221. *Proposed by Gregory Dresden, Washington and Lee University, Lexington, VA.*

Shown below (from left to right) are the graphs of $r = \sin 4\theta/3$ and $r = \sin 6\theta/5$, where every other adjacent region (starting from the outside) is shaded black. Find the total shaded area for any such graph $r = \sin(k+1)\theta/k$, where $k > 0$ is an odd integer and θ ranges from 0 to $2k\pi$.



Solution by Guiseppe Fera, Vincenza, Italy.

We prove that the total shaded area is $\frac{\pi}{2}$, regardless of k.

The symmetry center of the $2(k+1)$-petalled rose

$$r = \sin\left[\left(\frac{k+1}{k}\right)\theta\right]$$

is the pole of a polar coordinate system, and the polar axis passes through one of the common points to two black shaded regions on the border of the curve.

First, we evaluate the total shaded area of a petal. Consider the petal symmetric to the line $\theta = \frac{\pi}{2(k+1)}$. Looking at the solutions of

$$x = r\cos\theta > 0$$
$$y = r\sin\theta = 0$$

we get $k-1$ intersection point (other than the pole) between the polar axis and the curve, for $\theta = m\pi$, with $m = 1, 2, \ldots, k-1$. Their cartesian coordinates are $\left(\sin\frac{m\pi}{k}\right)_{m=1,2,\ldots,k-1}$. The identity $\sin\frac{m\pi}{k} = \sin\frac{(k-m)\pi}{k}$ for $m = 1, 2, \ldots, \frac{k-1}{2}$ shows that every intersection is double. Indeed, these intersection points (and the pole) are the start-points of the black shaded regions inside the petal. The slope of the tangent line to the curve at such points is less than $\frac{\pi}{2}$ for $m = 1, 2, \ldots, \frac{k-1}{2}$ and greater than $\frac{\pi}{2}$ for $m = \frac{k+1}{2}, \frac{k+3}{2}, \ldots, k-1$. Since the petal contains $\frac{k+1}{2}$ shaded regions, symmetric with respect to the line $\theta = \frac{\pi}{2(k+1)}$, the shaded half area of the petal is

$$S = \frac{1}{2}\int_0^{\frac{\pi}{2(k+1)}} r^2\,d\theta + \frac{1}{2}\sum_{m=1}^{\frac{k-1}{2}}\left[\int_{m\pi}^{m\pi+\frac{\pi}{2(k+1)}} r^2\,d\theta - \int_{\left(\frac{k-1}{2}+m\right)\pi}^{\left(\frac{k-1}{2}+m\right)\pi+\frac{\pi}{2(k+1)}} r^2\,d\theta\right].$$

Set $n = \frac{k-1}{2} + m$. The integration is elementary and gives

$$S = \frac{\pi - k\sin\left(\frac{\pi}{k}\right)}{8(k+1)} + \sum_{m=1}^{\frac{k-1}{2}} s,$$

where

$$s = \frac{k}{8(k+1)}\left[\sin\left(\frac{2m\pi}{k}\right) - \sin\left(\frac{(2m+1)\pi}{k}\right)\right.$$
$$\left. - \left(\sin\left(\frac{2n\pi}{k}\right) - \sin\left(\frac{(2n+1)\pi}{k}\right)\right)\right].$$

Reintroducing n and simplifying, we get

$$S = \frac{\pi - k\sin\left(\frac{\pi}{k}\right)}{8(k+1)} + \frac{k}{8(k+1)}\sum_{m=1}^{\frac{k-1}{2}}\left[\sin\left((2m-1)\frac{\pi}{k}\right) - \sin\left((2m+1)\frac{\pi}{k}\right)\right].$$

Using a prosthaphaeresis identity, we have

$$S = \frac{\pi - k\sin\left(\frac{\pi}{k}\right)}{8(k+1)} + \frac{k}{8(k+1)}\left[-2\sin\left(\frac{\pi}{k}\right)\right]\sum_{m=1}^{\frac{k-1}{2}}\cos\left(\frac{2m\pi}{k}\right).$$

Using the exponential representation of the cosine,

$$\cos\left(\frac{2m\pi}{k}\right) = \frac{\exp\left(\frac{2m\pi i}{k}\right) + \exp\left(\frac{-2m\pi i}{k}\right)}{2},$$

the sum becomes a geometric series, so the value of the sum is

$$\frac{1}{2}\left[\frac{\exp\left(\frac{\pi i(k+1)}{k}\right) - \exp\left(\frac{2\pi i}{k}\right)}{\exp\left(\frac{2\pi i}{k}\right) - 1} + \frac{\exp\left(\frac{-\pi i(k+1)}{k}\right) - \exp\left(\frac{-2\pi i}{k}\right)}{\exp\left(\frac{-2\pi i}{k}\right) - 1}\right].$$

Simplifying, this is $-\frac{1}{2}$ so that

$$S = \frac{\pi - k\sin\left(\frac{\pi}{k}\right)}{8(k+1)} + \frac{k\sin\left(\frac{\pi}{k}\right)}{8(k+1)} = \frac{\pi}{8(k+1)}.$$

Finally, since the rose has $2(k+1)$ petals, the total shaded area is $4(k+1)S = \frac{\pi}{2}$.

Also solved by J. A. Grzesik, Allwave Corp.; Paul Stockmeyer, C. of William & Mary; and the proposer.

Properties of a general parabola

1222. *Proposed by Kent Holing, Trondheim, Norway.*

Consider the parabola $f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ with real coefficients, $B \neq 0$ and $A, C > 0$.

1. Show that the parabola is nondegenerate if and only if $\beta = BE - CD \neq 0$.
2. Show that in the degenerate case, the parabola can be given by the formula $f(x, y) = Ax + By + D \pm \sqrt{\alpha_1} = 0$ for $\alpha_1 = D^2 - AF$ or (equivalently) by $f(x, y) = Bx + Cy + E \pm \sqrt{\alpha_2} = 0$ for $\alpha_2 = E^2 - CF$ and $\alpha_{1,2} \geq 0$.
3. When $\beta \neq 0$, show that $(A + C)(Bx_T + Cy_T) + BD + CE = 0$ for the coordinates x_T and y_T of the vertex T.

4. Using 3., show that $x_T = -\frac{\alpha_2}{2\beta} + At$ for $t = \frac{\beta}{2C(A+C)^2}$.
5. Show that the coordinates of the focus F of the parabola are $x_F = x_T + Ct$ and $y_F = y_T - Bt$.

Solution by Michel Bataille, Rouen, France.

Let $\mathcal{P}$ be the given parabola. The discriminant of the second degree part $Ax^2 + 2Bxy + Cy^2$ must vanish, hence, $B^2 = AC$, an equality that will be used freely in what follows. The equation of the parabola is equivalent to

$$(Bx + Cy)^2 + 2CDx + 2CEy + CF = 0$$

or

$$(Ax + By)^2 + 2DAx + 2EAy + AF = 0.$$

The equation $Bx + Cy = 0$ (or equivalently $Ax + By = 0$) gives the direction of the diameters of $\mathcal{P}$.

1. $\mathcal{P}$ is nondegenerate if and only if every diameter intersects the parabola in a unique point. Let d, with equation $Bx + Cy + k = 0$, be a diameter. From (1), a point (x, y) is in $d \cap \mathcal{P}$ if and only the two equations $Bx + Cy + k = 0$ and $2CDx + 2CEy + CF + k^2 = 0$ are satisfied. This system has a unique solution if and only if $2CD \cdot C - 2CE \cdot B \neq 0$, that is, if and only if $BE - CD \neq 0$.

2. If $\mathcal{P}$ degenerates into two parallel lines, then its equation can be written as $(Ax + By + p)(Ax + By + q) = 0$. Comparing with (1) leads to $pq = AF$, $p + q = 2D$ (note that $DB = EA$ because $CD = BE$) and p, q are solutions of the quadratic $X^2 - 2DX + FA = 0$. Thus, $\alpha_1 = D^2 - FA \geq 0$ and $\{p, q\} = \{D + \sqrt{\alpha_1}, D - \sqrt{\alpha_1}\}$. In a similar way, comparing (1) with $(Bx + Cy + p')(Bx + Cy + q') = 0$ gives $\alpha_2 \geq 0$ and $\{p', q'\} = \{E + \sqrt{\alpha_2}, E - \sqrt{\alpha_2}\}$. The required results follow.

3. The vector $\left(\frac{\partial f}{\partial x}(x_T, y_T), \frac{\partial f}{\partial y}(x_T, y_T)\right)$ is orthogonal to the tangent at the vertex T, hence, is collinear to the direction vector $(C, -B)$ of the diameters. It follows that $B\frac{\partial f}{\partial x}(x_T, y_T) + C\frac{\partial f}{\partial y}(x_T, y_T) = 0$.
   Since $\frac{\partial f}{\partial x}(x, y) = 2B(Bx + Cy) + 2CD$ and $\frac{\partial f}{\partial y}(x, y) = 2C(Bx + Cy) + 2CE$, an easy calculation yields $(A + C)(Bx_T + Cy_T) + BD + CE = 0$.

4. Let $\lambda = \frac{BD+CE}{A+C}$ so that $Bx_T + Cy_T + \lambda = 0$. Since the equation of $\mathcal{P}$ can be written as

   $$(Bx + Cy + \lambda)^2 + 2x(CD - \lambda B) + 2Cy(E - \lambda) + FC - \lambda^2 = 0,$$

   expressing that T is on $\mathcal{P}$ we obtain $2(CD - \lambda B)x_T - 2(E - \lambda)(\lambda + Bx_T) + FC - \lambda^2 = 0$ so that

   $$-2\beta x_T = 2\lambda E - FC - \lambda^2 = \alpha_2 - (\lambda - E)^2.$$

   Since an easy calculation gives $(\lambda - E)^2 = \frac{A\beta^2}{C(A+C)^2}$, we get $x_T = -\frac{\alpha_2}{2\beta} + At$.

5. Let δ be the line $C(x - x_T) - B(y - y_T) + C(A + C)t = 0$. The parabola with directrix δ and focus $F = (x_T + Ct, y_T - Bt)$ is the locus of all the points $P(x, y)$ such that $(d(P, \delta))^2 = PF^2$. Thus, to answer the question, it is sufficient to show that the equation $f(x, y) = 0$ is equivalent to

$$\frac{[C(x - x_T) - B(y - y_T) + C(A + C)t]^2}{C(A + C)} = (x - x_T - Ct)^2 + (y - y_T + Bt)^2$$

But (2) writes as $((B(y - y_T) - C(x - x_T))^2 - 2C(A + C)t((B(y - y_T) - C(x - x_T)) = (B^2 + C^2)((x - x_T)^2 + (y - y_T)^2) + 2C(A + C)t((B(y - y_T) - C(x - x_T))$, that is, $(Bx + Cy + \lambda)^2 + 4C(A + C)t[By - Cx + Cx_T + \frac{B}{C}(\lambda + Bx_T)] = 0$, hence, we have to show that

$$4C(A + C)t\left(By - Cx + (A + C)x_T + \frac{\lambda B}{C}\right) = 2x(CD - \lambda B) + 2Cy(E - \lambda) + FC - \lambda^2$$

for all x, y. Simple calculations give $CD - \lambda B = -2C^2(A + C)t$, $E - \lambda = 2(A + C)tB$ and a slightly longer one gives $FC - \lambda^2 = 4C(A + C)t\big((A + C)x_T + \frac{\lambda B}{C}\big)$ so we are done.

Also solved by Hongwei Chen, Christopher Newport U.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Which rectangular numbers are squares - again?

1223. *Don Redmond, Southern Illinois University, Carbondale, IL.*

Let h be a positive integer and define the nth rectangular number of order h, denoted by $R_h(n)$, as $R_h(n) = n(n + h)$. Determine all positive integer values of h for which the equation $R_h(n) = m^2$ has a solution for some positive integers n and m.

Solution by Kathleen Lewis, University of the Gambia, Brikama, Republic of the Gambia.

All positive integers except 1, 2, and 4 are possible values for h. First notice why these three values are excluded. When $h = 1$, $n(n + h) = n(n + 1) = n^2 + n$, which lies between n^2 and $(n + 1)^2$, so it cannot be a perfect square. The same problem occurs when $h = 2$ and $n(n + 2) = n^2 + 2n$. When $h = 4$, the integers n and $n + h$ have the same parity, so $n(n + h)$ also has the same parity as n^2. That means that $n(n + 4)$ cannot be equal to $(n + 1)^2$. But it's too small to be $(n + 2)^2$. Therefore, 4 is also an impossible choice for h.

To see that all other values of h are possible, consider the cases $h = 2k + 1$, $h = 4k + 2$ and $h = 4k + 4$, with $k \in \mathbb{N}$. All positive integers other than 1, 2 and 4 fall into one of these cases.

- If $h = 2k + 1$, let $n = k^2$. Then $n(n + h) = (k^2)(k^2 + 2k + 1) = [k(k + 1)]^2$.
- If $h = 4k + 2$, let $n = 2k^2$. Then $n(n + h) = 2k^2(2k^2 + 4k + 2) = 4k^2(k^2 + 2k + 1) = [2k(k + 1)]^2$.
- If $h = 4k + 4$, let $n = k^2$. Then $n(n + h) = k^2(k^2 + 4k + 4) = [k(k + 2)]^2$.

Editor's note: Bataille pointed out that this problem appeared as number 871 in the March 2008 issue and provided two new solutions. Stone and Hawkins, provided an algorithm for producing, for a given h, all pairs (n, m) such that $R_h(n) = m$. Indeed, writing $m = n + j$, with $1 \leq j < \frac{h}{2}$, and setting $n = \frac{j^2}{h-2j}$ produces a solution for each such n that is a positive integer.

Also solved by Michel Bataille, Rouen, France; Anthony Bevelacqua, U. of N. Dakota; Kyle Calderhead, Malone U.; John Christopher, California St. U., Sacramento; Eagle Problem Solvers, Georgia Southern U.; Habib Far, Lone Star C. - Montgomery; Dmitry Fleischman, Santa Monica, CA; Donald Hooley, Bluffton, Ohio; Tom Jager, Calvin U.; Graham Lord, Princeton, NJ; Matthew McMullen, Otterbein U.; Northwestern U. Math Problem Solving Group; Mark Sand, C. of Saint Mary; David Stone and John Hawkins, Georgia Southern U. (retired); Michael Vowe, Therwil, Switzerland; Owen Zhang, (student) MathILy summer math program; and the proposer. Two incomplete solutions were received.

A criterion for a group to be cyclic

1224. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let G be a finite group, and suppose that for any subgroups H and K of G, we have $|H \cap K| = \gcd(|H|, |K|)$. Prove that G is cyclic.

Solution by Anthony Bevelacqua, University of North Dakota.

Suppose $a, b \in G$ have order d. Then

$$|\langle a \rangle \cap \langle b \rangle| = \gcd(|\langle a \rangle|, |\langle b \rangle|) = d$$

and so $\langle a \rangle = \langle b \rangle$. Since a cyclic group of order d has exactly $\phi(d)$ generators, we see that G has exactly $\phi(d)$ elements of order d.

Let N be the order of G, and let N_d be the number of elements of order d in G. By the last paragraph we have either $N_d = 0$ or $N_d = \phi(d)$. Thus,

$$N = \sum_{d|N} N_d \leq \sum_{d|N} \phi(d)$$

Since $N = \sum_{d|N} \phi(d)$ for any positive integer N and $N_d \leq \phi(d)$ for each d, we must have $N_d = \phi(d)$ for each $d|N$. In particular, G must contain an element of order N, and so G is cyclic.

Also solved by Paul Budney, Sunderland, MA; Aran Bybee and Sam Lowery; Kevin Byrnes, Glen Mills, PA; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, U. of Maryland Global Campus; Eugene Herman, Grinnell C.; Tom Jager, Calvin U.; Joel Scholosberg, Bayside, NY; Ed Enochs, U. of Kentucky (retired) and David Stone, Georgia Southern U. (retired); and the proposer.

A reduced ring with all subrings chained is a field

1225. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

All rings R throughout are commutative with $1 \neq 0$ and all subrings S of R are unital (that is, $1 \in S$). Recall that a ring R is *chained* provided that for any ideals I and J of R, either $I \subseteq J$ or $J \subseteq I$.

1. Give an example of a ring R which is not a field with the property that every subring of R is chained.
2. Suppose now that R is *reduced*, that is, R has no nonzero nilpotents. Prove that if every subring of R is chained, then R is a field.

Solution by Anthony Bevelacqua, University of North Dakota.

$\mathbb{Z}_4$, the ring of integers modulo 4, is not a field, but it is chained as the only ideals in $\mathbb{Z}_4$ are $0\mathbb{Z}_4 \subseteq 2\mathbb{Z}_4 \subseteq \mathbb{Z}_4$.

We note that the following rings are not chained: $\mathbb{Z}$ (consider $2\mathbb{Z}$ and $3\mathbb{Z}$), $k[t]$ the ring of polynomials over a field k (consider $tk[t]$ and $(t+1)k[t]$), and $S \oplus T$ the direct sum of rings S and T (consider $S \oplus 0$ and $0 \oplus T$). As special cases of the last example, $\mathbb{Z}_m$, the ring of integers modulo m, if $m = st$ for relatively prime $s, t > 1$ and $k[t]/(g)$ if $g \in k[t]$ is the product of relatively prime polynomials of positive degree are not chained.

Now suppose R is reduced and every subring of R is chained. $Z = 1\mathbb{Z}$ is a subring of R isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_m$ for some $m \geq 2$. Since Z is chained we must $Z \cong \mathbb{Z}_{p^e}$ for some prime p and some $e \geq 1$, and since Z is reduced we must have $e = 1$. We can suppose $\mathbb{Z}_p$ is a subring of R.

$\mathbb{Z}_p[a]$ is a subring of R for any $a \in R$. Since the ring of polynomials over $\mathbb{Z}_p$ is not chained, a must be algebraic over $\mathbb{Z}_p$. Thus $\mathbb{Z}_p[a] \cong \mathbb{Z}_p[t]/(g)$ for some monic $g \in k[t]$ of positive degree. Since $\mathbb{Z}_p[a]$ is chained we have $g = \pi^e$ for a monic irreducible $\pi \in k[t]$ and some $e \geq 1$, and since $\mathbb{Z}_p[a]$ is reduced we have $e = 1$. Thus, $\mathbb{Z}_p[a] \cong k[t]/(\pi)$ is a field. Since every nonzero $a \in R$ is invertible, R is a field.

Also solved by Eugene Herman, Grinnell C.; Tom Jager, Calvin U.; and the proposer.

SOLUTIONS

The limit of a difference of harmonic sums

1211. *Proposed by Needet Batir, Nevşehir Haci Bektaş Veli University, Nevşehir, Turkey.*

Evaluate the following limit, where below, $H_0 = 0$ and for $n > 0$, H_n denotes the nth haromic number $\sum_{k=1}^{n} \frac{1}{k}$:

$$\lim_{n\to\infty} \left((H_n)^2 - \sum_{k=1}^{n} \frac{H_{n-k}}{k} \right).$$

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

First we establish that

$$\sum_{k=1}^{n} \frac{H_{n-k}}{k} = H_n^2 - H_n^{(2)},$$

where $H_n^{(2)} = \sum_{k=1}^{n} 1/k^2$.

The formula is clearly true for $n = 1$. Now suppose that the formula holds for some integer $N > 1$. Then, noting that $H_{m+1} = H_m + 1/(m+1)$ and $H_{m+1}^{(2)} = H_m^{(2)} + 1/(m+1)^2$,

$$\begin{aligned}
\sum_{k=1}^{N+1} \frac{H_{N+1-k}}{k} &= \sum_{k=1}^{N} \frac{H_{N-k+1}}{k} + \frac{H_0}{N+1} \\
&= \sum_{k=1}^{N} \frac{H_{N-k}}{k} + \sum_{k=1}^{N} \frac{1}{k(N-k+1)} \\
&= H_N^2 - H_N^{(2)} + \frac{1}{N+1} \sum_{k=1}^{N} \left(\frac{1}{k} + \frac{1}{N-k+1} \right) \\
&= H_N^2 - H_N^{(2)} + \frac{2H_N}{N+1} \\
&= \left(H_{N+1} - \frac{1}{N+1} \right)^2 - \left(H_{N+1}^{(2)} - \frac{1}{(N+1)^2} \right) + \frac{2H_N}{N+1} \\
&= H_{N+1}^2 - H_{N+1}^{(2)} - \frac{2\left(H_N + \frac{1}{N+1}\right)}{N+1} + \frac{2}{(N+1)^2} + \frac{2H_N}{N+1} \\
&= H_{N+1}^2 - H_{N+1}^{(2)}.
\end{aligned}$$

Therefore,

$$H_n^2 - \sum_{k=1}^{n} \frac{H_{n-k}}{k} = H_n^2 - (H_n^2 - H_n^{(2)}) = H_n^{(2)} \to \zeta(2) = \frac{\pi^2}{6} \quad \text{as } n \to \infty.$$

Also solved by Robert Agnew, Palm Coast, FL; Paul Bracken, U. of Texas at Austin; Brian Bradie,Christopher Newport U.; Bruce Burdick, Providence, RI; Hongwei Chen, Christopher Newport U.; Russ Gordon, Whitman C.; G. C. Greubel, Newport News, VA; Jacob Guerra, Salem St. U.; GWStat Problem Solving Group; Stephen Kaczkowski, South Carolina Governor's School for Science and Mathematics; Kee-Wai Lau, Hong Kong, China; Shing Hin Jimmy Pa; Henry Ricardo, Westchester Area Math Circle, Purchase, NY (2 additional solutions); Abhishek Sinha, Tata Institute of Fundamental Research, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, King Abdullah U. of Science and Technology; Michael Vowe, Therwil, Switzerland; Mark Wildon, Royal Holloway, Egham, UK; and the proposer.

Two trig sum identities

1212. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Let n be an odd natural number and let $\theta \in \mathbb{R}$ be such that $\cos(n\theta) \neq 0$. Prove the following:

$$\sum_{k=0}^{n-1} \frac{\sin\theta}{\sin^2\theta - \cos^2(\frac{k\pi}{n})} = -\frac{n\sin(n\theta)}{\cos\theta\cos(n\theta)}, \text{ and} \tag{1}$$

$$\sum_{k=0}^{n-1} \frac{(-1)^{k+1}\cos(\frac{k\pi}{n})}{\sin^2\theta - \cos^2(\frac{k\pi}{n})} = \frac{n\sin(\frac{n\pi}{2})}{\cos\theta\cos(n\theta)}. \tag{2}$$

Solution by Michel Bataille, Rouen, France.

We will apply the following formula: if $n \in \mathbb{N}$ and $x, y, x-y$ are not a multiple of π, then

$$\sum_{k=0}^{n-1} \frac{1}{\sin(\frac{x-k\pi}{n})\sin(\frac{y-k\pi}{n})} = \frac{n\sin(x-y)}{\sin(x)\sin(y)\sin(\frac{x-y}{n})} \tag{3}$$

(see a proof at the end).

Proof of (1). (1) is obvious if $\sin(\theta) = 0$ so we suppose $\sin(\theta) \neq 0$ in what follows. We notice that

$$\sin^2\theta - \cos^2(\frac{k\pi}{n}) = \frac{1-\cos(2\theta)}{2} - \frac{1+\cos(\frac{2k\pi}{n})}{2} = -\cos(\frac{k\pi}{n}+\theta)\cos(\frac{k\pi}{n}-\theta), \tag{4}$$

hence $\sin^2\theta - \cos^2(\frac{k\pi}{n}) = -\sin(\frac{x-k\pi}{n})\sin(\frac{y-k\pi}{n})$ with $x = n(\frac{\pi}{2}-\theta),\ y = n(\frac{\pi}{2}+\theta)$. Formula (3) yields

$$\sum_{k=0}^{n-1} \frac{1}{\sin^2\theta - \cos^2(\frac{k\pi}{n})} = -\frac{n\sin(-2n\theta)}{\sin(n(\frac{\pi}{2}-\theta))\sin(n(\frac{\pi}{2}+\theta))\sin(-2\theta)} = -\frac{n\sin(n\theta)}{\sin\theta\cos\theta\cos(n\theta)}$$

(note that, n being odd, $\sin(n(\frac{\pi}{2}\pm\theta)) = (-1)^{(n-1)/2}\cos(n\theta)$.) The identity (1) follows.

Proof of (2). First, we consider (3) with $x = y + n\frac{\pi}{2}$ and obtain

$$\sum_{k=0}^{n-1} \frac{1}{\cos(\frac{y-k\pi}{n})\sin(\frac{y-k\pi}{n})} = \frac{n\sin(n\frac{\pi}{2})}{(-1)^{(n-1)/2}\sin(y)\cos(y)}$$

or

$$\sum_{k=0}^{n-1} \frac{1}{\cos(\frac{\pi}{2} - \frac{2y}{n} + \frac{2k\pi}{n})} = \frac{n\sin(n\frac{\pi}{2})}{(-1)^{(n-1)/2}\sin(2y)}. \tag{5}$$

Now, using $2\cos(\frac{k\pi}{n})\cos(\theta) = \cos(\frac{k\pi}{n}+\theta) + \cos(\frac{k\pi}{n}-\theta)$ and (4), we see that we have to prove

$$S = \frac{2n\sin(n\frac{\pi}{2})}{\cos(n\theta)},$$

where $S = \sum_{k=0}^{n-1}(-1)^k\left(\frac{1}{\cos(\frac{k\pi}{n}+\theta)} + \frac{1}{\cos(\frac{k\pi}{n}-\theta)}\right)$. Setting $n = 2m+1$, we have

$$S = \sum_{j=0}^{m}\left(\frac{1}{\cos(\frac{2j\pi}{n}+\theta)} + \frac{1}{\cos(\frac{2j\pi}{n}-\theta)}\right) - \sum_{j=0}^{m-1}\left(\frac{1}{\cos(\frac{(2j+1)\pi}{n}+\theta)} + \frac{1}{\cos(\frac{(2j+1)\pi}{n}-\theta)}\right)$$

and

$$\frac{1}{-\cos(\frac{(2j+1)\pi}{n}+\theta)} + \frac{1}{-\cos(\frac{(2j+1)\pi}{n}-\theta)} = \frac{1}{\cos(\frac{2(m+j+1)\pi}{n}+\theta)} + \frac{1}{\cos(\frac{2(m+j+1)\pi}{n}-\theta)}$$

so that

$$S = \sum_{k=0}^{n-1}\left(\frac{1}{\cos(\frac{2k\pi}{n}+\theta)} + \frac{1}{\cos(\frac{2k\pi}{n}-\theta)}\right).$$

With the help of (5), we obtain

$$\sum_{k=0}^{n-1}\frac{1}{\cos(\frac{2k\pi}{n}+\theta)} = \frac{n\sin(n\frac{\pi}{2})}{(-1)^{(n-1)/2}\sin(n\frac{\pi}{2}-n\theta)} = \frac{n\sin(n\frac{\pi}{2})}{\cos(n\theta)}$$

and therefore

$$S = \frac{n\sin(n\frac{\pi}{2})}{\cos(n\theta)} + \frac{n\sin(n\frac{\pi}{2})}{\cos(n(-\theta))} = \frac{2n\sin(n\frac{\pi}{2})}{\cos(n\theta)},$$

as desired.

Proof of (3). From $\frac{\sin(x-y)}{\sin x\cdot\sin y} = \frac{2i}{e^{2iy}-1} - \frac{2i}{e^{2ix}-1}$ (easily checked) and the decomposition into partial fractions

$$\frac{1}{z^n-1} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\overline{w}^k}{z-\overline{w}^k} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{zw^k-1}$$

where $w = e^{-\frac{2\pi i}{n}}$ we deduce that

$$\frac{\sin(x-y)}{\sin x\cdot\sin y} = \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{2i}{e^{\frac{2i(y-k\pi)}{n}}-1} - \frac{2i}{e^{\frac{2i(x-k\pi)}{n}}-1}\right) = \frac{1}{n}\sum_{k=0}^{n-1}\frac{\sin\left(\frac{x-k\pi}{n}-\frac{y-k\pi}{n}\right)}{\sin\left(\frac{x-k\pi}{n}\right)\cdot\sin\left(\frac{y-k\pi}{n}\right)}$$

and therefore

$$\sum_{k=0}^{n-1}\frac{1}{\sin\left(\frac{x-k\pi}{n}\right)\cdot\sin\left(\frac{y-k\pi}{n}\right)} = \frac{n\sin(x-y)}{\sin x\cdot\sin y\cdot\sin\left(\frac{x-y}{n}\right)}$$

Also solved by Brian Bradie, Christopher Newport U.; Hongwei Chen, Christopher Newport U.; Shing Hin Jimmy Pak; Albert Stadler, Herrliberg, Switzerland; Michael Vowe, Therwil, Switzerland; and the proposer. One incomplete solution was received.

The limit of a product of powers of sums

1213. *Proposed by Rafael Jakimczuk, Universidad National de Lujá, Buenos Aires, Argentina.*

Let (a_n) be a sequence of positive integers, and for every positive integer n, define $P_n := (1+\frac{1}{a_1 n})^{a_1} \cdot (1+\frac{1}{a_2 n})^{a_2} \cdots (1+\frac{1}{a_n n})^{a_n}$. Find $\lim_{n\to\infty} P_n$.

Solution by Ulrich Abel, Technische Hochschule, Mittelhessen, Germany.

Let n and $a_1, a_2, a_3, \ldots$ be positive integers. The Bernoulli inequality $(1+x)^n \geq 1+nx$ $(x \geq -1)$ implies that $\left(1+\frac{1}{a_k n}\right)^{a_k} \geq 1+1/n$. On the other hand, the well-known inequality $(1+x/n)^n \leq e^x$ $(x \geq 0)$ implies that $\left(1+\frac{1}{a_k n}\right)^{a_k} \leq e^{1/n}$. Consequently,

$$\left(1+\frac{1}{n}\right)^n \leq \prod_{k=1}^{n}\left(1+\frac{1}{a_k n}\right)^{a_k} \leq e,$$

which implies

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{a_k n}\right)^{a_k} = e.$$

Several solvers pointed out that this problem, by a different proposer, appeared as problem 12256 in *The American Mathematical Monthly*.

Also solved by Robert Agnew, Palm Coast, FL; Michel Bataille, Rouen, France; Paul Bracken, U. of Texas, Edinburg; Brian Bradie,Christopher Newport U.; Hongwei Chen, Christopher Newport U.; Dmitri Fleischman, Santa Monica, CA; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, U. of Maryland Global Campus; Lixing Han, U. of Michigan - Flint; Jim Hartman, C. of Wooster; Eugene Herman, Grinnell C.; Walther Janous, Innsbruck, Austria; Stephen Kaczkowski, S. Carolina Governor's School for Science and Mathematics; Kee-Wai Lau, Hong Kong, China; Kelly McLenithan, Los Alamos, NM; Albert Natian, Los Angeles Valley C.; Edward Omey, KULeuven @ Campus Brussels; Shing Hin Jimmy Pak; Mark Sand, C. of Saint Mary; Randy Schwartz (emeritus), Schoolcraft C.; Abhishek Sinha, Tata Inst. of Fundamental Research, Mumbai, India; Albert Stadler, Herrliberg, Switzerland; Michael Vowe, Therwil, Switzerland; and the proposer. One incorrect solution was received.

A closed form expression for a sequence

1214. *Proposed by Luis Moreno, SUNY Broome Community College, Binghampton, NY.*

The following sequence can be found in the text *Intermediate Analysis* by John Olmsted: $(1, 2, 2\frac{1}{2}, 3, 3\frac{1}{3}, 3\frac{2}{3}, 4, 4\frac{1}{4}, 4\frac{2}{4}, 4\frac{3}{4}, 5, \ldots)$. Now let n be a positive integer. Find a closed-form expression for a_n, the nth term of the above sequence.

Solution by Habib Far, Lone Star College – Montgomery, Conroe, Texas.

We realize that $a_n = k+1$ when $n = T_k + 1$, where $T + k = \dfrac{k(k+1)}{2}$ is the triangular number for some positive integer k. If $T_{k+1} < n \leq T_{k+1}$, then

$$a_n = k + 1 + \frac{n - T_k - 1}{k+1}.$$

Let $n = T_j + 1$, for some positive integer j. Solve $j(j+1) = 2(n-1)$ yields $j = \dfrac{-1+\sqrt{8n-7}}{2}$. Let $k = \lfloor j \rfloor = \left\lfloor \dfrac{-1+\sqrt{8n-7}}{2} \right\rfloor$, where $\lfloor x \rfloor$ is the greatest integer function. Thus

$$a_n = k + 1 + \frac{n - T_k - 1}{k+1}.$$

Also solved by Ulrich Abel, Technische Hochschule, Mittelhessen, Germany; Robert Agnew, Palm Coast, FL; Ashland U Problem Solving Group; Michel Bataille, Rouen, France; Brian Beasley, Presbyterian C.; Hudson Bouw, Braxton Green, Dillon King (students), Taylor U.; Brian Bradie, Christopher Newport U.; Case Western Reserve U. Problem Solving Group; Hongwei Chen, Christopher Newport U.; John Christopher, California St. U.; Gregory Dresden, Washington & Lee U.; Skye Fisher, (student) U. of Arkansas at Little Rock; Dmitry Fleischman, Santa Monica, CA; Natacha Fontes-Merz, Westminster C.; Dominique Frost (student) U. of Arkansas at Little Rock; Rohan Dalal, (student) and Tommy Goebeler, The Episcopal Academy; Lixing Han, U. of Michigan - Flint and Xinjia Tang, Changzhou U., Changzhou, China; Walther Janous, Innsbruck, Austria; Kelly McLenithan, Los Alamos, NM; Northwestern U Math Problem Solving Group; Lawrence Peterson, U. of North Dakota; Bill Reil, Philadelphia, PA; Mark Sand, C. of St. Mary; Tyler Sanders, (student) U. of Arkansas at Little Rock; Randy Schwartz (emeritus), Schoolcraft C.; Doug Serfass, (student) U. of Arkansas at Little Rock; Vishwest Ravi Shrimali; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, King Abdullah U. of Science and Technology; Robert Vallin, Lamar U.; Michael Vowe, Therwil, Switzerland; Edward White and Roberta White, Frostburg, MD; and the proposer.

Rings for which no proper subring has an identity

1215. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a ring (assumed only to be associative but not to contain an identity unless stated). Recall that a *subring* of R is a nonempty subset of R closed under addition, negatives, and multiplication. Find all rings R with identity $1 \neq 0$ with the property that no proper, nontrivial subring of R has an identity (which need NOT be the identity of R).

Solution by Mark Wildon, Royal Holloway, Egham, UK.

Say that a ring R with unit element $1 \neq 0$ is *small* if no proper nontrivial subring of R has an identity.

The subring of R generated by 1 is $\{m1 : m \in \mathbb{Z}\}$. Clearly it contains the identity of R. Therefore if R is small, R is generated as an abelian group by 1. Hence R has $\mathbb{Z}$-rank 1 as an abelian group and so either $R = \mathbb{Z}$ or $R = \mathbb{Z}/N\mathbb{Z}$ for some $N \in \mathbb{N}$ with $N \geq 2$. Since $m^2 = m$ for $m \in \mathbb{Z}$ if and only if $m = 0$ or $m = 1$, the only possible identity in a subring of $\mathbb{Z}$ is 1. Hence $\mathbb{Z}$ is small. If N is composite, with $N = AB$ where $\gcd(A, B) = 1$ then, by the Chinese Remainder Theorem,

$$\frac{\mathbb{Z}}{N\mathbb{Z}} \cong \frac{\mathbb{Z}}{A\mathbb{Z}} \times \frac{\mathbb{Z}}{B\mathbb{Z}}$$

and $\{(x, 1) : x \in \mathbb{Z}/A\mathbb{Z}\}$ is a proper subring with identity of the right-hand side. (In this case the identity is not the identity of $\mathbb{Z}/N\mathbb{Z}$.) Hence $\mathbb{Z}/N\mathbb{Z}$ is small only if N is a power of a prime. In this case $\mathbb{Z}/N\mathbb{Z}$ *is* small, since $m^2 \equiv m \bmod p^a$ if and only if $m(m-1) \equiv 0 \bmod p^a$, and since m and $m-1$ are coprime integers, either $p^a \mid m$ which implies that $m \equiv 0 \bmod p^a$, or $p^a \mid m-1$, which implies that $m \equiv 1 \bmod p^a$. We conclude that the small rings are precisely $\mathbb{Z}$ and $\mathbb{Z}/p^a\mathbb{Z}$ for p a prime and $a \geq 1$.

Also solved by Anthony Bevelacqua, U. of N. Dakota; Paul Budney, Sunderland, MA; Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; and the proposer.

SOLUTIONS

Harmonic, Fibonacci, and triangular numbers

1206. *Proposed by Seán M. Stewart, Bomaderry, NSW, Australia.*

Let $H_n := \sum_{k=1}^{n} \frac{1}{k}$ denote the nth harmonic number, let F_n denote the nth Fibonacci number, where $F_0 := 0$, $F_1 := 1$, and $F_n := F_{n-1} + F_{n-2}$ for $n \geq 2$. Further, let T_n be the nth triangular number defined by $T_0 := 0$ and $T_n := n + T_{n-1}$ for $n \geq 1$, and let $\varphi := \frac{1+\sqrt{5}}{2}$ be the golden ratio. Prove the following:

$$\sum_{n=1}^{\infty} \frac{T_n H_n F_n}{2^n} = 52 \log(2) + \frac{232}{\sqrt{5}} \log(\varphi) + 73.$$

Solution by Hongwei Chen, Christopher Newport University, Newport News, Virginia.

Recall the generating function of the harmonic numbers:

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\log(1-x)}{1-x}.$$

Let

$$f(x) := -\frac{x \log(1-x)}{1-x}.$$

Differentiating

$$\sum_{n=1}^{\infty} H_n x^{n+1} = f(x)$$

twice leads to

$$\sum_{n=1}^{\infty} n(n+1) H_n x^{n-1} = f''(x). \tag{1}$$

Using this fact and $T_n = n(n+1)/2$ we find the generating function of $\{T_n H_n\}_{n=1}^{\infty}$:

$$\sum_{n=1}^{\infty} T_n H_n x^n = \frac{1}{2} x f''(x) := g(x).$$

Direct computation gives

$$g(x) = \frac{x}{2}\left(\frac{2}{(1-x)^2} + \frac{3x}{(1-x)^3} - \frac{2\log(1-x)}{(1-x)^2} - \frac{2x\log(1-x)}{(1-x)^3}\right).$$

Using the well-known Binet formula

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right),$$

and with some simplifications, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{T_n H_n F_n}{2^n} &= \frac{1}{\sqrt{5}}\left(g\left(\frac{\phi}{2}\right) - g\left(-\frac{1}{2\phi}\right)\right)\\ &= 73 - \frac{130-58\sqrt{5}}{5}\log\left(1+\frac{1}{2\phi}\right) - \frac{130+58\sqrt{5}}{5}\log\left(1-\frac{\phi}{2}\right) \quad (1)\end{aligned}$$

Notice that

$$\log\left(1+\frac{1}{2\phi}\right) + \log\left(1-\frac{\phi}{2}\right) = \log\left(1+\frac{1}{2\phi}\right)\left(1-\frac{\phi}{2}\right) = \log\left(\frac{1}{4}\right) = -2\log(2)$$

and

$$\log\left(1+\frac{1}{2\phi}\right) - \log\left(1-\frac{\phi}{2}\right) = \log\left(\frac{1+1/2\phi}{1-\phi/2}\right) = \log(\phi^4) = 4\log(\phi).$$

From (1) we consequently find

$$\sum_{n=1}^{\infty} \frac{T_n H_n F_n}{2^n} = 73 + 52\log(2) + \frac{232}{\sqrt{5}}\log(\phi),$$

as desired.

Also solved by Narendra Bhandari, Bajura, Nepal; Brian Bradie,Christopher Newport U.; Bruce Burdick, Providence, RI; Nandan Sai Dasireddy, Hyderabad, Telangana, India; Russ Gordon, Whitman C.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Volkhard Schindler, Berlin, Germany; Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lake Forest C.; and the proposer.

A sum of a product of sums

1207. *Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Establish the following:

$$\sum_{n=1}^{\infty}(2n-1)\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)=\zeta(2)+\zeta(3),$$

where for a positive integer k, we have $\zeta(k)=\sum_{n=1}^{\infty}\frac{1}{n^k}$.

Solution by Brian Bradie, Christopher Newport University, Newport News, Virginia.

First, write

$$\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)=\sum_{j=n}^{\infty}\frac{1}{j^2}\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}+\sum_{j=n}^{\infty}\frac{1}{j^3}\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}.$$

Next,

$$\begin{aligned}\sum_{n=1}^{\infty}(2n-1)\sum_{j=n}^{\infty}\frac{1}{j^2}\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}&=\sum_{j=1}^{\infty}\frac{1}{j^2}\sum_{n=1}^{j}(2n-1)\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}=\sum_{j=1}^{\infty}\sum_{\ell=j}^{\infty}\frac{1}{\ell^3}\\&=\sum_{\ell=1}^{\infty}\frac{1}{\ell^3}\sum_{j=1}^{\ell}1=\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}=\zeta(2),\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}(2n-1)\sum_{j=n}^{\infty}\frac{1}{j^3}\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}&=\sum_{j=1}^{\infty}\frac{1}{j^3}\sum_{n=1}^{j}(2n-1)\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}=\sum_{j=1}^{\infty}\frac{1}{j}\sum_{\ell=j+1}^{\infty}\frac{1}{\ell^2}\\&=\sum_{\ell=2}^{\infty}\frac{1}{\ell^2}\sum_{j=1}^{\ell-1}\frac{1}{j}=\sum_{\ell=2}^{\infty}\frac{H_\ell-\frac{1}{\ell}}{\ell^2}=\sum_{\ell=1}^{\infty}\frac{H_\ell}{\ell^2}-\sum_{\ell=1}^{\infty}\frac{1}{\ell^3}\\&=2\zeta(3)-\zeta(3)=\zeta(3),\end{aligned}$$

where $H_n=\sum_{j=1}^{n}\frac{1}{j}$ denotes the nth harmonic number, and we have used the well-known identity

$$\sum_{\ell=1}^{\infty}\frac{H_\ell}{\ell^2}=2\zeta(3).$$

Finally,

$$\sum_{n=1}^{\infty}(2n-1)\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)=\zeta(2)+\zeta(3).$$

Also solved by Narendra Bhandari, Bajura, Nepal; Paul Bracken, U. of Texas, Edinburgh; Bruce Burdick, Providence, RI; Hongwei Chen, Christopher Newport U.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Kee-Wai Lau, Hong Kong, China; Shing Hin Jimmy Pak; Seán Stewart, King Abdullay U. of Sci. and Tech., Thuwal, Saudi Arabia; and the proposer.

An integral of logarithms

1208. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovakia.*

Prove that

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x} dx = -\frac{5}{8}\zeta(3),$$

where as above, for a positive integer k, we have $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$.

Solution by Didier Pinchon, Toulouse, France.

Let I be the integral to evaluate. Using identity

$$\begin{aligned} \ln(1-x)\ \ln(1+x) &= \frac{1}{4}\left[(\ln(1-x)+\ln(1+x))^2 - (\ln(1-x)-\ln(1+x))^2\right] \\ &= \frac{1}{4}\left[\ln^2(1-x^2) - \ln^2\left(\frac{1-x}{1+x}\right)\right], \end{aligned}$$

it follows that $I = (I_1 - I_2)/4$, with

$$I_1 = \int_0^1 \frac{\ln^2(1-x^2)}{x}\, dx, \quad I_2 = \int_0^1 \frac{\ln^2\left(\frac{1-x}{1+x}\right)}{x}\, dx.$$

The substitutions $x = \sqrt{u}$ in I_1 and $x = (1-u)/(1+u)$ in I_2 give

$$I_1 = \frac{1}{2}\int_0^1 \int_0^1 \frac{\ln^2(u)}{1-u}\, du, \quad I_2 = 2\int_0^1 \int_0^1 \frac{\ln^2(u)}{1-u^2}\, du.$$

The dominated convergence theorem allows to permute the series expansion of $1/(1-u)$ (resp. $1/(1-u^2)$) with the integration in I_1 (resp. I_2), and therefore

$$I_1 = \frac{1}{2}\sum_{n\geq 0}\int_0^1 \ln^2(u)\, u^n\, du, \quad I_2 = 2\sum_{n\geq 0}\int_0^1 \ln^2(u)\, u^{2n}\, du.$$

For any nonnegative integer k, two successive integrations by parts provide the result

$$\int_0^1 \ln^2(u)\, u^n\, du = \frac{2}{(n+1)^3},$$

and it follows that

$$I_1 = \sum_{n\geq 0}\frac{1}{(n+1)^3} = \zeta(3),$$

$$I_2 = 4\sum_{n\geq 0}\frac{1}{(2n+1)^3} = 4\left[\sum_{n\geq 0}\frac{1}{(n+1)^3} - \sum_{n\geq 0}\frac{1}{(2n+2)^3}\right] = \frac{7}{2}\zeta(3).$$

In conclusion, $I = \frac{1}{4}(I_1 - I_2) = -\frac{5}{8}\zeta(3)$.

Several solvers pointed out that this problem, by a different proposer, appeared as problem 12256 in *The American Mathematical Monthly*.

Also solved by F. R. ATAEV, Uzbekistan; KHRISTO BOYADZHIEV, Ohio Northern U.; BRIAN BRADIE,Christopher Newport U.; BRUCE BURDICK, Providence, RI; HONGWEI CHEN, Christopher Newport U.; KYLE GATESMAN (student), Johns Hopkins U.; SUBHANKAR GAYEN, West Bengal, India; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MOUBINOOL OMARJEE, Lycée Henri IV, Paris, France; HENRY RICARDO, Westchester Area Math Circle; ALBERT STADLER, Herrliberg, Switzerland; SEÁN STEWART, King Abdullay U. of Sci. and Tech., Thuwal, Saudi Arabia; MICHAEL VOWE, Therwil, Switzerland; and the proposer. One incomplete solution was received.

The rank of a matrix

1209. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*
For non-negative integers i and j, define

$$a_{ij} := \begin{cases} i(i-1)\cdots(i-j+1) & \text{if } 1 \le j \le i, \\ 1 & \text{if } i = 0 \text{ and } j \ge 0, \text{ or } j = 0 \text{ and } i \ge 0, \text{ and} \\ 0 & \text{if } j > i \ge 1. \end{cases}$$

Now let m be a positive integer. Prove that every $m \times m$ submatrix of the infinite matrix $(a_{2i,j})$ with $0 \le j \le m-1$ and $i \ge 0$ has rank m and, in addition, that $\sum_{i=0}^{m}(-1)^i\binom{m}{i}a_{2k+2i,j} = 0$ for $0 \le j \le m-1$ and any $k \in \mathbb{N}$.

Solution by the proposer.

Introduce the polynomials

$$f_0(x) = 1,\ f_1(x) = x,\ f_2(x) = x(x-1), \ldots, f_j(x) = x(x-1)\cdots(x-j+1).$$

Then $a_{2i,j} = f_j(2i)$. Since for any j

$$x^j = f_j(x) + \sum_{n=0}^{j-1} c_n f_n(x)$$

for some constants c_n, it is clear that any matrix of the form

$$\big(f_j(x_i)\big) \text{ with } 0 \le j, i \le m-1, \text{ and where all } x_i \text{ are distinct,}$$

can be transformed into a Vandermonde matrix by elementary row operations, so its determinant must be different from zero.

For the second statement, start by observing that the identity

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}f(i) = 0$$

must be valid whenever $f(x)$ is a polynomial of degree at most $m-1$. Indeed, let us define $\Delta(f(x)) = f(x) - f(x+1)$, and note that

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}f(i) = \Delta^m(f(x))(0).$$

The difference operator decreases the degree of the polynomial, and the equation can be proved inductively, using Pascal's identity.

As we saw above, the function $i \to a_{2i,j}$ is a polynomial of degree j. Hence

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}a_{2k+2i,j} = 0 \text{ for } 0 \le j \le m-1.$$

This completes the solution.

No other solutions were received.

The existence of a countable commutative integral domain with a sum-free collection of ideals

1210. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring with identity, and let I and J be ideals of R. Recall that the *sum* of I and J is the ideal defined by $I + J := \{i + j : i \in I, j \in J\}$. Prove or disprove: there exists a countable commutative integral domain D with identity and a collection $\mathcal{S}$ of $2^{\aleph_0}$ ideals of D such that for all $I \neq J$ in $\mathcal{S}$, we have $I + J \notin \mathcal{S}$.

Solution by Anthony Bevelacqua, University of North Dakota.

Let $D = \mathbb{Z}[x_1, x_2, \ldots]$ be the polynomial ring in countably many indeterminates with coefficients in $\mathbb{Z}$. Since D is the countable union of the countable $\mathbb{Z}[x_1, \ldots, x_n]$ for each $n \in \mathbb{N}$, D is a countable commutative integral domain with identity.

For any $A \subseteq \mathbb{N}$ let I_A be the ideal of D generated by $\{x_i \mid i \in A\}$. For all $A, B \subseteq \mathbb{N}$ we have (i) $I_A = I_B$ if and only if $A = B$ and (ii) $I_A + I_B = I_{A\cup B}$. Thus it suffices to find a collection S of $2^{\aleph_0}$ subsets of $\mathbb{N}$ such that for all $A \neq B$ in S we have $A \cup B \notin S$.

It's well-known (see below for sketch of proof) that for any countable set X there exists a collection T of $2^{\aleph_0}$ subsets of X such that each $U \in T$ is infinite and for all $U \neq V$ in T we have $U \cap V$ is finite. Since each element of T is an infinite set, $U \cap V \notin T$. So there exists T a family of $2^{\aleph_0}$ subsets of $\mathbb{N}$ such that for all $U \neq V$ in T we have $U \cap V \notin T$. Now $S = \{\mathbb{N} - U \mid U \in T\}$ has the desired properties: S is a family of $2^{\aleph_0}$ subsets of $\mathbb{N}$ such that for all $A \neq B$ in S we have $A \cup B \notin S$.

Thus $D = \mathbb{Z}[x_1, x_2, \ldots]$ is a countable commutative integral domain with identity containing a collection $\mathcal{S} = \{I_A \mid A \in S\}$ of $2^{\aleph_0}$ ideals such that for all $I \neq J$ in S we have $I + J \notin \mathcal{S}$.

Sketch of a standard proof of above claim: Without loss of generality we can suppose $X = \mathbb{Q}$. There are $2^{\aleph_0}$ real irrational numbers. For each real irrational r let $(u_n)_{n=1}^{\infty}$ be a sequence of rational numbers converging to r, and let $U_r = \{u_n \mid n \in \mathbb{N}\}$. Each U_r is infinite and $U_r \cap U_s$ is finite for any distinct real irrationals r and s.

Also solved by Northwestern U. Math Problem Solving Group; and the proposer.

Correction: In the featured solution to problem 1195 in the January 2022 issue, two numerators were missing in the second line. The second line as provided by the solver should have been

$$\sum_{n=1}^{\infty}\sum_{k=n+2}^{\infty}\frac{h_n}{(n+1)k^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{h_n}{(n+1)(n+k+1)^2}.$$

The editor apologizes for the error.

Polynomials of degree n tangent to a circle at $n-1$ points

1196. *Proposed by Ferenc Beleznay, Mathleaks, Budapest, Hungary, and Daniel Hwang, Wuhan Britain-China School, Wuhan, China.*

Prove or disprove: for every positive integer n, there exists a polynomial of degree $n+1$ with real coefficients whose graph is tangent to some circle at n points.

Solution by Mark Wildon, Royal Holloway, Egham, UK.

Such polynomials exist. Shifting n, we shall prove that for each $n \in \mathbb{N}$ with $n \geq 3$ there exists a polynomial P_n of degree n with coefficients in the integers such that the graph of $P_n(x)$ is tangent to the unit circle at exactly $n-1$ points in the open interval $(-1, 1)$. For $n = 2$ we may simply take $P_2(x) = 1$, which is tangent to the unit circle at 0 and has degree 0.

To define the P_n for $n \geq 3$, we need the Chebyshev polynomials of the second kind. Recall that, in the usual notation, U_m is the unique polynomial with real coefficients of degree m such that $(\sin\theta)U_m(\cos\theta) = \sin(m+1)\theta$. For instance $U_0(x) = 1$, $U_1(x) = 2x$, and since $\sin 3\theta = -\sin^3\theta + 3\sin\theta\cos^2\theta = \sin\theta(-\sin^2\theta + 3\cos^2\theta) = \sin\theta(-1 + 4\cos^2\theta)$ we have $U_2(x) = 4x^2 - 1$. In fact each U_n has integer coefficients. For each $n \in \mathbb{N}$ with $n \geq 4$, define

$$P_n(x) = x^2 U_{n-2}(x) - 2xU_{n-3} + U_{n-4}.$$

As shown in [1, Theorem 5], the defining property of U_m and the relation $2\cos\theta\sin r\theta = \sin(r+1)\theta + \sin(r-1)\theta$ imply that if $n \geq 4$ then

$$\begin{aligned}
&(\sin\theta)P_n(\cos\theta)\\
&= (\cos^2\theta\sin\theta)U_{n-2}(\cos\theta) - 2(\cos\theta\sin\theta)U_{n-3}(\cos\theta) + (\sin\theta)U_{n-4}(\cos\theta)\\
&= \cos^2\theta\sin(n-1)\theta - 2\cos\theta\sin(n-2)\theta + \sin(n-3)\theta\\
&= (1-\sin^2\theta)\sin(n-1)\theta - \sin(n-1)\theta - \sin(n-3)\theta + \sin(n-3)\theta\\
&= -\sin^2\theta\sin(n-1)\theta.
\end{aligned}$$

Hence, $P_n(\cos\theta) = -\sin\theta\sin(n-1)\theta$ for each such n. Setting $P_3(x) = 2x^3 - 2x$ we have $P_3(\cos\theta) = 2\cos^3\theta - 2\cos\theta = 2(\cos^2\theta - 1)\cos\theta = -2\sin^2\theta\cos\theta = -\sin\theta\sin 2\theta$. Therefore,

$$P_n(\cos\theta) = -\sin\theta\sin(n-1)\theta \quad \text{if } n \geq 3. \tag{$\star$}$$

Since each U_m has integer coefficients, so does each P_n.

Observe that, by ($\star$),

$$(\cos\theta)^2 + P_n(\cos\theta)^2 = \cos^2\theta + \sin^2\theta\sin^2(n-1)\theta \leq \cos^2\theta + \sin^2\theta = 1.$$

Hence, the graph of $P_n(x)$ for $-1 \leq x \leq 1$ lies inside the closed unit disc. Moreover, we have $(\cos\theta)^2 + P_n(\cos\theta)^2 = 1$ if and only if $\sin^2(n-1)\theta = 1$, so if and only if $\theta = \frac{(2k-1)\pi}{n-1}$ for some $k \in \mathbb{N}$. Thus if $x = \cos\frac{(2k-1)\pi}{n-1}$ and $x \in (-1, 1)$, the graph of $P_n(x)$ is tangent to the unit circle.

To get distinct values of $\cos\theta$, we may assume that $\theta \in [0, \pi]$. If $n = 2m$ is even then there are $2m - 1$ distinct tangent points, obtained by taking $k = 1, \ldots, m - 1, m, m + 1, \ldots 2m - 1$ to get x-coordinates

$$\cos\frac{\pi}{2m-1}, \ldots, \cos\frac{(2m-3)\pi}{2m-1}, \cos\frac{(2m-1)\pi}{2m-1} = -1, -\cos\frac{2\pi}{2m-1}, \ldots, -\cos\frac{(2m-2)\pi}{2m-1}.$$

If $n = 2m + 1$ is odd, then there are $2m$ distinct tangent points, obtained by taking $k = 1, \ldots, m - 1, m$ to get x coordinates

$$\cos\frac{\pi}{2m}, \ldots, \cos\frac{(2m-3)\pi}{2m}, \cos\frac{(2m-1)\pi}{2m}$$

and then $k = m + 1, \ldots, 2m$ to get x coordinates

$$-\cos\frac{\pi}{2m}, \ldots, -\cos\frac{(2m-3)\pi}{2m}, -\cos\frac{(2m-1)\pi}{2m}.$$

This completes the proof.

Remark. We remark that since $P_n(1) = P_n(\cos 0) = 0$ and $P_n(-1) = P_n(\cos\pi) = 0$ by ($\star$), the graph of $P_n(x)$ meets the graph of the unit circle at $x = \pm 1$; of course since the unit circle has a vertical asymptote at these points, the graph is not tangent. Thus, P_n is tangent to the unit circle at $n - 1$ points and has two further intersection points. Since tangent points have multiplicity (at least) 2, this meets the bound in Bezout's Theorem, that the intersection multiplicity between the algebraic curves $y = P_n(x)$ and $x^2 + y^2 = 1$ of degrees n and 2, respectively, is $2n$, and shows that each tangent point has degree exactly 2.

References

[1] Janjić, M. (2008). On a class of polynomials with integer coefficients. *J. Integer Seq.* 11(5): Article 08.5.2, 9.

Also solved by the proposer. We received one incomplete solution.

Matrices with presistently unequal rows

1197. *Proposed by Valery Karachik and Leonid Menikhes, South Ural State University, Chelyabinsk, Russia*

Let A be an arbitrary $n \times m$ matrix that has no equal rows. Find a necessary sufficient condition relating n and m so that there exists a column of A, after removal of which, all rows remain different.

Solution by Eugene Herman, Grinnell College, Grinnell, Iowa.

The given property holds in a trivial sense when $n = 1$ or $m = 1$. In both cases, after a column has been removed there do not exist two rows that are equal. Otherwise, the necessary and sufficient condition is $2 \leq n \leq m$. Suppose first that $m + 1 = n \geq 2$. Let $A = [a_{ij}]$, where $a_{ij} = 0$ when $j \geq i$ and $a_{ij} = 1$ when $j < i$. If column j of A is removed then rows j and $j + 1$ are equal; hence the given property fails to hold. If $n \geq m + 2$, construct the first $m + 1$ rows of A as before and fill in the rest of the matrix so all rows are different.

Suppose $2 \leq n \leq m$ and suppose the given property does not hold. Thus, for each $j \in \{1, 2, \ldots, m\}$, there exists a pair of rows $P_j = \{r, s\}$ such that r and s are unequal but become equal when the jth entry is removed from each. We create an undirected graph as follows. Each vertex corresponds to a row, and so the number of vertices is n. The edges correspond to the sets P_j; specifically, (r, s) is an edge if and only if $\{r, s\} = P_j$ for some j. Hence the number of edges is m. No vertex is joined to itself by an edge and no two vertices are joined by more than one edge. We show that the graph contains no cycles. Suppose $(r_1, \ldots, r_k)$ is a cycle; that is, $r_1, \ldots, r_k$ are distinct vertices and $(r_1, r_2), \ldots, (r_{k-1}, r_k), (r_k, r_1)$ are edges. The edges correspond to different columns, which we may assume are columns 1 through k (by permuting columns, if necessary). Let $r_1 = (a_1, a_2, \ldots, a_m)$. Thus, $r_2 = (b_1, a_2, a_3, \ldots, a_m)$ where $b_1 \neq a_1$ and $r_3 = (b_1, b_2, a_3, \ldots, a_m)$ where $b_2 \neq a_2$, and so on until $r_k = (b_1, b_2, \ldots, b_{k-1}, a_k, \ldots, a_n)$ where $b_{k-1} \neq a_{k-1}$. Then (r_k, r_1) cannot be an edge since r_k and r_1 differ in in $k - 1$ entries and $k - 1 > 1$. Our graph is therefore a tree. In a tree, the number of vertices is always larger than the number of edges, and so $m < n$. This contradiction establishes our necessary and sufficient condition.

Also solved by the proposer.

The cardinality of a set of maximal ideals

1198. *Proposed by Alan Loper, The Ohio State University, Newark OH, and Greg Oman, The University of Colorado, Colorado Springs, CO.*

Let n be a nonnegative integer, and consider the ring $R := \mathbb{Q}[X_0, \ldots, X_n]$ of polynomials (via usual polynomial addition and multiplication) in the (commuting) variables $X_0, \ldots X_n$ with coefficients in $\mathbb{Q}$. It is well known that R is a Noetherian ring, and so every ideal of R is finitely generated. Since R is countable, and there are but countably many finite subsets of a countable set, we deduce that R has but countably many ideals and thus, in particular, countably many maximal ideals. Next, let $X_0, X_1, X_2, \ldots$ be a countably infinite collection of indeterminates. Observe that (to within isomorphism) $\mathbb{Q}[X_0] \subseteq \mathbb{Q}[X_0, X_1] \subseteq \mathbb{Q}[X_0, X_1, X_2] \subseteq \cdots$. Let $\mathbb{Q}[X_0, X_1, X_2, \ldots]$ be the union of the this increasing chain. How many maximal ideals does the ring $\mathbb{Q}[X_0, X_1, X_2, \ldots]$ have? (More precisely, what is the cardinality of the set of maximal ideals of $\mathbb{Q}[X_0, X_1, X_2, \ldots]$?)

Solution by Kenneth Schilling, University of Michigan-Flint, Flint, Michigan.

Since $\mathbb{Q}[X_0, X_1, X_2....]$ has countably many elements, it has at most $2^{\aleph_0}$ maximal ideals. We shall exhibit $2^{\aleph_0}$ maximal ideals, proving that this is the exact cardinality.

Let $p_0(t) = t$ and $p_1(t) = t - 1$. For each infinite sequence $\alpha : \mathbb{N} \to \{0, 1\}$, let I_α be the ideal of $\mathbb{Q}[X_0, X_1, X_2....]$ generated by the set of polynomials

$$\{p_{\alpha(k)}(X_k) : k = 1, 2, 3, ...\}.$$

Since $p_{\alpha(k)}(\alpha(k)) = 0$, for any $q\,(X_1, X_2, ..., X_n) \in I_\alpha$,

$$q\,(\alpha(0), \alpha(1), ..., \alpha(n)) = 0.$$

It follows that I_α is a proper ideal of $\mathbb{Q}[X_0, X_1, X_2....]$, and so is contained in a maximal ideal M_α.

Now consider any pair α, β of distinct infinite sequences from $\{0, 1\}$. For some k, $\{\alpha(k), \beta(k)\} = \{0, 1\}$, so $\{p_{\alpha(k)}(X_k), p_{\beta(k)}(X_k)\} = \{X_k, X_k - 1\}$. Therefore the ideal generated by $I_\alpha \cup I_\beta$ is the whole ring $\mathbb{Q}[X_0, X_1, X_2....]$. It follows that the union $M_\alpha \cup M_\beta$ of maximal ideals must also generate the whole ring, and so, in particular, $M_\alpha \neq M_\beta$.

We conclude that the set of ideals M_α over all infinite sequences $\alpha : \mathbb{N} \to \{0, 1\}$ is of cardinality $2^{\aleph_0}$, and the proof is complete.

Also solved by Paul Budney, Sunderland, MA; and the proposer. We received one incomplete solution.

An oscillating function with prescribed zeros

1199. *Proposed by Corey Shanbrom, Sacramento State University, Sacramento, CA.*

Find a smooth, oscillating function whose periods form a bi-infinite geometric sequence. More precisely, given a positive $\lambda \neq 1$, find a smooth function f on an open half-line whose root set $\mathcal{R}$ is given by

$$\mathcal{R} = \left\{ \cdots - \frac{1}{\lambda^3} - \frac{1}{\lambda^2} - \frac{1}{\lambda}. - \frac{1}{\lambda^2} - \frac{1}{\lambda}, -\frac{1}{\lambda}, 0, \right.$$
$$\left. 1, 1+\lambda, 1+\lambda+\lambda^2, 1+\lambda+\lambda^2+\lambda^3, \cdots \right\}.$$

Editor's note: The problem statement in the March 2021 issue omitted one of the zeros. The functions defined in the submitted solutions included this value in their root set.

Solution by Albert Natian, Los Angeles Valley College, Valley Glen, California..

Answer: $f(x) = \sin\left(\frac{\pi \ln[(\lambda-1)x+1]}{\ln \lambda}\right)$ defined on $\left([1-\lambda]^{-1}, \infty\right)$ if $\lambda > 1$ and defined on $\left(-\infty, [1-\lambda]^{-1}\right)$ if $\lambda < 1$.

Justification It's clear that $\sin\theta = 0 \iff \theta = n\pi,\ n \in \mathbb{Z}$. So

$$\begin{aligned}
f(x) = 0 &\iff \sin\left(\frac{\pi \ln[(\lambda-1)x+1]}{\ln\lambda}\right) = 0 \\
&\iff \frac{\pi \ln[(\lambda-1)x+1]}{\ln\lambda} = n\pi,\ n \in \mathbb{Z} \\
&\iff \ln[(\lambda-1)x+1] = n \ln\lambda,\ n \in \mathbb{Z} \\
&\iff \ln[(\lambda-1)x+1] = \ln\lambda^n,\ n \in \mathbb{Z} \\
&\iff (\lambda-1)x+1 = \lambda^n,\ n \in \mathbb{Z} \\
&\iff x = \frac{\lambda^n - 1}{\lambda - 1} \text{ if } n \geq 0,\ x = -\frac{1}{\lambda} \cdot \frac{\left(\frac{1}{\lambda}\right)^{-n} - 1}{\left(\frac{1}{\lambda}\right) - 1} \text{ if } n < 0,\ n \in \mathbb{Z} \\
&\iff x = \sum_{j=0}^{n-1} \lambda^j \text{ if } n \geq 0,\ x = -\sum_{j=1}^{-n} \left(\frac{1}{\lambda}\right)^j \text{ if } n < 0,\ n \in \mathbb{Z}.
\end{aligned}$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

A recurrence satisfied by a sequence with a given generating function

1200. *Proposed by Russ Gordon, Whitman College, Walla Walla, Washington, and George Stoica, St. John, New Brunswick, Canada*

Let c be an arbitrary real number. Prove that the sequence $(a_n)_{n\geq 0}$ defined by

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - cx + cx^2 - x^3}$$

satisfies $a_n(a_n - 1) = a_{n+1}a_{n-1}$ for all $n \geq 1$.

Solution 1 by Michel Bataille, Rouen, France.

Since $1 - cx + cx^2 - x^3 = (1-x)(1 + (1-c)x + x^2)$, the sequence (a_n) is the unique sequence satisfying

$$(1 + (1-c)x + x^2) \cdot \sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Multiplying out on the left, we obtain $a_0 = 1,\ a_1 + (1-c)a_0 = 1$ and for $n \geq 2$

$$a_n + a_{n-1}(1-c) + a_{n-2} = 1. \tag{1}$$

Now, we prove that $a_n(a_n - 1) = a_{n+1}a_{n-1}$ for all $n \geq 1$ by induction.
Since $a_1(a_1 - 1) = c(c-1)$ and (using (1)), $a_2 a_0 = a_2 = 1 - a_1(1-c) - a_0 = c(c-1)$, the relation holds for $n = 1$.
Assume that $a_n(a_n - 1) = a_{n+1}a_{n-1}$ for some integer $n \geq 1$. Then, we have

$$\begin{aligned}
a_n a_{n+2} &= a_n(1 - a_n - (1-c)a_{n+1}) \quad \text{(using (1))} \\
&= a_n(1 - a_n) - a_n a_{n+1}(1-c) \\
&= -a_{n+1}a_{n-1} - a_n a_{n+1}(1-c) \quad \text{(by assumption)} \\
&= -a_{n+1}(a_{n-1} + a_n(1-c)) \\
&= -a_{n+1}(1 - a_{n+1}) \quad \text{(using (1))},
\end{aligned}$$

hence $a_{n+1}(a_{n+1} - 1) = a_n a_{n+2}$. This completes the induction step and the proof.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

Denote the recurrence relation $a_n\,(a_n - 1) = a_{n+1}a_{n-1}$ by *.

- If $c = -1$, then

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x)^2},$$

  so that $a_n = \frac{1}{4}\,[1 + (-1)^n(2n+3)]$, and * holds.
- If $c = 3$, then

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1-x)^3},$$

  so that $a_n = \frac{(n+1)(n+2)}{2}$, and * again holds.

In what follows, we assume that $c \neq -1, 3$. Let $\alpha = \frac{c-1+\sqrt{(c-3)(c+1)}}{2}$, so that $\alpha \neq -1, 0, 1$. We have $c = \dfrac{1+\alpha+\alpha^2}{\alpha}$, and

$$\begin{aligned}\frac{1}{1-cx+cx^2-x^3} &= \frac{\alpha}{(1-x)(\alpha-x)(1-\alpha x)} \\ &= \frac{\alpha}{(1-\alpha)^2}\left(\frac{1}{(1+\alpha)(\alpha-x)} + \frac{\alpha^2}{(1+\alpha)(1-\alpha x)} - \frac{1}{1-x}\right).\end{aligned}$$

Hence

$$a_n = \frac{\alpha}{(1-\alpha)^2}\left(\frac{1}{(1+\alpha)\alpha^{n+1}} + \frac{\alpha^{n+2}}{1+\alpha} - 1\right) = \frac{\left(1-\alpha^{n+1}\right)\left(1-\alpha^{n+2}\right)}{(1+\alpha)\,(1-\alpha)^2\,\alpha^n},$$

and it is easy to check that * holds in this case as well.

Solution 3 by Graham Lord, Princeton, New Jersey.

That $a_0 = 1$ is immediate from the substitution $x = 0$ in the equation. The latter's first and second derivatives at 0 show $a_1 = c$ and $a_2 = c(c-1)$, respectively. Note, $c - 1 = \frac{a_2+a_0-1}{a_1}$ and $a_1(a_1-1) = a_2a_0$. For convenience, set $a_{-1} = 0$, so $a_0(a_0-1) = a_1a_{-1}$.

The equation's RHS denominator, $1 - cx + cx^2 - x^3$ factors into $(1-x)$ and $(1-(c-1)x+x^2)$. So multiplication of the equation through by the latter factor gives: $1 + \sum_{n=1}^{\infty}(a_n - (c-1)a_{n-1} + a_{n-2})x^n = \frac{1}{1-x} = 1 + x + x^2 + \dots$.

Hence for all $n \geq 1$, as the coefficients of x^n on both sides of this last equation are equal: $(a_n - (c-1)a_{n-1} + a_{n-2}) = 1$. Equivalently: $c - 1 = \frac{a_n+a_{n-2}-1}{a_{n-1}}$. That is, for any $n \geq 1$ the ratio, $\frac{a_n+a_{n-2}-1}{a_{n-1}}$ is constant, independent of n, and equal to $c-1$. In particular: $\frac{a_n+a_{n-2}-1}{a_{n-1}} = \frac{a_{n+1}+a_{n-1}-1}{a_n}$. The latter simplified is the sought after identity $a_n(a_n-1) = a_{n+1}a_{n-1}$.

Also solved by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule, Mittelhessen, Germany; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Kyle Calderhead, Malone U.; Hongwei Chen, Christopher Newport U.; FAU Problem Solving Group, Florida Atlantic U.; Geuseppe Fera, Vicenza, Italy; Dmitry Fleischman, Santa Monica, CA; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, U. of Maryland Global Campus (jointly); G. C. Greubel, Newport News, VA; GWstat Problem Solving Group, The George Washington U.; Eugene Herman, Grinnell C.; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Omran Kouba, Higher Inst. for Applied Sci. and Tech., Damascus, Syria. Northwestern U. Math Problem Solving Group; Carlos Shine, São Paulo, Brazil; Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lake Forest C.; Michael Vowe, Therwil, Switzerland; and the proposer.

It was brought to our attention that CMJ problem 1208 has already appeared as problem 12256 in the May 2021 issue of the Monthly (by a different proposer). Accordingly, we will not be featuring a solution to this problem. We apologize for the error.

SOLUTIONS

An equilateral triangle in an isosceles triangle

1191. *Proposed by Herb Bailey, Rose-Hulman Institute of Technology, Terre Haute, IN.*

An isosceles triangle has incenter I, circumcenter O, side length S, and base length W. Show that there is a unique value of $\frac{S}{W}$ so that there exists a point P on one of the two sides of length S such that triangle IOP is equilateral. Find this value. *Solution by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.*

The unique value is

$$\frac{S}{W} = \frac{1 + \sqrt{3 + 2\sqrt{7/3}}}{2} \approx 1.7303506.$$

Position the isosceles triangle ABC with $A = (0, a)$, $B = (W/2, 0)$ and $C = (-W/2, 0)$. Then $a^2 = S^2 - \frac{W^2}{4}$ and $a = \frac{\sqrt{4S^2 - W^2}}{2}$. The circumcenter O lies at the intersection of the perpendicular bisectors of AB and BC. The midpoint of AB is $(W/4, a/2)$ and the slope of AB is $\frac{-2a}{W}$, so the perpendicular bisector of AB has slope $\frac{W}{2a}$ and equation

$$y = \frac{a}{2} + \frac{W}{2a}\left(x - \frac{W}{4}\right).$$

Since the perpendicular bisector of BC is the y-axis, then the circumcenter O has y-coordinate

$$y_O = \frac{a}{2} - \frac{W^2}{8a} = \frac{4a^2 - W^2}{8a} = \frac{2S^2 - W^2}{2\sqrt{4S^2 - W^2}}.$$

Since the y-axis bisects $\angle A$, then the incenter I also lies on the y-axis; its y-coordinate is given by

$$y_I = \frac{Wa}{2S + W} = \frac{W\sqrt{4S^2 - W^2}}{2(2S + W)}.$$

Thus, the distance between I and O is given by

$$IO = y_O - y_I = \frac{2S^2 - W^2}{2\sqrt{4S^2 - W^2}} - \frac{W\sqrt{4S^2 - W^2}}{2(2S + W)} = \frac{S(S - W)}{\sqrt{4S^2 - W^2}}.$$

If a point P is equidistant from O and I, then its y-coordinate must be given by

$$\begin{aligned} y_P &= \frac{y_O + y_I}{2} \\ &= \frac{2S^2 - W^2}{4\sqrt{4S^2 - W^2}} + \frac{W\sqrt{4S^2 - W^2}}{4(2S + W)} \\ &= \frac{2S^2 - W^2 + W(2S - W)}{4\sqrt{4S^2 - W^2}} \\ &= \frac{S^2 + SW - W^2}{2\sqrt{4S^2 - W^2}}. \end{aligned}$$

If P also lies on AB, then its x-coordinate must be given by

$$\begin{aligned} x_P &= \frac{W}{2}\left(1 - \frac{y_P}{a}\right) \\ &= \frac{W}{2}\left(1 - \frac{S^2 + SW - W^2}{4S^2 - W^2}\right) \\ &= \frac{SW(3S - W)}{2(4S^2 - W^2)}. \end{aligned}$$

Thus, the square of the distance between O and P is

$$OP^2 = \left(\frac{IO}{2}\right)^2 + x_P^2$$

$$= \frac{S^2(S-W)^2}{4(4S^2-W^2)} + \frac{S^2W^2(3S-W)^2}{4(4S^2-W^2)^2}$$
$$= \frac{S^3\left(S^3-2S^2W+3SW^2-W^3\right)}{\left(4S^2-W^2\right)^2}.$$

If triangle IOP is equilateral, then $IO^2 = OP^2$; that is,

$$\frac{S^2(S-W)^2}{4S^2-W^2} = \frac{S^3\left(S^3-2S^2W+3SW^2-W^3\right)}{\left(4S^2-W^2\right)^2}$$
$$(S-W)^2\left(4S^2-W^2\right) = S^4-2S^3W+3S^2W^2-SW^3$$
$$3S^4-6S^3W+3SW^3-W^4=0.$$

Dividing by $W^4 \neq 0$ and letting $x = S/W$ gives the equation

$$3x^4-6x^3+3x-1=0.$$

Substituting $x = z + 1/2$ and multiplying by 16, we get

$$48z^4-72z^2-1=0,$$

so that

$$z^2 = \frac{72 \pm 16\sqrt{21}}{96} = \frac{3}{4} \pm \frac{\sqrt{21}}{6} = \frac{3 \pm 2\sqrt{7/3}}{4},$$

$$z = \frac{\pm\sqrt{3 \pm 2\sqrt{7/3}}}{2},$$

and

$$x = \frac{1 \pm \sqrt{3 \pm 2\sqrt{7/3}}}{2}.$$

Since $x = \frac{S}{W}$ is a positive real number, there is a unique solution:

$$\frac{S}{W} = \frac{1+\sqrt{3+2\sqrt{7/3}}}{2} \approx 1.7303506.$$

Also solved by Michel Bataille, Rouen, France; James Duemmel, Bellingham, WA; Jeffrey Groah, Lone Star C. - Montgomery; Eugene Herman, Grinnell C.; Elias Lampakis, Kiparissia, Greece; Volkhard Schindler, Berlin, Germany; Randy Schwartz, Schoolcraft C. (retired); Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lakeforest C.; Michael Vowe, Therwil, Switzerland; and the proposer.

Ubiquitous zero divisors without nontrivial nilpotent elements implies infinite

1192. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring (not assumed to have an identity). Recall that an element $x \in R$ is a *zero divisor* if there is some nonzero $y \in R$ such that $xy = 0$; x is nilpotent if $x^n = 0$ for some positive integer n (note that we do *not* require a zero divisor to be nonzero).

(a) Prove or disprove: there exists a finite commutative ring R for which

1. every element of R is a zero divisor, and
2. the only nilpotent element of R is 0.

(b) Does your answer change if "finite" is replaced with "infinite"?

Solution by Northwestern University Math Problem Solving Group.

1. The answer is *negative*, i.e., there is no finite commutative ring satisfying 1 and 2. If $R = \{0\}$ (the trivial ring), then 0 is not a zero divisor, sit fails to satisfy 1. Hence, we may assume that R is non-trivial, and the proof proceeds as follows.

   Let $S = \{x_1, x_2, ..., x_n\}$ be a maximal set (n maximum) of distinct non-zero elements of R with the property $x_i x_j = 0$ for every $i \neq j$. Denote $s = x_1 + x_2 + \cdots + x_n$ its sum. Then

   (a) We h ave $s \neq 0$ because otherwise $x_1 = -x_2 - \cdots - x_n$, hence $x_1^2 = -x_2x_1 - \cdots - x_nx_1 = 0$, contradicting the assumption that 0 is the only nilpotent element.

   (b) Since all elements of R are zero divisors, there must be a non-zero r such that $0 = rs = rx_1 + rx_2 + \cdots + rx_n$. Hence, for each $i = 1, ..., n$, we have

   $$rx_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \to (rx_i)^2 = r\,(rx_i)\,x_i = -r\sum_{\substack{j=1 \\ j\neq i}}^{n} rx_jx_i = 0 \to rx_i = 0.$$

   This implies that the set $S' = \{r, x_1, x_2, ..., x_n\}$ also has the property that every pair of distinct elements in it has product zero, but S' has $n + 1$ elements, contradicting the maximality of S.

2. For infinite rings, the answer is *affirmative*. An example is the ring R of infinite sequences of integers with finitely many non-zero elements (and term-wise addition and multiplication). This ring satisfies the required properties, as shown below.

   - Property 1: If $\{a_n\}_{n\in\mathbb{N}}$ is in R, then there will be some (in fact infinitely many) $m \in \mathbb{N}$ such that $a_m = 0$. Given a fixed m such that $a_m = 0$, let $b_m = 1$ and $b_n = 0$ for $n \neq m$. Then we have that $\{b_n\}_{n\in\mathbb{N}}$ is not zero, but $a_nb_n = 0$ for every n, so that $\{a_n\}_{n\in\mathbb{N}}$ is a zero divisor.
   - If $k \geq 1$, then, for each n, $a_n^k = 0$ if and only if $a_n = 0$. Hence the zero element of R, consisting of the sequence with all terms zero, is the only nilpotent element in R.

Also solved by Egle Bettio and Liceo Benedetti-Tommaseo, Venezia, Italy; Anthony Bevelacqua, U. of N. Dakota; Paul Budney, Sunderland, MA; Elias Lampakis, Kiparissia, Greece; and the proposer.

A function that is a polynomial over the rationals in each slot separately need not be a polynomial over $\mathbb{Q}^2$

1193. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be a function such that $y \to f(a, y)$ is a polynomial over $\mathbb{Q}$ for every $a \in \mathbb{Q}$ and $x \to f(x, b)$ is a polynomial over $\mathbb{Q}$ for every $b \in \mathbb{Q}$. Is it true that $f(x, y)$ is a polynomial in $(x, y) \in \mathbb{Q}^2$?

Solution by Paul Budney, Sunderland, Massachusetts.

Such functions exist which cannot be defined by a polynomial in $\mathbb{Q}[x, y]$. Let $r_1, r_2, \ldots$ be a faithfully-indexed sequence of the rationals. Define $f : \mathbb{Q}^2 \to \mathbb{Q}$ by

$$f(x, y) = \sum_{k=1}^{\infty} \prod_{i=1}^{k} (x - r_i)(y - r_i).$$

For each $(x, y) = (r_m, r_n) \in \mathbb{Q}^2$, this series has only finitely many non-zero terms, so it converes on $\mathbb{Q}^2$. For any rational $x = r_n$, if $n > 1$,

$$f(r_n, y) = \sum_{k=1}^{n-1} \prod_{i=1}^{k} (r_n - r_i)(y - r_i) \in \mathbb{Q}[y],$$

a polynomial of degree $n - 1$. If $n = 1$, $f(r_1, y)$. Similarly, for $n > 1$, $f(x, r_n) \in \mathbb{Q}[x]$, a polynomial of degree $n - 1$. Also, $f(x, r_1) = 0$. Now, if f is defined by a polynomial $f(x, y) \in \mathbb{Q}[x, y]$, we can choose a positive integer $n > \deg[f(x, y)] = d > 0$. But then $f(r_{n+1}, y)$ is a polynomial of degree n and also a polynomial of degree at most $d < n$. This is impossible since non-constant polynomials have only finitely many zeros. Thus $f(x, y)$ can't be defined by a polynomial in $\mathbb{Q}[x, y]$.

Also solved by Gerald Edgar, Denver, CO; Albert Natian, Los Angeles Valley C.; Kenneth Schilling, U. of Michigan - Flint; and the proposer. One incomplete solution and one incorrect solution were received.

A two-variable inequality over the integers

1194. *Proposed by Andrew Simoson, King University, Bristol, TN.*

Let a and b be positive integers with $a \geq b$. Prove the following:
(a) $\frac{b}{a+b} + \frac{a+b}{b} > \sqrt{5}$, and
(b) either $\frac{a}{a+b} + \frac{a+b}{a} > \sqrt{5}$ or $\frac{a}{b} + \frac{b}{a} > \sqrt{5}$.

Solution by Charlie Mumma, Seattle, Washington.

For convenience, set $c = (\sqrt{5} - 1)/2$, $d = (\sqrt{5} + 1)/2$, and $f(x) = x + 1/x$. Observe that f is strictly decreasing on $(0, 1)$, strictly increasing on $(1, \infty)$, and $f(c) = f(d) = \sqrt{5}$. Since $a \geq b$, $(a + b)/b \geq 2 > d$, which proves (a) $[f((a + b)/b) > f(d)]$. Next notice that when $a/b + b/a \leq \sqrt{5}$, $c \leq b/a \leq 1$. Hence $(a + b)/a = 1 + b/a \geq 1 + c = d$. If $a = b$, $a/(a + b) + (a + b)/a = 5/2 > \sqrt{5}$. However, for $b = ca$, $a/(a + b) + (a + b)/a = a/b + b/a = \sqrt{5}$. Thus (b) is true so long as a/b is not the golden ratio (a condition less stringent than the requirement that both a and b be integers).

Also solved by Ulrich Abel, Technische Hochschule Mittelhessen, Germany and Georg Arends, Eschweiler, Germany (jointly); Farrukh Rakhimjanovich Ataev, Westminster International U., Tashkent, Uzbekistan;

Michel Bataille, Rouen, France; Brian Beasley, Presybterian C.; Brian Bradie, Christopher Newport U.; Kyle Calderhead, Malone U.; John Christopher, California St. U., Sacramento; Christopher Newport U. Problem Solving Seminar; Matthew Creek, Assumption U.; Richard Daquila, Muskingum U.; Eagle Problem Solvers, Georgia Southern U.; Habin Far, Lone Star C. - Montgomery; Dmitry Fleischman, Santa Monica, CA; Davide Fusi, U. of South Florida Beaufort; Russ Gordon, Whitman C.; Lixing Han, U. of Michigan - Flint and Xinjia Tang, Chang Zhou U.; Eugene Herman, Grinnell C.; Donald Hooley, Bluffton, OH; Tom Jager, Calvin U.; A. Bathi Kasturiarachi, Kent St. U. at Stark; Elias Lampakis, Kiparissia, Greece; Kee-Wai Lau, Hong Kong, China; Seungheon Lee, Yonsei U.; Graham Lord, Princeton, NJ; Rhea Malik; Northwestern U. Math Problem Solving Group; Ángel Plaza and Francisco Perdomo, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain; Mark Sand; Randy Schwartz, Schoolcraft C. (retired); Albert Stadler, Herrliberg, Switzerland; Enrique Treviño, Lake Forest C.; Michael Vowe, Therwil, Switzerland; Roy Willits; Lienhard Wimmer; and the proposer.

A sum of harmonic sums

1195. *Proposed by Marián Štofka, Slovak University of Technology, Bratislava, Slovak Republic.*

Prove the following:

$$\sum_{k=1}^{\infty} \frac{H_k}{k+1}\left(\frac{\pi^2}{6} - H_{k+1,2}\right) = \frac{\pi^4}{90},$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ is the kth harmonic number and $H_{k,2} = \sum_{i=1}^{k} \frac{1}{i^2}$ is the kth generalized harmonic number.

Solution by Russ Gordon, Whitman College, Walla Walla, WA.

Since $\frac{1}{6}\pi^2 = \sum_{k=1}^{\infty}\left(1/k^2\right)$, we can express the given sum as

$$\sum_{\infty}^{n=1} \sum_{k=n+2}^{\infty} \frac{}{(n+1)k^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{}{(n+1)(n+k+1)^2}.$$

Using integration by parts, it is not difficult to verify that

$$\int_0^1 -x^{n-1}\ln x\,dx = \frac{1}{n^2} \text{ and } \int_0^1 x^{n-1}(\ln x)^2\,dx = \frac{2}{n^3}$$

for each positive integer n. We also make note of the following Macluarin series:

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{1}{n}x^n \text{ and } \frac{(\ln(1-x))^2}{2x} = \sum_{n=1}^{\infty} \frac{h_n}{n+1}x^n.$$

Using this information, we find that

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{h_n}{(n+1)(n+k+1)^2} &= \sum_{n=1}^{\infty} \frac{h_n}{n+1} \sum_{k=1}^{\infty} \int_0^1 -x^{n+k}\ln x\,dx \\
&= \sum_{n=1}^{\infty} \frac{h_n}{n+1} \int_0^1 \frac{-x^{n+1}}{1-x}\ln x\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{-\ln x}{1-x} \sum_{n=1}^{\infty} \frac{h_n}{n+1} x^{n+1}\, dx \\
&= \int_0^1 \frac{-\ln x}{1-x} \cdot \frac{(\ln(1-x))^2}{2}\, dx \\
&= \frac{1}{2} \int_0^1 \frac{-\ln(1-x)(\ln x)^2}{x}\, dx \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} (\ln x)^2\, dx \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{2}{n^4} \\
&= \frac{\pi^4}{90},
\end{aligned}
$$

the desired result.

Also solved by Michel Bataille, Rouen, France; Gerald Bilodeau, Boston Latin School; Khristo Boyadzhiev, Ohio Northern U.; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Bruce Burdick, Roger Williams U.; Hongwei Chen, Christopher Newport U.; Lixing Han, U. of Michigan-Flint and Xinjia Tang, Chang Zhou U.; Eugene Herman, Grinnell C.; Omran Kouba, Higher Inst. for Applied Sci. and Tech., Damascus, Syria. Elias Lampakis, Kiparissia, Greece; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, Bomaderry, NSW, Australia; Michael Vowe, Therwil, Switzerland; and the proposer.

Editor's note: The name of James Brenneis was omitted from the list of solvers of problem 1183 in the November 2021 issue. We apologize for the omission.

SOLUTIONS

A continued fraction given by Fibonacci

1186. *Proposed by Gregory Dresden, Washington and Lee University, Lexington, VA and ZhenShu Luan (high school student), St. George's School, Vancouver, BC, Canada.*

Find a closed-form expression for the continued fraction $[1, 1, \ldots, 1, 3, 1, 1, \ldots, 1]$, which has n ones before, and after, the middle three.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

In order to get the desired expression, we recall the following elegant way of evaluating the convergents of a continued fraction. [See, for instance, https://de.wikipedia.org/wiki/Kettenbruch, particularly the paragraph "matrixdarstellung."] We have to evaluate the product

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

Let F_n be the nth Fibonacci number. From the familiar representation

$$[1, 1, ..., 1] = \frac{F_{n+1}}{F_n},$$

(with n 1's), we get

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},$$

whence

$$\begin{bmatrix} 1 & \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix};$$

that is

$$\begin{bmatrix} 1 & \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$
$$= \begin{bmatrix} F_{n+1} \cdot (3F_{n+1} + 2F_n) & F_{n+1} \cdot F_{n-1} + F_n\,(3F_{n+1} + F_n) \\ F_{n+1} \cdot F_{n-1} + 3F_n F_{n+1} + F_n^2 & F_n\,(2F_{n-1} + 3F_n) \end{bmatrix}.$$

This leads to the desired closed-form expression of [1, ..., 1, 3, 1, ..., 1]:

$$\frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+1} \cdot F_{n-1} + 3F_n \cdot F_{n+1} + F_n^2} = \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+1}\,(F_{n+1} - F_n) + F_n\,(3F - n + 1 + F_n)}$$
$$= \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+1}^2 + 2F_{n+1} \cdot F_n + F_n^2}$$
$$= \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{(F_{n+1} + F_n)^2}$$
$$= \frac{F_{n+1}\,(3F_{n+1} + 2F_n)}{F_{n+2}^2}$$
$$= \frac{F_{n+1}\,(F_{n+1} + 2F_{n+2})}{F_{n+2}^2}.$$

This and

$$F_{n+1} + 2F_{n+2} = F_{n+3} + F_{n+2} = F_{n+4}$$

yield the closed-form result

$$\frac{F_{n+1}F_{n+4}}{F_{n+2}^2}.$$

Also solved by Brian Beasley, Presbyterian C.; Anthony Bevelacqua, U. of N. Dakota; Brian Bradie, Christopher Newport U.; James Brenneis, Penn State - Shenango; Hongwei Chen, Christopher Newport U.;Giuseppe Fera, Vicenza, Italy; Eugene Herman, Grinnell C.; Donald Hooley, Bluffton, OH; Joel Iiams, U. of N. Dakota; Harris Kwong, SUNY Fredonia; Seungheon Lee, Yonsei U.; Carl Libis, Columbia Southern U.; Graham Lord, Princeton, NJ; Ioana Mihaila, Cal Poly Pomona; Missouri State U. Problem Solving Group; Northwestern U. Math Problem Solving Group; Randy Schwartz, Schoolcraft C. (retired); Albert Stadler, Herrliberg, Switzerland; Paul Stockmeyer, C. of William and Mary; David Terr, Oceanside, CA; Enrique Treviño, Lakeforest C.; Michael Vowe, Therwil, Switzerland; and the proposer.

A limit of maxima

1187. *Proposed by Reza Farhadian, Lorestan University, Khorramabad, Iran.*

Let $\alpha > 1$ be a fixed real number, and consider the function $M\colon [1, \infty) \to \mathbb{N}$ defined by $M(x) = \max\{m \in \mathbb{N}\colon m! \leq \alpha^x\}$. Prove the following:

$$\lim_{n\to\infty} \frac{\sqrt[n]{M(1)M(2)\cdots M(n)}}{M(n)} = e^{-1}.$$

Solution by Randy Schwartz, Schoolcraft College (retired), Ann Arbor, Michigan.

From the definition of the function M, we have $[M(n)+1]! > \alpha^n$ for $\alpha > 1$, so $\lim_{n\to\infty} M(n) = \infty$, and thus $\lim_{n\to\infty} \ln M(n) = \infty$. Also from the definition, we have

$$[M(n)]! \le \alpha^n \Rightarrow \ln([M(n)]!) \le n \ln\alpha \Rightarrow \frac{\ln([M(n)]!)}{n} \le \ln\alpha,$$

and thus

$$\lim_{n\to\infty} \frac{\ln([M(n)]!)}{n} \le \ln\alpha. \tag{1}$$

Applying Stirling's approximation to (1) leads to

$$\lim_{n\to\infty} \frac{\left(M(n)+\frac{1}{2}\right)\ln M(n) - M(n) + \frac{1}{2}\ln 2\pi}{n} \le \ln\alpha$$

$$\lim_{n\to\infty} \left[\frac{M(n)}{n}(\ln M(n) - 1) + \frac{\ln M(n)}{2n} + \frac{\ln 2\pi}{2n}\right] \le \ln\alpha$$

$$\lim_{n\to\infty} \left[\frac{M(n)}{n}(\ln M(n) - 1) + \frac{\ln M(n)}{2n}\right] \le \ln\alpha$$

The last term inside the brackets is nonnegative and, from the foregoing, the factor $\ln M(n) - 1$ increases without bound; thus, $\frac{M(n)}{n}$ must vanish, since otherwise the above limit could not be a finite number such as $\ln\alpha$. Thus, we have established

$$\in_{n\to\infty} \frac{M(n)}{n} = 0.$$

We can deduce more the definition of the function M:

$$[M(n)+1]!\alpha^n$$

$$[M(n)+1]M(n)! > \alpha^n$$

$$[M(n)]! > \frac{\alpha^n}{M(n)+1}$$

$$\ln([M(n)!) > n\ln\alpha - \ln[M(n)+1]$$

$$\frac{\ln([M(n)!])}{n} > \ln\alpha - \frac{\ln[M(n)+1]}{n}$$

$$\lim_{n\to\infty} \frac{\ln([M(n)]!)}{n} \ge \ln\alpha,$$

and combining this with (1) yields

$$\lim_{n\to\infty} \frac{\ln)[M(n)!])}{n} = \ln\alpha$$

and then

$$\lim_{n\to\infty} \frac{\ln)[M(n)!])}{n} = 1. \tag{2}$$

Using Stirling again, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln([M(n)!)}{M(n)\ln M(n)} &= \lim_{n\to\infty} \frac{\left(M(n)+\frac{1}{2}\right)\ln M(n) - M(n) + \frac{1}{2}\ln 2\pi}{M(n)\ln M(n)} \\
&= \lim_{n\to\infty} \left[\frac{M(n)+\frac{1}{2}}{M(n)} - \frac{1}{\ln M(n)} + \frac{\ln 2\pi}{2M(n)\ln M(n)}\right] \\
&= 1 - 0 + 0 = 1.
\end{aligned}$$

and combining this with (2) yields

$$\lim_{n\to\infty} \frac{M(n)\ln M(n)}{n\ln\alpha} = 1. \tag{3}$$

We can now calculate the requested value, L. We have

$$L = \lim_{n\to\infty} \frac{\sqrt[n]{\prod_{h=1}^{n} M(h)}}{M(n)} = \lim_{n\to\infty} \sqrt[n]{\prod_{h=1}^{n} \frac{M(h)}{M(n)}},$$

and then

$$\ln L = \lim_{n\to\infty} \sum_{h=1}^{n} \frac{1}{n} \ln\left[\frac{M(h)}{M(n)}\right].$$

There are many repeated terms in the above summation. The interval between $(j-1)!$ and $j!$, involving as it does a multiplication by j, encloses approximately $\log_\alpha j$ powers of α, each one of them associated with the same value of the function M. In other words, the number of integer solutions of $M(n) = j$ is asymptotically $\log_\alpha j = \frac{\ln j}{\ln \alpha}$. Using that as a weighting factor to gather the repeated terms, we can rewrite the above summation as

$$\begin{aligned}
\ln L &= \lim_{n\to\infty} \sum_{j=1}^{M(n)} \frac{1}{n} \cdot \frac{\ln j}{\ln\alpha} \ln\left[\frac{j}{M(n)}\right] \\
&= \lim_{n\to\infty} \sum_{j=1}^{M(n)} \frac{(\ln j)^2 - \ln j \cdot \ln M(n)}{n\ln\alpha} \\
&= \lim_{n\to\infty} \sum_{j=1}^{M(n)} \frac{(\ln j)^2 - \ln j \cdot \ln M(n)}{M(n)\ln M(n)}, \text{ using (3),}
\end{aligned}$$

and thus

$$\ln L = \lim_{n\to\infty} \left[\frac{1}{M(n)\ln M(n)} \sum_{j=1}^{M(n)} (\ln j)^2 - \frac{1}{M(n)} \sum_{j=1}^{M(n)} \ln j\right]. \tag{4}$$

Using inscribed and circumscribed rectangles, we have

$$\int_1^k \ln x\,dx < \sum_{j=1}^{k} \ln j < \int_2^{k+1} \ln x\,dx$$

$$\sum_{j=1}^{k} \ln j \approx \int_{1}^{k} \ln x \, dx = k \ln k - k + 1$$

$$\lim_{n\to\infty} \frac{1}{k} \sum_{j=1}^{k} \ln j = \ln k - 1,$$

and similarly

$$\sum_{j=1}^{k} (\ln j)^2 \approx \int_{1}^{k} (\ln x)^2 \, dx = k(\ln k)^2 - 2k \ln k + 2k - 2$$

$$\lim_{n\to\infty} \frac{1}{k \ln k} \sum_{j=1}^{k} (\ln j)^2 = \ln k - 2.$$

Applying these to (4) yields

$$\ln L = \lim_{n\to\infty} [(\ln M(n) - 2) - (\ln M(n) - 1)] = -1,$$

and thus

$$L = e^{-1}.$$

Also solved by Dmitry Fleischman, Santa Monica, CA; Lixing Han, U. of Michigan-Flint and Xinjia Tang, Chang Zhou U.; Albert Stadler, Herrliberg, Switzerland; and the proposer.

A recursively defined sequence of trigonometric functions

1188. *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.*

Let $\{f_n(x)\}_{n\geq 1}$ be the sequence of functions recursively defined by $f_n(x) = \int_0^{f_{n-1}(x)} \sin t dt$, with initial condition $f_1(x) = \int_0^x \sin t dt$. For each $n \in \mathbb{N}$, find the value of p_n such that $L_n = \lim\limits_{x\to 0} \frac{f_n(x)}{x^{p_n}} \in \mathbb{R}\backslash\{0\}$ and the corresponding value L_n. Prove also that $\log_2(L_n^{-1}) = 3\log_2(L_{n-1}^{-1}) - 2\log_2(L_{n-2}^{-1})$ for $n \geq 3$.

Solution by Michael Vowe, Therwil, Switzerland.

We have

$$f_1(x) = \int_0^x \sin t \, dt = 1 - \cos x = \frac{x^2}{2!} + O\left(x^4\right)$$

and hence $p_1 = 2$, $L_1 = \frac{1}{2}$. Further

$$\begin{aligned} f_2(x) &= 1 - \cos(1 - \cos x) \\ &= \left(\frac{1-\cos x}{2!}\right)^2 - \left(\frac{1-\cos x}{4!}\right)^4 + \cdots = \frac{x^4}{2!4} + O\left(x^6\right), \end{aligned}$$

which means that $p_2 = 4$, $L_2 = \frac{1}{8}$.

Since

$$f_n(x) = 1 - \cos\left(f_{n-1}(x)\right), \; p_1 = 2, \; L_1 = \frac{1}{2},$$

we obtain

$$p_n = 2p_{n-1} = 2 \cdot 2p_{n-2} = \cdots = 2^{n-1} p_1 = 2^n$$

and

$$L_n = \frac{1}{2!}\left(L_{n-1}\right)^2 = \frac{1}{2!} \cdot \frac{1}{(2!)^2}\left(L_{n-1}\right)^4 = \cdots = \frac{1}{2!^{1+2+4+\cdots+2^{n-2}}}\left(L_1\right)^{2^{n-1}}$$
$$= \frac{1}{2^{2^{n-1}-1}} \cdot \frac{1}{2^{2^{n-1}}} = \frac{1}{2^{2^n-1}}.$$

Now

$$3\log_2\left(L_{n-1}^{-1}\right) - 2\log_2\left(L_{n-2}^{-1}\right) = 3\left(2^{n-1}-1\right) - 2\left(2^{n-2}-1\right)$$
$$= 2 \cdot 2^{n-1} - 1 = 2^n - 1 = \log_2 2^{2^n-1} = \log_2\left(L_n^{-1}\right).$$

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN BRADIE, Christopher Newport U.; PAUL BUDNEY, Sunderland, MA; HONGWEI CHEN, Christopher Newport U.; CHRISTOPHER NEWPORT U. PROBLEM SOLVING SEMINAR; GERALD EDGAR, Denver, CO; LIXING HAN, U. of Michigan-Flint; JUSTIN HAVERLICK, State U. of New York at Buffalo; EUGENE HERMAN, Grinnell C.; CHRISTOPHER JACKSON, Coleman, Florida; ELIAS LAMPAKIS, Kiparissia, Greece; ALBERT NATIAN, Los Angeles Valley C.; MARK SAND, C. of Saint Mary; RANDY SCHWARTZ, Schoolcraft C. (retired); ALBERT STADLER, Herrliberg, Switzerland; SEÁN STEWART, Bomaderry, NSW, Australia; and the proposer. One incomplete solution and one incorrect solution were received.

A sum of harmonic sums

1189. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*

Evaluate the following sum:

$$\sum_{n=1}^{\infty} \frac{H_{n+1} + H_n - 1}{(n+1)(n+2)},$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number.

Solution by Robert Agnew, Palm Coast, Florida.

The sum

$$S = \sum_{n=1}^{\infty} \frac{H_{n+1} + H_n - 1}{(n+1)(n+2)}$$

can be written as

$$S = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}\left(-1 + \frac{1}{n+1} + 2 \cdot \sum_{k=1}^{n} \frac{1}{k}\right)$$

$$= -\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)} + 2 \cdot \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \sum_{k=1}^{n} \frac{1}{k}.$$

Evaluating each of these sums in turn gives

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{2};$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)} &= \sum_{n=1}^{\infty} \left(-\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{(n+1)^2} \right) \\
&= -\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\
&= -\frac{1}{2} + \left(\frac{\pi^2}{6} - 1 \right) \\
&= -\frac{3}{2} + \frac{\pi^2}{6};
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \sum_{k=1}^{n} \frac{1}{k} &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=k}^{\infty} \frac{1}{(n+1)(n+2)} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=k}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \quad = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\
&= 1.
\end{aligned}$$

Hence

$$S = \frac{\pi^2}{6}.$$

Also solved by Arkady Alt, San Jose, CA; Farrukh Rakhimjanovich Ataev, Westminster International U., Tashkent, Uzbekistan; Michel Bataille, Rouen, France; Necdet Batir, Nevşehir Haci Bektaş Veli U.; Khristo Boyadzhiev, Ohio Northern U.; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Hongwei Chen, Christopher Newport U.; Geon Choi, Yonsei U.; Nandan Sai Dasireddy, Hyderabad, India; Bruce Davis, St. Louis Comm. C. at Florissant Valley; Giuseppe Fera, Vicenza, Italy; Subhankar Gayen, West Bengal, India; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, U. of Maryland Global Campus; GWStat Problem Solving Group; Lixing Han, U. of Michigan - Flint and Xinjia Tang, Chang Zhou U.; Eugene Herman, Grinnell C.; Walther Janous, Innsbruck, Austria; Kee-Wai Lau, Hong Kong, China; Seungheon Lee, Yonsei U.; Graham Lord, Princeton, NJ; Missouri State U. Problem Solving Group; Shing Hin Jimmy Pa; Ángel Plaza and Francisco Perdomo, Universidad de Las Palmas de Gran Canaria, Las Palmas, Spain; Rob Pratt, Apex, NC; Arnold Saunders, Arlington, VA; Volkhard Schindler, Berlin, Germany; Randy Schwartz, Schoolcraft C. (retired); Allen Schwenk, Western Michigan U. Albert Stadler, Herrliberg, Switzerland; Marián Ŝtofka, Slovak U. of Technology; Enrique Treviño, Lake Forest C.; Michael Vowe, Therwil, Switzerland; and the proposer.

A second-order differential equation

1190. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Find all twice differentiable functions $y = y(x)$ such that $(y + x)y'' = y'(y' + 1)$.

Solution by Eugene Herman, Grinnell College, Grinnell, Iowa.

Substituting $z(x) = y(x) + x$ into the differential equation yields $zz'' = (z' - 1)z'$. This has solutions $z = k$ and $z = x + k$. Other than these, we have

$$\frac{d}{dx}\left(\frac{z' - 1}{z}\right) = \frac{zz'' - (z' - 1)z'}{z^2} = 0$$

and so $z' - 1 = cz$, where $c \neq 0$. Separating variables yields $z = \frac{ke^{cx}-1}{c}$. Therefore, the solutions for y are

$$k - x, \quad k, \quad \frac{ke^{cx} - 1}{c} - x \text{ (where } c \neq 0).$$

Editor's note: Solvers exercised various degrees of care in ensuring the existence of an interval on which one could safely avoid dividing by zero. In the interests of space, we have not incorporated that analysis here.

Also solved by Robert Agnew, Palm Coast, FL; Arkady Alt, San Jose, CA; Tomas Barajas, U. of Arkansas at Little Rock; Michel Bataille, Rouen, France; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Hongwei Chen, Christopher Newport U.; Richard Daquila, Muskingham U.; Bruce Davis, St. Louis Comm. C. at Florissant Valley; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, U. of Maryland Global Campus; Anna DePoyster, Missie Bogard, Rylee Buck, and Chanty Gray, (students) U. of Arkansas, Little Rock; Raymond Greenwell, Hofstra U.; Lixing Han, U. of Michigan-Flint and Xinjia Tang, Chang Zhou U.; Justin Haverlick, State U. of New York at Buffalo; Logan Hodgson; Walther Janous, Innsbruck, Austria; Harris Kwong, SUNY Fredonia; Seungheon Lee, Yonsei U.; William Littlejohn, Jason Pearson, and Cole Stillman (students) U. of Arkansas, Little Rock; James Magliano, Union Country C. (emeritus); Albert Natian, Los Angeles C.; Randy Schwartz, Schoolcraft C. (retired); Albert Stadler, Herrliberg, Switzerland; Seán Stewart, Bomaderry, NSW, Australia; Nora Thornber, Raritan Valley Comm. C.; and the proposer.

SOLUTIONS

Two limits of integrals

1181. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $k > 0$ be a real number. Calculate the following:

1. $L := \lim_{n\to\infty} \int_0^1 \left(\frac{\sqrt[n]{x}+k-1}{k}\right)^n dx$, and
2. $\lim_{n\to\infty} n\left(\int_0^1 \left(\frac{\sqrt[n]{x}+k-1}{k}\right)^n dx - L\right)$.

Solution by Seán Stewart, Bomaderry, NSW, Australia.

We will show that for $k > 0$,

1. $L = \lim\limits_{n\to\infty} \displaystyle\int_0^1 \left(\frac{\sqrt[n]{x}+k-1}{k}\right)^n dx = \frac{k}{k+1}$, and
2. $\lim\limits_{n\to\infty} n\left[\displaystyle\int_0^1 \left(\frac{\sqrt[n]{x}+k-1}{k}\right)^n dx - L\right] = \frac{k(k-1)}{(k+1)^3}$.

We first find an asymptotic expansion for the term

$$J = \left(\frac{x^{\frac{1}{n}} + k - 1}{k} \right)^n,$$

for large n. For $x \in (0, 1)$, from the Maclaurin series expansion for the exponential function as $y \to 0$ we have

$$\exp(y \log x) = 1 + y \log(x) + \frac{1}{2} y^2 \log^2(x) + \mathcal{O}(y^3).$$

Setting $y = \frac{1}{n}$ then as $n \to \infty$, we have

$$\exp\left(\frac{1}{n} \log x \right) = x^{\frac{1}{n}} = 1 + \frac{\log(x)}{n} + \frac{\log^2(x)}{2n^2} + \mathcal{O}\left(\frac{1}{n^3} \right).$$

Thus

$$\frac{x^{\frac{1}{n}} - 1}{k} = \frac{\log(x)}{nk} + \frac{\log^2(x)}{2n^2 k} + \mathcal{O}\left(\frac{1}{n^3} \right).$$

Now

$$\log J = n \log \left(1 + \frac{x^{\frac{1}{n}} - 1}{k} \right) = n \log \left[1 + \left\{ \frac{\log(x)}{nk} + \frac{\log^2(x)}{2n^2 k} + \mathcal{O}\left(\frac{1}{n^3} \right) \right\} \right]. \quad (1)$$

From the Maclaurin series expansion for $\log(1 + x)$, as $x \to 0$, we have

$$\log(1 + x) = x - \frac{x^2}{2} + \mathcal{O}(x^3).$$

Using this result we can write (1) as

$$\begin{aligned} \log J &= n \left[\frac{\log(x)}{nk} + \frac{(k-1)\log^2(x)}{2n^2 k^2} + \mathcal{O}\left(\frac{1}{n^3} \right) \right] \\ &= \log\left(x^{\frac{1}{k}} \right) + \frac{(k-1)\log^2(x)}{2nk^2} + \mathcal{O}\left(\frac{1}{n^2} \right). \end{aligned}$$

Thus

$$\begin{aligned} J = e^{\log J} &= \exp\left[\log\left(x^{\frac{1}{k}} \right) + \frac{(k-1)\log^2(x)}{2nk^2} + \mathcal{O}\left(\frac{1}{n^2} \right) \right] \\ &= x^{\frac{1}{k}} \exp\left[\frac{(k-1)\log^2(x)}{2nk^2} + \mathcal{O}\left(\frac{1}{n^2} \right) \right]. \end{aligned} \quad (2)$$

From the Maclaurin series expansion for the exponential function, as $x \to 0$, we have

$$e^x = 1 + x + \mathcal{O}(x^2).$$

Using this result we can write (2) as

$$J = x^{\frac{1}{k}} \left[1 + \frac{(k-1)\log^2(x)}{2nk^2} + \mathcal{O}\left(\frac{1}{n^2}\right) \right]$$
$$= x^{\frac{1}{k}} + \frac{(k-1)x^{\frac{1}{k}}\log^2(x)}{2nk^2} + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{3}$$

as $n \to \infty$ and is the asymptotic expansion we sought for the term J.

Let

$$I_n = \int_0^1 \left(\frac{\sqrt[n]{x} + k - 1}{k} \right)^n dx.$$

From the result given for the asymptotic expansion in (3), an asymptotic expansion for the integral I_n as $n \to \infty$ is

$$I_n = \int_0^1 x^{\frac{1}{k}}\, dx + \frac{k-1}{2nk^2} \int_0^1 x^{\frac{1}{k}} \log^2(x)\, dx + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{4}$$

The first of the integrals to the right of the equality is elementary. The result is

$$\int_0^1 x^{\frac{1}{k}}\, dx = \frac{k}{k+1}.$$

For the second of the integrals to the right of the equality, enforcing a substitution of $x \mapsto x^k$ produces

$$\int_0^1 x^{\frac{1}{k}} \log^2(x)\, dx = k^3 \int_0^1 x^k \log^2(x)\, dx.$$

Integrating by parts twice leads to

$$\int_0^1 x^{\frac{1}{k}} \log^2(x)\, dx = \frac{2k^3}{(k+1)^3}.$$

Thus (4) becomes

$$I_n = \frac{k}{k+1} + \frac{k(k-1)}{n(k+1)^3} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{5}$$

Using the result given in (5), we are now in a position to answer the questions asked in each part. For the first part

$$L = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \left[\frac{k}{k+1} + \frac{k(k-1)}{n(k+1)^3} + \mathcal{O}\left(\frac{1}{n^2}\right) \right] = \frac{k}{k+1},$$

as announced. And for the second part.

$$\lim_{n\to\infty} n(I_n - L) = \lim_{n\to\infty} n \left[\left\{ \frac{k}{k+1} + \frac{k(k-1)}{n(k+1)^3} + \mathcal{O}\left(\frac{1}{n^2}\right) \right\} - \frac{k}{k+1} \right]$$

$$= \lim_{n\to\infty}\left[\frac{k(k-1)}{(k+1)^3} + \mathcal{O}\left(\frac{1}{n}\right)\right]$$
$$= \frac{k(k-1)}{(k+1)^3},$$

as announced.

Also solved by Robert Agnew, Palm Coast, FL (part 1 only); Paul Brracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Hongwei Chen, Christopher Newport U.; James Duemmel, Bellingham, WA; Giuseppe Fera, Vicenza, Italy; Dmitry Fleischman, Santa Monica, CA (part 1 only); Russ Gordon, Whitman C.; Walther Janous, Innsbruck, Austria (part 1 only); Albert Stadler, Herrliberg, Switzerland; and the proposer. One incorrect solution was received.

The edge of convergence

1182. *Proposed by Adam Hammett, Cedarville University, Cedarville, OH.*

Let $c \in \mathbb{R}$, let $\{a_k\}_{k\geq 1}$ be a sequence of real numbers satisfying $a_k - a_{k-1} \geq a_{k+1} - a_k \geq 0$ for all $k \geq 2$, and introduce the power series

$$\chi(c, \{a_k\}, x) := \sum_{n\geq 2}(a_{n-1} - c)\frac{(-1)^n}{x^n}.$$

1. Find a real number $r > 0$ such that $\chi(c, \{a_k\}, x)$ converges absolutely for $x > r$ and all choices of c and $\{a_k\}$, but $\chi(c, \{a_k\}, r)$ diverges for for some choice of c or $\{a_k\}$, and
2. Prove that there exists a function $f(c, \{a_k\}) \geq r$ and a threshold value c^* such that $\chi(c, \{a_k\}, x) > 0$ for each $c < c^*$ and $x > f(c, \{a_k\})$, and $\chi(c, \{a_k\}, x) < 0$ for each $c > c^*$ and $x > f(c, \{a_k\})$. Give an explicit formula for $f(c, \{a_k\})$ and value for c^*.

Solution by the proposer.

Since the constant sequence $\{a_k\} = \{1\}$ satisfies the sequence condition, and $\chi(0, \{1\}, 1)$ diverges, it becomes clear that $r \geq 1$. Let's show that we actually have $r = 1$. For this, assuming $x > 0$, note that by the triangle inequality and the condition on $\{a_k\}$ we have

$$\begin{aligned}
\sum_{n\geq 2}|a_{n-1} - c|\frac{1}{x^n} &= \sum_{n\geq 2}|a_{n-1} - a_{n-2} + a_{n-2} - a_{n-3} + \cdots + a_2 - a_1 + a_1 - c|\frac{1}{x^n}\\
&\leq \sum_{n\geq 2}(a_{n-1} - a_{n-2} + a_{n-2} - a_{n-3} + \cdots + a_2 - a_1 + |a_1 - c|)\frac{1}{x^n}\\
&\leq \sum_{n\geq 2}\frac{(n-1)M(c, \{a_k\})}{x^n}, \qquad M(c, \{a_k\}) := \max\{a_2 - a_1, |a_1 - c|\}.
\end{aligned}$$

Applying the ratio test to this last series, we obtain

$$\frac{nM(c, \{a_k\})/x^{n+1}}{(n-1)M(c, \{a_k\})/x^n} = \left(\frac{n}{n-1}\right)\frac{1}{x} \to \frac{1}{x}, \qquad n \to \infty,$$

and so absolute convergence of $\chi(c, \{a_k\}, x)$ is guaranteed for $1/x < 1$, i.e. $x > 1$. So $r = 1$ and part (a) is solved. Consequently, below we will assume $x > 1$.

Now on to (b). By appropriately "shifting" the terms in the series $\Psi(c, \{a_k\}, x)$ and taking advantage of their alternating nature, we can remove c from all but one term, making our analysis far simpler. To this end, introduce $\psi(c, \{a_k\}, x) = (x^2 + x)\chi(c, \{a_k\}, x)$ and note that

$$\begin{aligned}
\psi(c, \{a_k\}, x) &= \left(x^2 + x\right) \sum_{n \geq 2} (a_{n-1} - c) \frac{(-1)^n}{x^n} \\
&= (a_1 - c) - \frac{(a_2 - c)}{x} + \frac{(a_3 - c)}{x^2} - \frac{(a_4 - c)}{x^3} + \cdots \\
&\qquad\qquad + \frac{(a_1 - c)}{x} - \frac{(a_2 - c)}{x^2} + \frac{(a_3 - c)}{x^3} - \cdots \\
&= (a_1 - c) + \sum_{n \geq 2} (a_n - a_{n-1}) \frac{(-1)^{n-1}}{x^{n-1}}. \qquad (6)
\end{aligned}$$

Since $x^2 + x > 0$ for $x > 1$, it follows that $\psi(c, \{a_k\}, x)$ and $\chi(c, \{a_k\}, x)$ have the same sign for $x > 1$. So let's analyze $\psi(c, \{a_k\}, x)$ as defined in (6), which will involve a careful case analysis for various c-values.

To start, what if consecutive terms of the sequence $\{a_k\}$ are ever equal? If, say, $a_m = a_{m+1}$ for some minimal $m \geq 1$, then the sequence condition implies

$$0 = a_{m+1} - a_m \geq a_{k+1} - a_k \geq 0, \qquad k \geq m,$$

that is $a_k = a_{k+1}$ for all $k \geq m$. So the sequence $\{a_k\}$ is constant from the mth term onward, and hence in this case $\psi(c, \{a_k\}, x)$ is a finite polynomial:

$$\psi(c, \{a_k\}, x) = (a_1 - c) + \sum_{2 \leq n \leq m} (a_n - a_{n-1}) \frac{(-1)^{n-1}}{x^{n-1}}. \qquad (7)$$

Here, the sum in (7) may well be empty (i.e. $m = 1$), and this would correspond to the case where $\{a_k\}$ is a constant sequence. If the sum is nonempty with at least two terms (i.e. $m \geq 3$), then the magnitude of the ratio of consecutive terms in the sum is

$$\frac{(a_{n+1} - a_n)/x^n}{(a_n - a_{n-1})/x^{n-1}} = \frac{a_{n+1} - a_n}{a_n - a_{n-1}} \left(\frac{1}{x}\right) < 1$$

for $2 \leq n < m$ and $x > 1$, since $(a_{n+1} - a_n)/(a_n - a_{n-1}) \leq 1$ due to the sequence condition. Hence, this alternating sum has terms that decrease in magnitude, and so

$$a_1 - c - \frac{(a_2 - a_1)}{x} \leq \psi(c, \{a_k\}, x) \leq a_1 - c \qquad \text{for } x > 1, \quad c \in \mathbb{R}. \qquad (8)$$

So from the right-hand side of (8), it follows that

$$\psi(c, \{a_k\}, x) \leq a_1 - c < 0 \qquad \text{for } c > a_1, \quad x > 1.$$

Consequently we have $\chi(c, \{a_k\}, x) < 0$ for $c > a_1$ and $x > 1$.

And what happens when $c < a_1$? Notice that for $x > 1$ we have

$$a_1 - c - \frac{(a_2 - a_1)}{x} > 0 \quad \iff \quad (a_1 - c)x - (a_2 - a_1) > 0 \quad \iff \quad x > \frac{a_2 - a_1}{a_1 - c}. \tag{9}$$

Here, the last algebraic manipulation requires that $c < a_1$ in order to safely divide through and preserve the direction of the inequality. So, invoking the left-hand side inequality in (8) we see that

$$\psi(c, \{a_k\}, x) \geq a_1 - c - \frac{(a_2 - a_1)}{x} > 0 \qquad \text{for } x > \max\left\{1, \frac{a_2 - a_1}{a_1 - c}\right\}, \quad c < a_1.$$

This means that $\chi(c, \{a_k\}, x) > 0$ for $x > \max\{1, (a_2 - a_1)/(a_1 - c)\}$ and any fixed $c < a_1$.

Finally, it remains to check the case where the sequence has all consecutive terms differing. Clearly, the condition on $\{a_k\}$ implies that the sequence is non-decreasing, and so in this case we would have a strictly increasing sequence $a_1 < a_2 < \cdots$. But careful examination of the argument just given for an eventually constant sequence shows that the same analysis goes through. So in summary, given a sequence $\{a_k\}$ satisfying our condition we've shown that for fixed $c > a_1$, $\chi(c, \{a_k\}, x) < 0$ for $x > 1$, and that for fixed $c < a_1$, $\Psi(c, \{a_k\}, x) > 0$ for $x > \max\{1, (a_2 - a_1)/(a_1 - c)\}$. For the sake of simplicity, it is worth noting that this latter condition on x reduces to $x > 1$ for $c \leq a_1 - (a_2 - a_1)$, and $x > (a_2 - a_1)/(a_1 - c)$ for $c \in (a_1 - (a_2 - a_1), a_1)$. In other words, our threshold value $c^* = a_1$, and

$$f(c, \{a_k\}) = \begin{cases} \frac{a_2 - a_1}{a_1 - c} & , \text{ for } c \in (a_1 - (a_2 - a_1), a_1) \\ 1 & , \text{ for } c \notin (a_1 - (a_2 - a_1), a_1) \end{cases}$$

Moreover, by taking, for example, $\{a_k\} = \{1, 2, 2, 2, \ldots\}$ we see that the left-hand bound in (8) is actually equality, and hence the algebraic manipulation in (9) involves $\psi(c, \{a_k\}, x)$ itself. This means that our choice of $f(c, \{a_k\})$ is optimal.

No other solutions were received.

Circular sums

1183. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Let n be an odd positive integer. Suppose that the integers $1, 2, \ldots, 2n$ are placed around a circle in arbitrary order.

1. Show that there exist n of these numbers, placed in successive locations around the circle, having sum S_1 satisfying $S_1 \geq n^2 + \frac{n+1}{2}$,
2. Show that there exist n of these numbers, placed in successive locations around the circle, having sum S_2 satisfying $S_2 \leq n^2 + \frac{n-1}{2}$, and
3. Show that it is possible to place the $2n$ numbers around the circle in such a way that the sum of every n of these numbers, placed in successive locations around the circle, has sum S_3 satisfying $n^2 + \frac{n-1}{2} \leq S \leq n^2 + \frac{n+1}{2}$.

Solution by Andie Rawson (undergraduate), Smith College.

Let $x_1, x_2, \ldots, x_{2n}$ be an arbitrary ordering of the integers $1, 2, \ldots, 2n$ around a circle. Then

$$\sum_{i=1}^{2n} x_i = 1 + 2 + \cdots + 2n = 2n^2 + n.$$

Then let S_i be the sum of n numbers placed in successive locations around the circle starting from x_i. That is,

$$S_1 = x_1 + x_2 + \cdots + x_n$$
$$S_2 = x_2 + x_3 + \cdots + x_{n+1}$$
$$\cdot$$
$$\cdot$$
$$S_{2n} = x_{2n} + x_1 + \cdots + x_{n-1}$$

As each x_i occurs in n sums,

$$\sum_{i=1}^{2n} S_i = n(x_1 + x_2 + \cdots + x_{2n}) = 2n^3 + n^2$$

The mean of the $S_i's$ is then $\overline{S} = n^2 + \frac{n}{2}$.

1. As n is odd, $\overline{S}$ is not an integer. Thus at least one of the S_i satisfies the inequality $S_i \geq \lceil n^2 + \frac{n}{2} \rceil$, so there exist n numbers in successive locations with sum S satisfying $S \geq n^2 + \frac{n+1}{2}$.
2. Again as $\overline{S}$ is not an integer, at least one of the S_i satisfies the inequality $S_i \leq \lfloor n^2 + \frac{n}{2} \rfloor$, so there exist n numbers in successive locations with sum S satisfying $S \leq n^2 + \frac{n-1}{2}$.
3. Consider the ordering where

$$x_i = \begin{cases} i & i \in 1, 3, \ldots, 2n-1 \\ i + (n+1) & i \in 2, 4, \ldots, n-1 \\ i - (n-1) & i \in n+1, n+3, \ldots, 2n \end{cases}.$$

Then

$$\begin{aligned} S_1 &= 1 + 3 + \cdots + n + (n+3) + (n+5) + \cdots + 2n \\ &= \frac{n+1}{2}\frac{n+1}{2} + \frac{n-1}{2}\frac{3n+3}{2} \\ &= n^2 + \frac{n-1}{2}. \end{aligned}$$

For $i \in 1, 2, \ldots, n$ we have that

$$S_{i+1} = S_i + x_{n+i} - x_i = S_i + (-1)^{i+1},$$

and for $i \in n+1, n+2, \ldots, 2n-1$ we have that

$$S_{i+1} = S_i + x_{i-n} - x_i = S_i + (-1)^{i+1}.$$

Therefore for all even i, $S_{even} = S_1 + 1 = n^2 + \frac{n+1}{2}$ and for all odd i, $S_{odd} = S_1 = n^2 + \frac{n-1}{2}$. So every sum S of n successive numbers in this ordering satisfies $n^2 + \frac{n-1}{2} \leq S \leq n^2 + \frac{n+1}{2}$

Also solved by Levent Batakci and Paramjyoti Mohapatra, Case Western Reserve U.; Brian Beasley, Presbyterian C.; Cal Poly Pomona Problem Solving Group; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, U. of Maryland Global Campus; Eugene Herman, Grinnell C.; Walther Janous, Innsburck, Austria; Missouri State U. Problem Solving Group; Mooez Muhammad, (student) Bloor Collegiate Inst.; Albert Natian, Los Angeles Valley C.; Northwestern U. Math Problem Solving Group; Joel Schlosberg, Bayside, NY; Philip Straffin; Texas State U. Problem Solving Group; Janet Lai-Yu Wang and Nicole Yuen-Yi Pa; and the proposer.

A double integral of a product

1184. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*

Evaluate the following integral:

$$\int_0^\infty \int_0^\infty \frac{\sin x \, \sin(x+y)}{x(x+y)} dx dy.$$

Solution by the Missouri State University Problem Solving Group.

We will show that, more generally,

$$\int_0^\infty \int_0^\infty f(x) f(x+y) \, dy \, dx = \frac{1}{2} \left(\int_0^\infty f(t) \, dt \right)^2.$$

Since it is well known that

$$\int_0^\infty \frac{\sin t}{t} \, dt = \frac{\pi}{2},$$

the value of the original integral is $\pi^2/8$.

Letting $u = x$ and $v = x + y$, reversing the order of integration, and then reversing the roles of u and v, we have

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty f(x) f(x+y) \, dy \, dx \\ &= \int_0^\infty \int_u^\infty f(u) f(v) \, dv \, du \\ &= \int_0^\infty \int_0^v f(u) f(v) \, du \, dv \\ &= \int_0^\infty \int_0^u f(u) f(v) \, dv \, du. \end{aligned}$$

Therefore

$$\begin{aligned} 2I &= \int_0^\infty \int_u^\infty f(u) f(v) \, dv \, du + \int_0^\infty \int_0^u f(u) f(v) \, dv \, du \\ &= \int_0^\infty \int_0^\infty f(u) f(v) \, dv \, du \end{aligned}$$

$$= \left(\int_0^\infty f(t)\, dt \right)^2,$$

and the result follows.

We note that similar techniques show that

$$\int_0^\infty \cdots \int_0^\infty f(x_1) f(x_1 + x_2) \ldots f(x_1 + x_2 + \ldots + x_n)\, dx_n \ldots\, dx_1$$
$$= \frac{1}{n!} \left(\int_0^\infty f(t)\, dt \right)^n.$$

Also solved by U. Abel and V. Kushnirevych, Technische Hochschule Mittelhesen, Germany; Radouan Boukharfane, Kaust, Thuwal, KSA; Khristo Boyadzhiev, Ohio Northern U.; Paul Bracken, U. of Texas, Edinburg; Brian Bradie, Christopher Newport U.; Hongwei Chen, Christopher Newport U.; Bruce Davis, St. Louis Comm. C. at Florissant Valley; Giuseppe Fera, Vicenza, Italy; Lixing Han, U. of Michigan-Flint; Eugene Herman, Grinnell C.; Walther Janous, Innsbruck, Austria; John Kampmeyer, (student), Elizabethtown C.; Kee-Wai Lau, Hong Kong, China; Moubinool Omarjee, Lycé e Henri IV, Paris, France; Volkhard Schindler, Berlin, Germany; Albert Stadler, Herrliberg, Switzerland; Justin Turner, (Ph. D student) U. of Arkansas at Little Rock; Stan Wagon, Macalester C.; and the proposer.

The non-existence of 'special' rings

1185. *Proposed by Greg Oman, University of Colorado, Colorado Springs, Colorado Springs, CO.*

Suppose that S is a commutative ring with identity 1. A subring R of S is called *unital* if $1 \in R$. For the purposes of this problem, call S *special* if S has the following properties:

(a) S has a proper unital subring,

(b) there exists a prime ideal of S which is not maximal, and

(c) if R is any proper unital subring of S, then every prime ideal of R is maximal.

Prove the existence of a special ring or show that no such ring exists.

Solution by Anthony Bevelacqua, U. of North Dakota.

Assume such a ring S exists. Then S contains a prime but not maximal ideal P. Since $Z = \mathbb{Z} \cdot 1_S$ has no proper unital subrings we have $Z \subsetneq S$. Since $Z \cap P$ is a prime (and therefore maximal) ideal of Z we must have $Z \cap P = pZ$ for some prime p. Hence $\mathbb{Z}_p \cong Z/pZ$, the field of p elements, embeds in S/P.

Suppose $a \in S$ and $Z[a] \subsetneq S$. Then $Z[a] \cap P$ is a prime (and hence maximal) ideal of $Z[a]$. Thus $Z[a]/(Z[a] \cap P)$ is a field, and since $Z[a]/(Z[a] \cap P)$ naturally embeds in S/P, we see that $\overline{a} = a + P$ is either zero or a unit in S/P. Therefore if $a \in S$ is such that $\overline{a}$ is a nonzero nonunit in S/P then $S = Z[a]$.

Now S/P is an integral domain but not a field so there exists $a \in S$ such that $\overline{a}$ is a nonzero nonunit in S/P. Thus we have $S = Z[a]$. Since $\overline{a}^2$ is another nonzero nonunit we must have $S = Z[a^2]$ as well.

Whenever $S = Z[w]$ for some $w \in S$ we have $S/P = \mathbb{Z}_p[\overline{w}]$. Thus $\mathbb{Z}_p[\overline{a}] = \mathbb{Z}_p[\overline{a}^2]$, and so $\overline{a}$ must be algebraic over $\mathbb{Z}_p$. Now $S/P = \mathbb{Z}_p[\overline{a}]$ is an integral domain algebraic over $\mathbb{Z}_p$. Hence S/P is a field, and so P is maximal, a contradiction. Therefore no such ring S exists.

Also solved by the proposer.

An inequality involving the trace

1176. *Proposed by Xiang-Qian Chang, MCPHS University, Boston, MA.*

Let $A_{n\times n}$ be an $n \times n$ positive semidefinite Hermitian matrix. Prove that the following inequality holds for any pair of integers $p \geq 1$ and $q \geq 0$:

$$\frac{\mathrm{Tr}(A^p) + \mathrm{Tr}(A^{p+1}) + \cdots + \mathrm{Tr}(A^{p+q})}{\mathrm{Tr}(A^{p+1}) + \mathrm{Tr}(A^{p+2}) + \cdots + \mathrm{Tr}(A^{p+q+1})} \leq \frac{r_A}{\mathrm{Tr}(A)},$$

where r_A is the rank of A and Tr is the trace function.

Solution by Michel Bataille, Rouen, France.

We assume that A is a non-zero matrix.
The matrix A is similar to a diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0)$ where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the positive eigenvalues of A. Since similar matrices have the same rank and the same trace, we have $k = r_A$ and $\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. Also, for any positive integer m, A^m is similar to D^m, hence $\mathrm{Tr}(A^m) = \lambda_1^m + \lambda_2^m + \cdots + \lambda_k^m$. Without loss of generality, we suppose that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. Then, from Chebychev's inequality, we have

$$(\lambda_1^m + \lambda_2^m + \cdots + \lambda_k^m)(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \leq k(\lambda_1^{m+1} + \lambda_2^{m+1} + \cdots + \lambda_k^{m+1})$$

so that

$$\mathrm{Tr}(A^m) \leq \frac{k}{\mathrm{Tr}(A)}\mathrm{Tr}(A^{m+1}).$$

It is immediately deduced that

$$\begin{aligned}&\mathrm{Tr}(A^p) + \mathrm{Tr}(A^{p+1}) + \cdots + \mathrm{Tr}(A^{p+q})\\ &\qquad \leq \frac{k}{\mathrm{Tr}(A)}(\mathrm{Tr}(A^{p+1}) + \mathrm{Tr}(A^{p+2}) + \cdots + \mathrm{Tr}(A^{p+q+1})),\end{aligned}$$

and the required result follows (since $k = r_A$).

Also solved by James Duemmel, Bellingham, WA; Dmitry Fleischman, Santa Monica, CA; Jim Hartman, The College of Wooster; Justin Haverlick, Keene Valley, NY; Eugene Herman, Grinnell C.; Koopa Koo, Hong Kong STEAM Academy; Omran Kouba, Damascus, Syria; Elias Lampakis, Kiparissia, Greece; Pi'ilani Noguchi; Northwestern U. Math Problem Solving Group; Sunghee Park, Seoul, Korea; Michael Vowe, Therwil, Switzerland; and the proposer.

Small maximal ideals

1179. *Proposed by Greg Oman, University of Colorado, Colorado Springs, Colorado Springs, CO.*

Let R be a ring, and let I be an ideal of R. Say that I is *small* provided $|I| < |R|$ (i.e., I has a smaller cardinality than R). Suppose now that R is an infinite commutative ring with identity that is not a field. Suppose further that R possesses a small maximal ideal M_0. Prove the following:

1. there exists a maximal ideal M_1 of R such that $M_1 \neq M_0$, and
2. M_0 is the *unique* small maximal ideal of R.

Solution by Anthony Bevelacqua, University of North Dakota, Grand Forks, ND.

We will need the following basic result about cardinality: If A or B is infinite then $|A \times B| = \max(|A|, |B|)$.

Since R is not a field there exists a non-zero non-unit $a \in R$. Let $Ra = \{ra \mid r \in R\}$ and $R[a] = \{r \in R \mid ra = 0\}$. It's clear that both Ra and $R[a]$ are ideals of R. Since a is a non-unit we have $1 \notin Ra$, and since a is not zero we have $1 \notin R[a]$. Thus both Ra and $R[a]$ are proper ideals of R. The map $R \to Ra$ given by $r \mapsto ra$ is a ring epimorphism with kernel $R[a]$ so, by the first isomorphism theorem, we have $Ra \cong R/R[a]$. Hence $|R| = |Ra \times R[a]| = \max(|Ra|, |R[a]|)$. Thus R possesses a proper ideal I of cardinality $|R|$. Let M_1 be a maximal ideal of R containing I. Then $|I| \leq |M_1| \leq |R|$ so M_1 has cardinality $|R|$. Since $|M_0| < |R|$ we have $M_1 \neq M_0$. Thus we've shown 1.

Now assume M_0 and N are distinct small maximal ideals of R. Then, since they are distinct maximal ideals, we have $R = M_0 + N$. Since $M_0 + N = \{x + y \mid (x, y) \in M_0 \times N\}$ and R is infinite we have M_0 or N is infinite. Now

$$|R| \leq |M_0 \times N| = \max(|M_0|, |N|) < |R|,$$

a contradiction. This establishes 2.

Also solved by Paul Budney, Sunderland, MA; Eagle Problem Solvers, Georgia Southern U.; Elias Lampakis, Kiparissia, Greece; and the proposer.

Ideals in ideals

1180. *Proposed by Luke Harmon, University of Colorado, Colorado Springs, Colorado Springs, CO.*

In both parts, R denotes a commutative ring with identity. Prove or disprove the following:

1. there exists a ring R with infinitely many ideals with the property that every nonzero ideal of R is a subset of but finitely many ideals of R, and
2. there exists a ring R with infinitely many ideals with the property that every proper ideal of R contains (as a subset) but finitely many ideals of R.

Solution by Bill Dunn, Lone Star College Montgomery, Conroe, TX.

For 1, let R be the ring of integers. Every ideal I of R is principal, $I = (n)$, for some positive integer n. Suppose I is nonzero, $n \neq 0$. Then I is a subset of any other ideal $J = (m)$ if and only if m divides n. Because there are only finitely many positive integer divisors of n, there are only finitely many ideals of R that contain I.

For 2, suppose such a ring R existed. Because R has infinitely many ideals, it must have infinitely many proper ideals. Also, R must be Artinian because, by hypothesis on every proper ideal containing but finitely many ideals of R, any decreasing sequence of ideal must terminate.

However, an Artinian ring has only finitely many maximal ideals. Because every proper ideal is contained in some maximal ideal, one of these maximal ideals must contain infinitely many ideals of R, contradicting the hypothesis.

Therefore, such a ring R does not exist.

Also solved by Anthony Bevelacqua, U. of N. Dakota; Paul Budney, Sunderland, MA; Eagle Problem Solvers, Georgia Southern U.; and the proposer.

SOLUTIONS

Roots of a cubic equation

1171. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, and c be the roots of the equation $x^3 - 2x^2 - x + 1 = 0$, with $a < b < c$. Find the value of the expression $(\frac{a}{b})^2 + (\frac{b}{c})^2 + (\frac{c}{a})^2$.

Solution by Robert Doucette, McNeese State University, Lake Charles, LA.

Let $S = x + y$ and $P = x \cdot y$, where

$$x = \left(\frac{a}{b}\right)^2 + \left(\frac{b}{c}\right)^2 + \left(\frac{c}{a}\right)^2,$$

and

$$y = \left(\frac{b}{a}\right)^2 + \left(\frac{a}{c}\right)^2 + \left(\frac{c}{b}\right)^2.$$

Since $abc = -1$,

$$\begin{aligned} S =& a^4c^2 + b^4a^2 + c^4b^2 + b^4c^2 + a^4b^2 + c^4a^2 \\ =& \left(a^2 + b^2 + c^2\right)\left(a^2b^2 + b^2c^2 + c^2a^2\right) - 3 \end{aligned}$$

and

$$\begin{aligned} P &= \left(a^4c^2 + b^4a^2 + c^4b^2\right)\left(b^4c^2 + a^4b^2 + c^4a^2\right) \\ &= \left(a^6c^6 + a^6b^6 + b^6c^6\right) + \left(a^6 + b^6 + c^6\right) + 3 \\ &= \left(a^2b^2 + b^2c^2 + c^2a^2\right)^3 + \left(a^2 + b^2 + c^2\right)^3 - 6S - 9. \end{aligned}$$

We also have $a + b + c = 2$ and $ab + bc + ca = -1$, so that

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 6,$$

and

$$a^2b^2 + b^2c^2 + c^2a^2 = (ab + bc + ca)^2 - 2abc(a + b + c) = 5.$$

Therefore $S = 6 \cdot 5 - 3 = 27$, and $P = 5^3 + 6^3 - 6 \cdot 27 - 9 = 170$. The system $x + y = 27$, $xy = 170$ has two solutions: $(x, y) = (10, 17)$ and $(x, y) = (17, 10)$.

Letting $p(x) = x^3 - 2x^2 - x + 1$, we find that $p(-1)p(-0.8)$, $p(0)p(1)$, and $p(2)p(3)$ are all negative. By the intermediate value theorem, $y > c^4a^2 > 2^4(0.8)^2 > 10$. Therefore $x = 10$ is the desired value.

Also solved by Robert Agnew, Palm Coast, FL; Yagub Aliyev, ADA U., Baku, Azerbaijan; Hatef Arshagi, Guilford Tech. Comm. C.; Farrukh Rakhimjanovich Ataev, WIUT, Uzbekistan; Dione Bailey, Elsie Campbell, and Charles Diminnie (jointly), Angelo St. U.; Michel Bataille, Rouen, France; Rich Bauer, Shoreline, WA; Anthony Bevelacqua, U. of N. Dakota; Brian Bradie, Christopher Newport U.; James Brenneis; Scott Brown, Auburn U. Montgomery; Jiakang Chen; John Christopher, California St. U., Sacramento; Satvik Dasariraju, (student), Lawrenceville S., Princeton, NJ; Gregory Dresden, Washington & Lee U.; James Duemmel, Bellingham, WA; G. A. Edgar, Ohio St. U.; Michael Goldenberg, Baltimore Polytechnic

Inst. and Mark Kaplan, Towson U. (jointly); G. C. Greubel, Newport News, VA; Lixing Han, U. of Michigan - Flint; Justin Haverlick, Keene Valley, New York; Eugene Herman, Grinnell C.; Timmy Hodges and Sean Parnell (jointly); Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Benjamin Klein, Davidson C.; Sushanth Satish Kumar, Portola H. S.; Elias Lampakis, Kiparissia, Greece; Kee-Wai Lau, Hong Kong, China; Math for America Teachers (2 solutions); Missouri State Problem Solving Group; Donald Moore, Wichita, KS; Bob Newcomb, U. of Maryland; Joel Schlosberg, Bayside, NY; Randy Schwartz, Schoolcraft C.; Ioannis Sfikas, Athens, Greece; Seán Stewart, Bomaderry, NSW, Australia; Georges Vidiani, Les Dijon, France; Michael Vowe, Therwil, Switzerland; Stan Wagon, Macalester C.; and the proposer. We received two incorrect solutions.

Asymptotic behavior of the solution of a first-order differential equation

1172. *Proposed by Xiang-Qian Chang, MCPHS University, Boston, MA.*

Suppose that a function $y = y(x)$ satisfies the following first-order differential equation:

$$y' + x^6 - x^4 - 2yx^3 - 3x^2 + yx + y^2 - 1 = 0,$$

with initial value $y(0) = \sqrt{\frac{\pi}{2}}$. Show that $y(x) \sim \frac{1+x^4}{x}$ as x tends to infinity.

Solution by Kee-Wai Lau, Hong Kong, China.

By the substitution $z = y - x^3 + \frac{x}{2}$, we transform the differential equation to

$$z' = -z^2 + \frac{x^2}{4} + \frac{3}{2}, \tag{1}$$

with initial value $z(0) = \sqrt{\frac{\pi}{2}}$. To show that $y(x) \sim \dfrac{1+x^4}{x}$, it suffices to show that

$$z(x) \sim \frac{x}{2} + \frac{1}{x}. \tag{2}$$

A particular solution to (1) is $z = \frac{x}{2} + \frac{1}{x}$. By using formula $a°$ on p. 7 of reference [1] , we readily obtain the exact solution

$$z = \frac{x}{2} + \frac{\left(\sqrt{\frac{\pi}{2}} + \int_0^x e^{-t^2/2}\,dt\right) e^{x^2/2}}{\left(\sqrt{\frac{\pi}{2}} + \int_0^x e^{-t^2/2}\,dt\right) x e^{x^2/2} + 1},$$

and (2) follows.

Reference

[1] Polyanin, A. D., Zaitsev, V. F. (2003). *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed. Boca Raton, London, New York: Chapman & and Hall, CRC Press.

Also solved by U. Abel and V. Kushnirevych, Technische Hochschule Mittelhessen, Germany; Robert Agnew, Palm Coast, FL; Michel Bataille, Rouen, France; Paul Bracken, U. of Texas, Edinburg; William Chang, U. of Southern California; G. C. Greubel, Newport News, VA; Elias Lampakis, Kiparissia, Greece; Ioannis Sfikas, Athens, Greece; and the proposer.

An infinite integral domain has the same cardinality as the set of units of an integral domain which is integral over it

1173. *Proposed by Greg Oman, University of Colorado Colorado Springs, Colorado Springs, CO.*

All rings in this problem are assumed commutative with identity. Now, let R and S be rings and suppose that R is a subring of S (we assume that the identity of R is the identity of S). An element $s \in S$ is *integral over* R if s is a root of a monic polynomial $f(x) \in R[x]$. If we set $\overline{R} := \{s \in S : s \text{ is integral over } R\}$, then it is well-known that $\overline{R}$ is a subring of S containing R. The ring $\overline{R}$ is called the *integral closure of* R *in* S. In case $\overline{R} = S$, then we say that S *is integral over* R. For a ring R, let $R^\times$ denote the multiplicative group of units of R. Prove or disprove: for every infinite integral domain D_1, there exists an integral domain D_2 such that D_2 is integral over D_1 and $|D_2^\times| = |D_1|$ (that is, the set of units of D_2 has the same cardinality as that of D_1).

Solution by Anthony Bevelacqua, University of North Dakota.

Let F be the quotient field of D_1. Since D_1 is infinite we have $|D_1 - \{0\}| = |D_1|$ and so $|D_1 \times (D_1 - \{0\})| = |D_1|^2 = |D_1|$. Since $D_1 \times (D_1 - \{0\}) \to F$ given by $(a, b) \mapsto a/b$ is surjective, we have $|F| \le |D_1 \times (D_1 - \{0\})|$. Thus $|F| \le |D_1|$.

Let Ω be the algebraic closure of F and let D_2 be the integral closure of D_1 in Ω. Then D_2 is integral over D_1. Since Ω is an algebraic extension of F and F is infinite we have $|\Omega| \le |F|$. Indeed, for each $d \ge 1$ the set of elements of Ω with minimal polynomial over F of degree d has cardinality $\le d|F|^d = |F|$, and so $|\Omega| \le \aleph_0 |F| = |F|$. Combining this with the first paragraph we have $|\Omega| \le |D_1|$.

Now for each $a \in D_1$, $x^2 + ax - 1$ has a root $u_a \in \Omega$, and, since $x^2 + ax - 1$ is monic, we have $u_a \in D_2$. Since $a \in D_1 \subseteq D_2$ we have $u_a + a \in D_2$ as well. Thus $u_a(u_a + a) = 1$ so $u_a \in D_2^\times$. We note that if $u_a = u_b$ for $a, b \in D_1$ then

$$u_a^2 + au_a - 1 = u_b^2 + bu_b - 1 \Rightarrow a = b.$$

Thus $|D_1| = |\{u_a : a \in D_1\}| \le |D_2^\times|$.

Finally we have

$$|D_1| \le |D_2^\times| \le |\Omega| \le |D_1|$$

where the first inequality is given by the previous paragraph, the second follows from $D_2^\times \subseteq \Omega$, and the last is given by the second paragraph. Therefore D_2 is integral over D_1 and $|D_2^\times| = |D_1|$.

Also solved by Tom Jager, *Calvin U.; and the proposer.*

Criterion for convergence of an infinite product

1174. *Proposed by George Stoica, New Brunswick, Canada.*

Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be complex numbers which are not integers. Prove that the infinite product below converges if and only if $\sum_{i=1}^k a_i = \sum_{i=1}^k b_i$. What is the value of the product?

$$\prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2) \cdots (n - a_k)}{(n - b_1)(n - b_2) \cdots (n - b_k)}$$

Solution by Eugene Herman, Grinnell College, Grinnell, Iowa.

The gamma function identity $\Gamma(1+z) = z\Gamma(z)$ holds for all complex numbers z that are not integers. Hence

$$\Gamma(1+z)\prod_{n=1}^{m}(n+z) = \Gamma(m+1+z).$$

Therefore

$$\frac{\prod_{i=1}^{k}\Gamma(1-a_i)}{\prod_{i=1}^{k}\Gamma(1-b_i)} \cdot \prod_{n=1}^{m}\frac{(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_k)} = \frac{\prod_{i=1}^{k}\Gamma(m+1-a_i)}{\prod_{i=1}^{k}\Gamma(m+1-b_i)}.$$

Furthermore,

$$\lim_{n\to\infty}\frac{\Gamma(n+z)}{\Gamma(n)n^z} = 1$$

for all complex numbers z that are not integers. Therefore the mth partial product of the given infinite product converges as $m \to \infty$ if and only if the following expression converges:

$$\frac{\prod_{i=1}^{k}(m+1)^{-a_i}}{\prod_{i=1}^{k}(m+1)^{-b_i}} = (m+1)^{\sum_{i=1}^{k} b_i - \sum_{i=1}^{k} a_i}.$$

Therefore, a necessary and sufficient condition for convergence of the product is $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$. Also, the limit is

$$\frac{\prod_{i=1}^{k}\Gamma(1-b_i)}{\prod_{i=1}^{k}\Gamma(1-a_i)}.$$

Editor's note: Janous and Lampakis pointed out that this problem and its solution are known, with both of these solvers providing reference [1] and Lampakis also providing reference [2].

References

[1] Whittaker, E. T., Watson, G. N. (1927). *Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions, with an Account of the Principal Transcendental Functions*, 4th ed. Cambridge: Cambridge University Press, p. 238.

[2] Nimbran, A. S. (2016). Interesting infinite products of rational functions motivated by Euler. *Math. Stud.* 85(1–2): 122, Theorem 3.1.

Also solved by MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, U. of Texas, Edinburg; JAMES DUEMMEL, Bellingham, WA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ELIAS LAMPAKIS, Kiparissia, Greece; MICHAEL VOWE, Therwil, Switzerland; and the proposer. There were three solutions that were either incomplete or incorrect

Nonexistence of a sign-preserving field isomorphism between distinct proper subfields of the reals

1175. *Proposed by George Stoica, New Brunswick, Canada.*

Let F_1 and F_2 be distinct proper subfields of the field $\mathbb{R}$ of real numbers. Is there a field isomorphism $f: F_1 \to F_2$ preserving signs, that is, for all real x: $x \in F_1$ and $x > 0$ if and only if $f(x) \in F_2$, $f(x) > 0$?

Solution by Northwestern University Math Problem Solving Group.

First note that every subfield of $\mathbb{R}$ contains the field of rational numbers $\mathbb{Q}$. This follows from the fact that every subfield of $\mathbb{R}$ contains 1, and $\mathbb{Q}$ is the subfield of $\mathbb{R}$ generated by 1. On the other hand, every isomorphism f between subfields of $\mathbb{R}$ restricted to $\mathbb{Q}$ is the identity on $\mathbb{Q}$, i.e., if $r \in \mathbb{Q}$, then $f(r) = r$. This can be proved as follows: f(0) = 0; f(1) = 1; for integers n, $f(n) = f(1 + \cdots + 1) = nf(1)$, $f(-n) = -f(n) = -n$; and for integers m and n, with $n \neq 0$, $f(m/n) = f(m)/f(n) = m/n$.

Next, since F_1 and F_2 are distinct, f cannot be the identity on F_1, so there is some $u \in F_1$ such that $f(u) \neq u$. Assume $u < f(u)$ (the case $u > f(u)$ is analogous). Since the rational numbers are dense in the reals, there is some number $r \in \mathbb{Q}$ such that $u < r < f(u)$; hence,

$$u - r < 0 < f(u) - r = f(u) - f(r) = f(u - r).$$

Letting $x = u - r$, we have $x < 0$ and $f(x) > 0$, implying that f does not preserve signs.

Also solved by Anthony Bevelacqua, U. of N. Dakota; Paul Budney, Sunderland, MA; William Chang, U. of Southern California; Dmitry Fleischman, Santa Monica, CA; Eugene Herman, Grinnell C.; Tom Jager, Calvin C.; Sushanth Sathish Kumar, Portola High S.; Elias Lampakis, Kiparissia, Greece; Missouri State Problem Solving Group; Lawrence Peterson, U. of N. Dakota; Stephen Scheinberg, Corona del Mar, CA; and the proposer.

Solutions

The largest divisor of $n^k - n$ **February 2020**

2086. *Proposed by David M. Bradley, University of Maine, Orono, ME.*

Let $f(k)$ denote the largest integer that is a divisor of $n^k - n$ for all integers n. For example, $f(2) = 2$ and $f(3) = 6$. Determine $f(k)$ for all integers $k > 1$.

Solution by the Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.
To simplify notation, we write $g_k(n) = n^k - n$.

First, we prove two lemmas.

Lemma 1. *For every $k > 1$, $f(k)$ is square-free, i.e., if p is a prime then p^2 does not divide $f(k)$.*

Proof. Note that $g_k(p) = p(p^{k-1} - 1)$. If p^2 divided $g_k(p)$ then p would divide $p^{k-1} - 1$. But this would imply that p divides 1, giving a contradiction. ■

Lemma 2. *If p is a prime, then p divides $f(k)$ if and only if $p - 1$ divides $k - 1$.*

Proof. ($\Leftarrow$) If $k - 1 = (p - 1)\ell$ for some $\ell \geq 1$, then

$$g_k(n) = n((n^\ell)^{p-1} - 1).$$

If p divides n then it divides $g_k(n)$ too. If p does not divide n then by Fermat's little theorem p divides $(n^\ell)^{p-1} - 1$. Hence p divides $g_k(n)$ for every n, and this implies that p divides $f(k)$.

($\Rightarrow$) Assume a prime p divides $f(k)$. This means that p divides $g_k(n) = n(n^{k-1} - 1)$ for every n. Pick n to be a primitive root modulo p (which, by a well-known result in number theory, always exists). Then $1, n, n^2, \ldots, n^{p-2}$, are distinct modulo p. Since p does not divide n, it must divide $n^{k-1} - 1$. Using the Euclidean algorithm we write $k - 1 = (p - 1)\ell + i$, with $\ell \geq 0$, $0 \leq i < p - 1$. By Fermat's little theorem $n^{p-1} \equiv 1 \pmod{p}$, hence

$$n^{k-1} = n^{(p-1)\ell+i} \equiv n^i \pmod{p}.$$

Since p divides $n^{k-1} - 1$ we have $n^{k-1} \equiv 1 \pmod{p}$, hence $n^i \equiv 1 \pmod{p}$. Since $1, n, \ldots, n^{p-2}$ are distinct modulo p, we must have $i = 0$. Therefore $k - 1 = (p - 1)\ell$, i.e., $p - 1$ divides $k - 1$. ■

Lemmas 1 and 2 allow us to determine $f(k)$:

$$f(k) = \prod_{\substack{d \mid k-1 \\ d+1 \text{ is prime}}} (d + 1).$$

Example: To compute $f(19)$ we find the divisors of $19 - 1 = 18 : 1, 2, 3, 6, 9, 18$, add 1 to each of them: $2, 3, 4, 7, 10, 19$, then multiply the primes appearing on this list: $2 \cdot 3 \cdot 7 \cdot 19 = 798$. Thus $f(19) = 798$.

Editor's Note. It is immediate that $f(2j) = 2$. The proposer points out that by the von Staudt–Clausen theorem, $f(2j + 1)$ is the denominator of B_{2j}, the $2j$th Bernoulli number.

Also solved by Elijah Bland & Brooke Mullins, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, William Chang, John Christopher, Prithwijit De & B. Sury (India), Joseph DiMuro, Dmitry Fleischman, George Washington University Problems Group, Justin Haverlick, Omran Kouba (Syria), Sushanth Satish Kumar, Elias Lampakis (Greece), László Lipták, José Heber Nieto (Venezuela), Joel Schlosberg, Randy K. Schwartz, Doga Can Sertbas (Turkey), Jacob Siehler, John H. Smith, Albert Stadler (Switzerland), David Stone & John Hawkins, Edward White & Roberta White, and the proposer. There was one incomplete or incorrect solution.

A limit involving a recursively defined sequence **February 2020**

2087. *Proposed by Florin Stanescu, Şerban Cioiculescu School, Găeşti, Romania.*

Consider the sequence defined by $x_1 = a > 0$ and

$$x_n = \ln\left(1 + \frac{x_1 + x_2 + \cdots + x_{n-1}}{n - 1}\right) \quad \text{for } n \geq 2.$$

Compute $\lim_{n\to\infty} x_n \ln n$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is 2. A simple induction argument shows that $x_n > 0$ for all $n \geq 1$. Now, let $S_n = x_1 + x_2 + \cdots + x_n$ and define $\sigma_n = S_n/n$. Using the well-known inequality $\ln(1+x) \leq x$ which is valid for $x > -1$ (with equality if and only if $x = 0$), we conclude that

$$S_n - S_{n-1} = x_n = \ln\left(1 + \frac{S_{n-1}}{n-1}\right) \leq \frac{S_{n-1}}{n-1}$$

or equivalently $\sigma_n \leq \sigma_{n-1}$ for $n \geq 2$. So, the sequence $(\sigma_n)_{n\geq 1}$ is positive and decreasing, and since $x_n = \ln(1+\sigma_{n-1})$ the sequence $(x_n)_{n\geq 1}$ is also positive decreasing. Let $\ell = \lim_{n\to\infty} x_n$. By Cezáro's lemma we know that $\ell = \lim_{n\to\infty} \sigma_n$ and the equality $x_n = \ln(1+\sigma_{n-1})$ implies that $\ell = \ln(1+\ell)$, and consequently $\ell = 0$.

Now, because

$$\lim_{x\to 0} \ln(1+x)/x = 1$$

we conclude that $\lim_{n\to\infty} x_n/\sigma_{n-1} = 1$ On the other hand

$$\sigma_n = \sigma_{n-1} - \frac{1}{n}(\sigma_{n-1} - x_n) = \sigma_{n-1} - \frac{\sigma_{n-1} - \ln(1+\sigma_{n-1})}{n}.$$

But $\ln(1+x) = x - (1/2)x^2 + O(x^3)$ for small x, so

$$\sigma_n = \sigma_{n-1} - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right).$$

In particular, σ_n, σ_{n-1}, x_n, and x_{n+1} are all equivalent as $n \to \infty$. Now

$$1 + \sigma_n = (1+\sigma_{n-1})\left(1 - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right)\right).$$

So

$$x_{n+1} = x_n + \ln\left(1 - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right)\right) = x_n - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right).$$

Hence

$$n\left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right) = \frac{1}{2}\frac{\sigma_{n-1}^2}{x_n x_{n+1}} + O(\sigma_{n-1}).$$

Thus

$$\lim_{n\to\infty} n\left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right) = \frac{1}{2}.$$

Consequently, the Stolz–Cesáro theorem implies that

$$\lim_{n\to\infty} \frac{1}{H_n}\cdot\frac{1}{x_n} = \frac{1}{2},$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the nth harmonic number. Finally, recalling that $H_n = \ln n + O(1)$ we conclude that $\lim_{n\to\infty} x_n \ln n = 2$, as claimed.

Also solved by Robert A. Agnew, Brian Bradie, Robert Calcaterra, Hongwei Chen, Kee-Wai Lau (Hong Kong), Albert Stadler (Switzerland), and the proposer.

A Fibonacci sum **February 2020**

2088. *Proposed by Mircea Merca, University of Craiova, Romania.*

Let n and t be nonnegative integers. Prove that

$$\sum_{k=0}^{2n}(-1)^k F_{tk} F_{2tn-tk} = -\frac{F_t}{L_t} F_{2tn},$$

where F_i denotes the ith Fibonacci number and L_i denotes the ith Lucas number.

Solution by G. C. Greubel, Newport News, VA.
More generally let

$$\phi_n = \frac{\mu^n - \nu^n}{\mu - \nu} \qquad \text{and} \qquad \theta_n = \mu^n + \nu^n,$$

where $\mu + \nu = a$ and $\mu\,\nu = -b$. Note that when $a = b = 1$, $\phi_n = F_n$ and $\theta_n = L_n$ by the Binet formulas.

We have

$$(\mu - \nu)^2\, \phi_{tk}\phi_{t(2n-k)} = \theta_{2tn} - \mu^{2tn}\left(\frac{\nu}{\mu}\right)^{tk} - \nu^{2tn}\left(\frac{\mu}{\nu}\right)^{tk}.$$

Using the sums

$$\sum_{k=0}^{2n}(-1)^k = 1$$

$$\sum_{k=0}^{2n}(-1)^k\left(\frac{\nu}{\mu}\right)^{tk} = \frac{\mu^t}{\theta_t}\left(1 + \left(\frac{\nu}{\mu}\right)^{t(2n+1)}\right)$$

$$\sum_{k=0}^{2n}(-1)^k\left(\frac{\mu}{\nu}\right)^{tk} = \frac{\nu^t}{\theta_t}\left(1 + \left(\frac{\mu}{\nu}\right)^{t(2n+1)}\right)$$

we find that

$$\begin{aligned}\sum_{k=0}^{2n}(-1)^k\,\phi_{tk}\,\phi_{t(2n-k)} &= \frac{1}{(\mu-\nu)^2}\left(\theta_{2tn} - \frac{2\,\theta_{t(2n+1)}}{\theta_t}\right)\\ &= -\frac{1}{\theta_t}\,\frac{1}{(\mu-\nu)^2}\,\left(2\,\theta_{t(2n+1)} - \theta_t\,\theta_{2tn}\right) = -\frac{\phi_t}{\theta_t}\,\phi_{2tn}.\end{aligned}$$

Letting $a = b = 1$ gives the desired result.

A similar argument shows that

$$\sum_{k=0}^{2n}\phi_{tk}\,\phi_{t(2n-k)} = \frac{(2\,n\,\phi_t\,\theta_{2tn} - \theta_t\,\phi_{2tn})}{(a^2 + 4\,b)\,\phi_t}$$

and hence

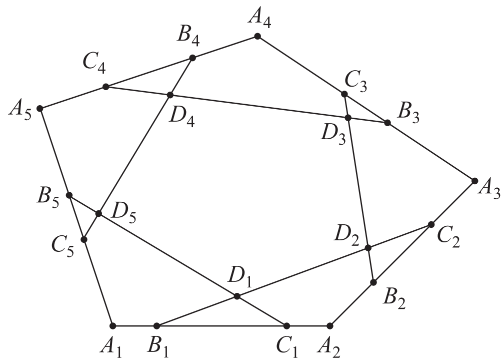
$$\sum_{k=0}^{2n} F_{tk}\, F_{t(2n-k)} = \frac{(2\,n\,F_t\,L_{2tn} - L_t\,F_{2tn})}{5\,F_t}.$$

Also solved by Michel Bataille (France), Brian Bradie, Robert Calcaterra, Dmitry Fleishman, Harris Kwong, Abhisar Mittal, José Heber Nieto (Venezuela), Angel Plaza (Spain), Albert Stadler (Switzerland), Michael Vowe (Switzerland), and the proposer.

A product of ratios for nested polygons **February 2020**

2089. *Proposed by Rick Mabry, LSU Shreveport, Shreveport, LA.*

Let $A_1, A_2, \ldots, A_n$ be the vertices of a convex n-gon in the plane. Identifying the indices modulo n, define the following points: Let B_i and C_i be vertices on $\overline{A_iA_{i+1}}$ such that $A_iB_i = C_iA_{i+1} < A_iA_{i+1}/2$ and let D_i be the intersection of $\overline{B_{i-1}C_i}$ and $\overline{B_iC_{i+1}}$. Prove that $\prod_{i=1}^{n}(B_iD_i)/(D_iC_i) = 1$.



Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
Let $\beta_i = \angle C_iB_iD_i$ and $\gamma_i = \angle D_iC_iB_i$. Applying the law of sines to triamgles $\triangle B_iC_iD_i$ and $\triangle B_{i-1}A_iC_i$ leads to

$$\frac{B_iD_i}{D_iC_i} = \frac{\sin\gamma_i}{\sin\beta_i} \qquad \text{and} \qquad \frac{B_{i-1}A_i}{A_iC_i} = \frac{\sin\gamma_i}{\sin\beta_{i-1}}.$$

Also, $A_iB_i = C_iA_{i+1}$ implies that $A_iC_i = B_iA_{i+1}$. Using these equations, we obtain

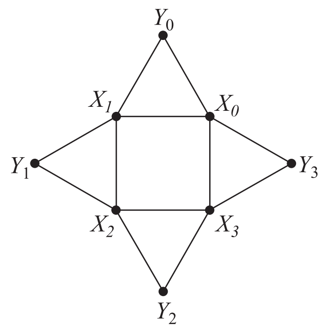
$$\begin{aligned}
\prod_{i=1}^{n}\frac{B_iD_i}{D_iC_i} &= \prod_{i=1}^{n}\frac{\sin\gamma_i}{\sin\beta_i} = \frac{\prod_{i=1}^{n}\sin\gamma_i}{\prod_{i=1}^{n}\sin\beta_i} = \frac{\prod_{i=1}^{n}\sin\gamma_i}{\prod_{i=1}^{n}\sin\beta_{i-1}}\\
&= \prod_{i=1}^{n}\frac{\sin\gamma_i}{\sin\beta_{i-1}} = \prod_{i=1}^{n}\frac{B_{i-1}A_i}{A_iC_i} = \frac{\prod_{i=1}^{n}B_{i-1}A_i}{\prod_{i=1}^{n}A_iC_i}\\
&= \frac{\prod_{i=1}^{n}B_{i-1}A_i}{\prod_{i=1}^{n}B_iA_{i+1}} = \frac{\prod_{i=1}^{n}B_{i-1}A_i}{\prod_{i=1}^{n}B_{i-1}A_i} = 1.
\end{aligned}$$

Also solved by Robert Calcaterra, William Chang, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), George Washington University Problems Group, Joel Schlosberg, and the proposer.

Matchings in a certain family of graphs **February 2020**

2090. *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA.*

Recall that a *matching* of a graph is a set of edges that do not share any vertices. For example, C_4, the cyclic graph on four vertices (i.e., a square), has seven matchings: the empty set, single edges (four of these), or pairs of opposite edges (two of these).
Let G_n be the (undirected) graph with vertices x_i and y_i, $0 \leq i \leq n-1$, and edges $\{x_i, x_{i+1}\}$, $\{x_i, y_i\}$, and $\{y_i, x_{i+1}\}$, $0 \leq i \leq n-1$, where the indices are to be taken modulo n. For example, G_4 is shown below. Determine the number of matchings of G_n.



Solution by the George Washington University Problems Group, George Washington University, Washington, DC.
The answer is 3^n. To see this, let $S = \{-1, 0, 1\}^n$, a set whose cardinality is clearly 3^n. We show that there is a bijection ϕ from S to the set of matchings of G_n. Let $a = (a_1, \dots, a_n)$ be an element of S. We define $\phi(a)$ as follows:

$$\{x_i, x_{i+1}\} \in \phi(a) \text{ if and only if } a_i = 1 \text{ and } a_{i+1} = -1,$$

$$\{x_i, y_i\} \in \phi(a) \text{ if and only if } a_i = 1 \text{ and } a_{i+1} \neq -1, \text{ and}$$

$$\{x_{i+1}, y_i\} \in \phi(a) \text{ if and only if } a_i \neq 1 \text{ and } a_{i+1} = -1.$$

We now check that $\phi(a)$ is indeed a matching. The edges incident to y_i are not both in $\phi(a)$, since $\{x_i, y_i\} \in \phi(a)$ requires $a_i = 1$ but $\{x_{i+1}, y_i\} \in \phi(a)$ requires $a_i \neq 1$. Also, among the four edges incident to x_i, at most one can be chosen for $\phi(a)$, since including $\{x_i, x_{i-1}\}$, $\{x_i, y_{i-1}\}$, $\{x_i, y_i\}$, and $\{x_i, x_{i+1}\}$ require, respectively, the four mutually exclusive conditions (1) $a_i = -1$ and $a_{i-1} = 1$, (2) $a_i = -1$ and $a_{i-1} \neq 1$, (3) $a_i = 1$ and $a_{i+1} \neq -1$, and (4) $a_i = 1$ and $a_{i+1} = -1$.

Given a matching M, there is a unique $a \in S$ so that M is $\phi(a)$. To see this, let $a_i = 1$ if M contains $\{x_i, x_{i+1}\}$ or $\{x_i, y_i\}$, let $a_i = -1$ if M contains $\{x_{i-1}, x_i\}$ or $\{x_i, y_{i-1}\}$, and let $a_i = 0$ if x_i is not the endpoint of any edge in M. This element $a \in S$ is the only element in $\phi^{-1}(M)$. Hence ϕ is bijective.

Also solved by Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Robert Calcaterra, Jiakang Chen, Eddie Cheng; Serge Kruk; Li Li & László Lipták (jointly), José H. Nieto (Venezuela), Kishore Rajesh, Edward Schmeichel, John H. Smith, and the proposer. There was one incomplete or incorrect solution.

SOLUTIONS

A Nilpotent Commutator

12339 [2022, 686]. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Let A and B be complex n-by-n matrices for which $A^2 + xB^2 = y(AB - BA)$, where x is a positive real number and y is a real number such that $(1/\pi)\cos^{-1}\big((y^2 - x)/(y^2 + x)\big)$ is irrational. Prove that $(AB - BA)^n$ is the zero matrix.

Solution by Kyle Gatesman, Fairfax, VA. Let $U = A + i\sqrt{x}B$ and $V = A - i\sqrt{x}B$. Note that $y \pm i\sqrt{x} \neq 0$ because y is real and x is positive. Since

$$UV = A^2 + xB^2 - i\sqrt{x}(AB - BA) = (y - i\sqrt{x})(AB - BA)$$

and

$$VU = A^2 + xB^2 + i\sqrt{x}(AB - BA) = (y + i\sqrt{x})(AB - BA),$$

we have

$$VU = \frac{y + i\sqrt{x}}{y - i\sqrt{x}}\, UV = \frac{y^2 - x + 2yi\sqrt{x}}{y^2 + x}\, UV.$$

Let $(y + i\sqrt{x})/(y - i\sqrt{x}) = \cos\theta + i\sin\theta = e^{i\theta}$. The spectrum of VU is $e^{i\theta}$ times that of UV. By hypothesis, θ is not a rational multiple of π, so $e^{in\theta} \neq 1$ for all nonzero integers n.

It is well known for complex n-by-n matrices U and V, that UV and VU have the same characteristic polynomial. Hence any eigenvalue of UV or VU is an eigenvalue of the other. Thus the spectrum of UV is invariant under multiplication by $e^{i\theta}$. Since the complex numbers $e^{i\theta}, e^{2i\theta}, e^{3i\theta}, \dots$ are distinct and the spectrum of UV has cardinality at most n, we conclude that the only eigenvalue of UV is zero. It follows that the characteristic polynomial of $AB - BA$ is λ^n. By the Cayley–Hamilton Theorem, $(AB - BA)^n$ is the zero matrix.

Also solved by C. P. Anil Kumar (India), S. Bhadra, E. A. Herman, O. P. Lossers (Netherlands), M. Omarjee (France), R. Stong, L. Zhou, and the proposer.

A Nascent Delta Function

12340 [2022, 686]. *Proposed by Antonio Garcia, Strasbourg, France.* Let $g\colon [0,1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx = Cg(1/2)$$

for some constant C (independent of g), and determine the value of C.

Solution by Missouri State University Problem Solving Group, Missouri State University, Springfield, MO. Substituting $u = n(2x-1)$ and letting $\chi_{[-n,n]}$ denote the characteristic function of $[-n, n]$ gives

$$\frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{g\left(\frac{1}{2} + \frac{u}{2n}\right) \chi_{[-n,n]}(u)}{\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n} \, du.$$

Since g is continuous, we may choose a $K > 0$ such that $|g(x)| \le K$ on $[0, 1]$. Further, for $n \ge 2$, the binomial theorem gives

$$\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n \ge 2\left(1 + \binom{n}{2}\frac{u^2}{n^2}\right) \ge 2\left(1 + \frac{u^2}{4}\right).$$

Therefore for $n \ge 2$,

$$\frac{1}{2} \left| \frac{g\left(\frac{1}{2} + \frac{u}{2n}\right) \chi_{[-n,n]}(u)}{\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n} \right| \le \frac{K}{4 + u^2}.$$

This upper bound has finite integral, so the dominated convergence theorem applies, and we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{g\left(\frac{1}{2} + \frac{u}{2n}\right)}{\left(1 + \frac{u}{n}\right)^n + \left(1 - \frac{u}{n}\right)^n} \, du \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{g(1/2)}{e^u + e^{-u}} \, du \\
&= \frac{1}{2} g(1/2) \arctan(e^u) \Big|_{-\infty}^{\infty} = \frac{\pi}{4} g(1/2).
\end{aligned}$$

Also solved by M. Aassuka (France), A. Berkane (Algeria), S. Bhadra (India), H. Chen (US), W. J. Cowieson, M.-C. Fan (China), K. Gatesman, R. Guadalupe (Philippines), E. A. Herman, N. Hodges (UK), F. Holland (Ireland), E. J. Ionascu, S. Kaczkowski, O. Kouba (Syria), C. Krattenthaler (Germany), G. Lavau (France), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), F. Masroor, R. Mortini (Luxembourg) & R. Rupp (Germany), M. Omarjee (France), D. Pascuas (Spain), P. Perfetti (Italy), K. Schilling, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), E. I. Verriest, J. Vukmirović (Serbia), J. H. Yan (China), and the proposer.

A Product Inequality

12341 [2022, 686]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let $x_1, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^n x_i^2 \le n$, and let $S = \sum_{i=1}^n x_i$. Prove

$$\prod_{i=1}^n \left(1 + \frac{1}{x_i x_{i+1}}\right)^{x_i^2} \ge 2^{S^2/n},$$

where x_{n+1} is taken to be x_1.

Solution by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy. We prove the more general inequality

$$\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2} \ge \left(1 + \frac{n}{T}\right)^{S^2/n}, \tag{$*$}$$

where $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are positive real numbers, $S = \sum_{i=1}^n x_i$, and $T = \sum_{i=1}^n y_i$. The required inequality follows from $(*)$ by letting $y_i = x_i x_{i+1}$ and noting that, by the rearrangement inequality,

$$T = \sum_{i=1}^n y_i = \sum_{i=1}^n x_i x_{i+1} \le \sum_{i=1}^n x_i^2 \le n.$$

To prove $(*)$, we compute

$$\begin{aligned}\log\left(\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2}\right) &= \sum_{i=1}^n x_i^2 \log\left(1 + \frac{1}{y_i}\right) \\ &= \sum_{i=1}^n x_i^2 \int_0^1 \frac{dt}{y_i + t} = \int_0^1 \sum_{i=1}^n \frac{x_i^2}{y_i + t}\, dt.\end{aligned}$$

For $0 \le t \le 1$, the Cauchy–Schwarz inequality implies

$$S^2 = \left(\sum_{i=1}^n \sqrt{y_i + t} \cdot \frac{x_i}{\sqrt{y_i + t}}\right)^2 \le \sum_{i=1}^n (y_i + t) \cdot \sum_{i=1}^n \frac{x_i^2}{y_i + t} = (T + nt) \sum_{i=1}^n \frac{x_i^2}{y_i + t},$$

so

$$\sum_{i=1}^n \frac{x_i^2}{y_i + t} \ge \frac{S^2}{T + nt}.$$

Therefore

$$\log\left(\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2}\right) = \int_0^1 \sum_{i=1}^n \frac{x_i^2}{y_i + t}\, dt \ge \int_0^1 \frac{S^2}{T + nt}\, dt = \frac{S^2}{n} \log\left(1 + \frac{n}{T}\right).$$

Inequality $(*)$ follows.

Also solved by P. Bracken, W. J. Cowieson, O. P. Lossers (Netherlands), S. Patra, A. Stadler (Switzerland), R. Stong, and the proposer.

Characterizing Cyclic Quadrilaterals

12343 [2022, 785]. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a convex quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, and $BD = f$. Prove that $ABCD$ is a cyclic quadrilateral (i.e., the four vertices lie on a circle) if and only if

$$\frac{f^2 - e^2}{ac + bd} = \frac{\left(a^2 - c^2\right)\left(b^2 - d^2\right)}{(ab + cd)(ad + bc)}.$$

Solution by Prithwijit De, Mumbai, India. Denote the angles of $ABCD$ at the four vertices by A, B, C, and D. Let

$$T_1 = \cos A + \cos C = \frac{d^2 + a^2 - f^2}{2ad} + \frac{b^2 + c^2 - f^2}{2bc},$$

$$T_2 = \cos B + \cos D = \frac{a^2 + b^2 - e^2}{2ab} + \frac{c^2 + d^2 - e^2}{2cd}.$$

Algebraic manipulation yields

$$2abcd\big((ab + cd)T_1 - (ad + bc)T_2\big) = (ac + bd)(a^2 - c^2)(b^2 - d^2) - (ab + cd)(ad + bc)(f^2 - e^2).$$

It therefore suffices to show that $ABCD$ is cyclic if and only if

$$(ab + cd)T_1 - (ad + bc)T_2 = 0.$$

By the sum-to-product formula for the cosine function and the fact that $B + D = 2\pi - (A + C)$, we have

$$(ab + cd)T_1 - (ad + bc)T_2 = 2\left((ab + cd)\cos\left(\frac{A - C}{2}\right) + (ad + bc)\cos\left(\frac{B - D}{2}\right)\right)\cos\left(\frac{A + C}{2}\right).$$

Since $|A - C|$ and $|B - D|$ are less than π, $\cos((A - C)/2)$ and $\cos((B - D)/2)$ are strictly positive. Hence $(ab + cd)T_1 - (ad + bc)T_2 = 0$ if and only if $\cos((A + C)/2) = 0$, which happens if and only if $A + C = \pi$, which is equivalent to $ABCD$ being cyclic.

Also solved by G. Fera (Italy), O. Geupel (Germany), M. Goldenberg & M. Kaplan, N. Hodges (UK), O. P. Lossers (Netherlands), C. R. Pranesachar (India), C. Schacht, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, Fejéntaláltuka Szeged Problem Solving Group (Hungary), and the proposer.

Linear Combinations of Powers That Are Not Perfect Squares

12346 [2022, 785]. *Proposed by Nguyen Quang Minh, Hwa Chong Institution, Bukit Timah, Singapore.* Prove that there are infinitely many integers A such that, for every nonzero integer x and distinct positive odd integers m and n, the integer $x^m + Ax^n$ is not a perfect square.

Solution by Yury J. Ionin, Central Michigan University, Mount Pleasant, MI. We claim that the infinite family consisting of the negatives of primes congruent to 3 modulo 8 satisfies the requirements of the problem.

Let $A = -p$ for such a prime p. Factoring out the perfect square $x^{\min\{m,n\}-1}$, we see that it suffices to show that no $x^m - px^n$ is a perfect square when m and n are odd and either $m = 1$ or $n = 1$. Suppose otherwise.

First consider $m = 1$ and set $k = (n - 1)/2$. With $x - px^n = x(1 - px^{2k})$, both factors are negative. Since also $1 - px^{2k}$ is relatively prime to x, both $-x$ and $px^{2k} - 1$ must be squares. Modulo p, the equation $px^{2k} - 1 = a^2$ for a positive integer a reduces to $a^2 \equiv -1$. However, when $p \equiv 3 \pmod 8$ (indeed, whenever $p \equiv 3 \pmod 4$) the value -1 is not a square modulo p, a contradiction.

Now consider $n = 1$ and set $k = (m - 1)/2$, so $x^m - px = x(x^{2k} - p)$. The greatest common divisor of x and $x^{2k} - p$ is 1 or p. Since $x^m - px$ is a square, we have either (i) $x = \pm a^2$ and $x^{2k} - p = \pm b^2$ or (ii) $x = \pm pa^2$ and $x^{2k} - p = \pm pb^2$, for some integers a and b.

Note that squares are congruent to 0, 1, or 4 modulo 8, and recall that $p \equiv 3 \bmod 8$. In case (i), if a is odd, then $x^{2k} - p \equiv 6 \pmod 8$. If a is even, then $x^{2k} - p \equiv 5 \pmod 8$. In both subcases, this value cannot be a square or its negative, so we move on to case (ii). Substituting for x and simplifying, we have $p^{2k-1}a^{4k} - 1 = \pm b^2$. The left side is positive. However, again because -1 cannot be a square modulo p, the alternative $p^{2k-1}a^{4k} - 1 = b^2$ is also impossible.

Editorial comment. All solvers had roughly similar approaches. We generalize some of their families. Using the fact that -2 is a quadratic nonresidue for primes p congruent to 5 or 7 modulo 8, one can show that the family $A = p^r$ satisfies the condition of the problem for such primes p and even r. Another family is given by $A = p^r$, where p is a prime congruent to 7 modulo 16 and r is odd. This can be proved by the method of descent.

Also solved by J. Boswell & C. Curtis, W. J. Cowieson, K. Gatesman, P. W. Lindstrom, R. Stong, R. Tauraso (Italy), H. von Eitzen (Germany), and the proposer.

A Functional Equation With Piecewise Linear Solutions

12347 [2022, 786]. *Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bîrlad, Romania.* Let a and b be real numbers with $0 < a < 1 < b$. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We show that there are exactly four solutions, given by

$$f(x) = ax, \quad f(x) = bx, \quad f(x) = \begin{cases} ax, & \text{if } x \geq 0, \\ bx, & \text{if } x < 0, \end{cases} \quad \text{and} \quad f(x) = \begin{cases} bx, & \text{if } x \geq 0, \\ ax, & \text{if } x < 0. \end{cases}$$

Clearly these four functions are solutions. Now let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$. For all $x \in \mathbb{R}$,

$$x = \frac{(a+b)f(x) - f(f(x))}{ab}.$$

This implies that $x = y$ if $f(x) = f(y)$, so f is one-to-one. Since f is continuous, it follows that f is monotonic, and consequently $f \circ f$ is increasing. Moreover, the equality

$$f(x) = \frac{f(f(x)) + abx}{a+b} \tag{1}$$

shows that f is increasing. Since $f(0) = 0$, the sign of $f(x)$ is the same as the sign of x. By (1), we have $f(x) > abx/(a+b)$ for all $x > 0$ and $f(x) < abx/(a+b)$ for all $x < 0$. This implies that $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Hence f is bijective.

Let $g = f^{-1}$. Applying the functional equation to $g(g(x))$ leads to

$$g(g(x)) - \left(\frac{1}{a} + \frac{1}{b}\right) g(x) + \frac{1}{ab}x = 0.$$

Thus g satisfies the same functional equation as f, but with a and b replaced by $1/a$ and $1/b$.

Suppose $x > 0$. We define two sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ by $x_0 = x$, $y_0 = f(x)$, and $x_{n+1} = f(x_n)$ and $y_{n+1} = g(y_n)$ when $n \geq 0$. By the functional equations of f and g, $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ satisfy the following second-order linear recurrence relations:

$$x_0 = x, \qquad x_1 = f(x), \qquad x_{n+2} - (a+b)x_{n+1} + abx_n = 0,$$

$$y_0 = f(x), \qquad y_1 = x, \qquad y_{n+2} - \left(\frac{1}{a} + \frac{1}{b}\right) y_{n+1} + \frac{1}{ab} y_n = 0.$$

Solving these recurrence relations, we find that for all $n \geq 0$,

$$x_n = \frac{f(x) - bx}{a - b} a^n + \frac{f(x) - ax}{b - a} b^n, \tag{2}$$

$$y_n = \frac{f(x) - bx}{a - b} a^{1-n} + \frac{f(x) - ax}{b - a} b^{1-n}. \tag{3}$$

We now consider two cases. If $f(x) \leq x$, then because f is increasing, we have $x_n \geq x_{n+1} > 0$ for all n. Thus the sequence $(x_n)_{n\geq 0}$ is nonincreasing and bounded below, so it must be convergent. Since $b > 1$, the coefficient of b^n in (2) must be zero, which implies that $f(x) = ax$.

On the other hand, if $f(x) > x$, then similar reasoning shows that the sequence $(y_n)_{n\geq 0}$ converges, the coefficient of a^{1-n} in (3) is zero, and $f(x) = bx$.

Thus for all $x > 0$, either $f(x) = ax$ or $f(x) = bx$, so $f(x)/x$ can take only the two values a and b on $(0, \infty)$. However, since f is continuous, it cannot take both values. We conclude that either $f(x) = ax$ for all $x > 0$ or $f(x) = bx$ for all $x > 0$.

Applying the above analysis for $x > 0$ to the function $-f(-x)$, we conclude that either $f(x) = ax$ for all $x < 0$ or $f(x) = bx$ for all $x < 0$. Thus there are no solutions other than the four listed earlier.

Also solved by J. Boswell & C. Curtis, H. Chen (China), W. J. Cowieson, H. von Eitzen (Germany), D. Henderson, N. Hodges (UK), O. P. Lossers (Netherlands), R. Mortini (Luxembourg), K. Schilling, R. Stong, R. Tauraso (Italy), and the proposer.

A Variation on the Josephus Problem

12348 [2022, 786]. *Proposed by Erik Vigren, Uppsala, Sweden, and Hans Rullgård, Kungälv, Sweden.* We have n people in a circle, numbered from 1 to n clockwise. They are removed one at a time as follows, until just one remains. At each step, remove the nth person among those remaining, where the count starts at the lowest-numbered person remaining and proceeds clockwise. Let $W(n)$ be the number of the last person remaining. For example, with $n = 5$, we remove in order the people numbered 5, 1, 3, and 2, and so $W(5) = 4$. (This is a variation of the classic Josephus problem.)
(a) What is $W(10^{12})$?
(b) For $n \geq 5$, show that $W(n) = n - 4$ if and only if $n/2$ is a Sophie Germain prime (i.e., $n/2$ and $n + 1$ are prime).
(c) Find the smallest even number that does not equal $W(n)$ for any n.

Composite solution by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy, and the proposers.
(a) By reversing the procedure, we show $W\left(10^{12}\right) = 671{,}046{,}354{,}072$. As in the problem statement, the *number* of a person is that person's original index and remains unchanged. The *position* of a person at a given time is that person's index among the remaining people; it counts the remaining people with smaller numbers (plus 1).

Consider the point in the process when m people remain. In the next step, skipping $n - 1$ people means passing through the entire list r times before stopping at the person to be removed, where $r = \lfloor (n-1)/m \rfloor$. The person removed will be in position $n - rm$. We say that removals whose associated value of r are the same occur in the same *round*, and we label this round with the value r. For example, in round 0 we remove person n, and in round 1 we remove all the remaining odd-numbered people, starting with person 1. The rounds occur in increasing order, but the round numbers are not consecutive. For example, when $n = 9$ there is no round 3, because $\lfloor 8/3 \rfloor = 2$ and $\lfloor 8/2 \rfloor = 4$. Rather than reversing the procedure one removal at a time, the computation is quicker if we reverse it one round at a time. This will also be useful in part **(c)**.

Now consider the time when a round has just been finished and k rounds remain to be completed. Let m_k denote the number of people remaining at this time, and let p_k denote the position at this time of the person P who will be the last person remaining. Thus $m_0 = 1$ and $p_0 = 1$, since P is never removed. For $k \geq 1$, let r_k denote the number of the round about to start. By definition, $r_k = \lfloor (n-1)/m_k \rfloor$.

The last removal in round r_{k+1} occurs with $m_k + 1$ people remaining, so

$$r_{k+1} = \lfloor (n-1)/(m_k+1) \rfloor. \tag{1}$$

When $r_{k+1} > 0$, the number of people remaining at the start of round r_{k+1} is the largest m such that $r_{k+1} = \lfloor (n-1)/m \rfloor$; that is,

$$m_{k+1} = \lfloor (n-1)/r_{k+1} \rfloor. \tag{2}$$

During round r_{k+1}, when m people remain, the person in position $n - r_{k+1}m$ will be removed. This position strictly increases throughout round r_{k+1} as m decreases from m_{k+1} to $m_k + 1$. Meanwhile, the position of P decreases from p_{k+1} to p_k. Since P reaches p_k, the position of P must decrease on the step that starts with m people remaining if and only if

$$n - r_{k+1}m \leq p_k. \tag{3}$$

By (2), we have $(n-1)/r_{k+1} < m_{k+1} + 1$, which yields $n - r_{k+1}(m_{k+1}+1) < 1$. Also, the definition of r_k implies $(n-1)/m_k \geq r_k \geq r_{k+1} + 1$, from which we obtain $n - r_{k+1}m_k \geq m_k + 1$. Together, these inequalities yield

$$n - r_{k+1}(m_{k+1}+1) < 1 \leq p_k < m_k + 1 \leq n - r_{k+1}m_k.$$

It follows that there is some integer j with $0 \leq j \leq m_{k+1} - m_k$ such that

$$n - r_{k+1}(m_{k+1} - (j-1)) \leq p_k < n - r_{k+1}(m_{k+1} - j).$$

By (3), there will then be exactly j steps during round r_{k+1} on which the position of P decreases by 1. Therefore

$$p_{k+1} = p_k + j = p_k + \left\lfloor \frac{p_k + r_{k+1}\,(m_{k+1}+1) - n}{r_{k+1}} \right\rfloor. \tag{4}$$

We now have a recursive procedure, starting from $m_0 = p_0 = 1$. Given m_k and p_k, we use m_k to compute r_{k+1} by (1), r_{k+1} to compute m_{k+1} by (2), and then all of $\{p_k, r_{k+1}, m_{k+1}\}$ to compute p_{k+1} by (4). We run the recursion until reaching k such that m_k equals $n-1$. The original position (and number) of P is then p_k. In the particular instance $n = 10^{12}$, we obtain $k = 1999997$, leading to $W(n)$ as claimed.

(b) Assume $n \geq 5$. Because all people with odd numbers will have been removed by the end of round 1, $W(n)$ is an even number less than n. In particular, $n-4$ is removed by then if n is odd, so we need only consider even n. When n is even, the person with the larger number will be removed when only two people remain. Therefore $W(n) = n-4$ if and only if the last two people are numbered $n-4$ and $n-2$.

Suppose that m people remain, where $m \leq n/2 - 1$. Recall that n is removed first and then all odd numbers. If both $n-4$ and $n-2$ remain, then they occupy positions $m-1$ and m. To avoid removing either, n must not be congruent to $m-1$ or m modulo m. That is, we avoid removing person $n-2$ if and only if n is not divisible by any number from 3 to $n/2 - 1$, meaning that $n/2$ is prime. Similarly, we avoid removing person $n-4$ if and only if $n-1$ is not divisible by any number from 3 to $n/2 - 1$, meaning that $n+1$ is prime.

(c) We show that the smallest even number that does not equal $W(n)$ for any n is 34. The table below gives the smallest value of n yielding each value of $W(n)$ less than 34, by explicit computation.

$W(n)$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
n	3	5	7	16	11	13	50	17	19	76	23	56	248	29	31	424

We need only consider $n > 34$ and show that in all cases person 34 is removed at some point in the process. We have observed that person n is removed in round 0, and all smaller odd numbers are removed in round 1. Person 34 is then in position 17.

Since round r is defined as $\{m\colon \lfloor (n-1)/m \rfloor = r\}$, the number of people remaining when round r ends is $\min\{m\colon \lfloor (n-1)/m \rfloor = r\} - 1$. This number is $\lfloor (n-1)/(r+1) \rfloor$. Let a_{r+1} be the integer such that

$$\lfloor (n-1)/(r+1) \rfloor = (n - a_{r+1})/(r+1).$$

The first person removed in round $r+1$ is in position a_{r+1} at the start of the round. For each subsequent removal in round $r+1$, the removed element pushes the round-starting position of the next person removed up by $r+2$. That is, the key additional observation is that positions at the start of round $r+1$ of the people removed in round $r+1$ are

$$a_{r+1}, \quad a_{r+1} + r + 2, \quad a_{r+1} + 2r + 4, \dots.$$

For even n, those removed in round 2 start the round in positions $2, 5, 8, 11, 14, 17, \dots$. Hence we may assume n is odd.

For odd n, those removed in round 2 start the round in positions $1, 4, 7, 10, 13, 16, \dots$. Thus after round 2, person 34 is in position 11.

When $n \equiv 3 \pmod 6$, those removed in round 3 start the round in positions 3, 7, 11, 15, $\dots$, so we may forbid this case.

When $n \in \{1, 5, 7, 11\} \pmod{12}$, getting $(n - a_3)/3$ to be an integer requires $a_3 \in \{1, 2\}$. Those removed in round 3 start the round in positions $1, 5, 9, 13, \dots$, or positions $2, 6, 10, 14, \dots$. In both cases, person 34 ends round 3 in position 8.

When $n \in \{7, 11\} \pmod{12}$, we have $a_4 = 3$, and those starting round 4 in positions 3, 8, $\dots$ are removed. Hence we may forbid this case.

When $n \in \{1, 5\} \pmod{12}$, we have $a_4 = 1$, and those starting round 4 in positions 1, 6, $\dots$ are removed. Hence person 34 occupies position 6 at the end of round 4. Since $a_5 \in \{1, 2, 3, 4, 5\}$, round 5 removes exactly one person from the first five positions, so person 34 ends round 5 in position 5.

When $n \equiv 5 \pmod{12}$, we have $a_6 = 5$, so round 6 removes person 34.

Hence we may assume $n \equiv 1 \pmod{12}$. If also $n \geq 73$, then at least 12 people remain at the end of round 5. When the number of people remaining is in $\{12, 6, 4, 3, 2\}$, the person occupying the first position at that time will be removed. This means that person 34, who is already as early as position 5 when at least 12 people remain, is removed while a person still remains.

To complete the proof, it remains only to check explicitly that $W(n) \neq 34$ when $n \in \{37, 49, 61\}$.

Editorial comment. Reasoning like that for part **(b)** shows that $W(n) = n - 1$ if and only if n is an odd prime. Round r actually eliminates one or more people if $(n-1)/(r+1) < \lfloor (n-1)/r \rfloor$. This holds for all r with $r \leq r^*$, where $r^* = \lfloor (\sqrt{4n-3} - 1)/2 \rfloor$. Thereafter, at most one person is removed per round. As a result, the number of rounds in which people are removed is $r^* + \lfloor (n-1)/(r^*+1) \rfloor$.

Also solved by O. P. Lossers (Netherlands). Parts **(b)** and **(c)** also solved by K. Schilling and Eagle Problem Solvers.

A Lobachevsky-type Formula

12351 [2022, 886]. *Proposed by Seán Stewart, King Abdullah University of Science and*

Technology, Thuwal, Saudi Arabia. Evaluate

$$\int_0^\infty \frac{\ln\left(\cos^2 x\right)\sin^3 x}{x^3\left(1+2\cos^2 x\right)}\,dx.$$

Solution by Mohammed Aassila, Strasbourg, France. Let I denote the requested integral. We prove that

$$I = -\frac{\pi}{4}\left(\ln 2 + \frac{\ln(1+\sqrt{3})}{\sqrt{3}}\right).$$

We have

$$\begin{aligned}
I &= \frac{1}{2}\int_{-\infty}^\infty \frac{\ln\left(\cos^2 x\right)\sin^3 x}{x^3\left(1+2\cos^2 x\right)}\,dx = \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_{k\pi}^{(k+1)\pi} \frac{\ln\left(\cos^2 x\right)\sin^3 x}{x^3\left(1+2\cos^2 x\right)}\,dx \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty}\int_0^\pi \frac{(-1)^k\ln\left(\cos^2 x\right)\sin^3 x}{(x+k\pi)^3\left(1+2\cos^2 x\right)}\,dx \\
&= \frac{1}{2}\int_0^\pi\left(\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(x+k\pi)^3}\right)\frac{\ln\left(\cos^2 x\right)\sin^3 x}{1+2\cos^2 x}\,dx,
\end{aligned}$$

where the final interchange of integration and summation can be justified by the dominated convergence theorem.

To evaluate the summation in the last formula, we start with the equation

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{x+k\pi} = \frac{1}{\sin x}.$$

(See I. S. Gradshteyn, I. M. Ryzhik (2007), *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, equation 1.422.6.) Differentiating twice, we get

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{(x+k\pi)^3} = \frac{1+\cos^2 x}{2\sin^3 x},$$

so this gives

$$\begin{aligned}
I &= \frac{1}{4}\int_0^\pi \frac{(1+\cos^2 x)\ln\left(\cos^2 x\right)}{1+2\cos^2 x}\,dx = \int_0^{\pi/2}\frac{(1+\cos^2 x)\ln\left(\cos x\right)}{1+2\cos^2 x}\,dx \\
&= \frac{1}{2}\int_0^{\pi/2}\ln(\cos x)\,dx + \frac{1}{2}\int_0^{\pi/2}\frac{\ln\left(\cos x\right)}{1+2\cos^2 x}\,dx.
\end{aligned}$$

Both of these integrals are special cases of equation 4.385.3 in Gradshteyn and Ryzhik:

$$\int_0^{\pi/2}\frac{\ln(\cos x)}{b^2\sin^2 x + a^2\cos^2 x}\,dx = \frac{\pi}{2ab}\ln\left(\frac{b}{a+b}\right)$$

for $a, b > 0$. Applying this with $b = 1$ and both $a = 1$ and $a = \sqrt{3}$ leads to the claimed answer.

Editorial comment. As several solvers noted, the beginning of this argument proves a Lobachevsky-type result: For any continuous function $f(x)$ that is periodic with period π,

$$\int_{-\infty}^\infty \frac{\sin^3 x}{x^3} f(x)\,dx = \frac{1}{2}\int_0^\pi (1+\cos^2 x) f(x)\,dx.$$

Also solved by T. Amdeberhan, A. Berkane (Algeria), P. Bracken, B. Bradie, C. Burnette, H. Chen (US), B. E. Davis, M. L. Glasser, G. C. Greubel, N. Hodges (UK), W. Janous (Austria), L. Kempeneers & J. V. Casteren (Belgium), O. Kouba (Syria), K. Nelson, M. Omarjee (France), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), Y. Zhang (China), and the proposer.

CLASSICS

C25. *L*et $w_0, w_1, \ldots$ be the sequence of *Fibonacci words*, defined by $w_0 = 0$, $w_1 = 1$, and, for $n \geq 2$, $w_n = w_{n-2}w_{n-1}$, the concatenation of w_{n-2} and w_{n-1}. Thus the sequence begins 0, 1, 01, 101, 01101, 10101101, 0110110101101, ... Show that, for $n \geq 3$, removing the first two symbols from w_n yields a palindrome.

The Tennis Ladder

C24. *Due to Colin L. Mallows.* Over the history of a certain tennis club, every player has played at least one match against every other player. Matches are played one at a time, and after each match a ranking of the players in the club is computed as follows. Starting with the most recent match and working backwards through time, use the match results to build up a partial order. Ignore any match that is inconsistent with more recent results. The final result is guaranteed to be a linear order, since any incomparability between a pair of players is resolved when a match between them is encountered. This linear order becomes the new club ranking. Prove or disprove: A player cannot rise in the club ranking by intentionally losing a match.

Solution. The assertion is false. Suppose that the results of the last nine matches among six players are as follows, where we write $a > b$ for a match where player a defeats player b and we list the matches from oldest to most recent.

$$2>3,\ \ 6>1,\ \ 2>4,\ \ 1>2,\ \ 6>4,\ \ 4>5,\ \ 3>4,\ \ 3>6,\ \ 5>6$$

The ranking at this moment is $1 > 2 > 3 > 4 > 5 > 6$, with player 3 in third place. However, if player 3 loses the next match to player 5, the ranking becomes $5 > 3 > 6 > 1 > 2 > 4$, with player 3 in second place. So player 3 ranks higher after losing.

Editorial comment. The problem appeared as E3240 [1987, 996; 1989, 530] in this MONTHLY. The problem statement has two interpretations. The strong form asks if a player can rank higher immediately after throwing a match. The weak form asks if a player can rank higher today by deciding to forfeit a match that took place in the past. No solution to the strong form of the problem was received from the MONTHLY readership other than the proposer's solution, which involved seven players. The example here involves six players. This raises the question of whether there is an example with five players.

One can show that any time a player defeats a lower-ranked opponent (or loses to a higher-ranked opponent), the ranking remains unchanged. However, reversing the outcome of each match in the example above shows that defeating a higher-ranked opponent can lower one's overall ranking.

Say that a ranking algorithm *respects duality* if changing all wins to losses reverses the resulting ranking. A familiar algorithm for ranking tennis club members is as follows: If a lower-ranked player A defeats a higher-ranked player B, the new ranking is formed by replacing B with A in the prior ranking and moving B and all the players ranked between A and B down one spot. If a higher-ranked player defeats a lower-ranked player, the ranking remains unchanged. One concern with this usual algorithm is that it fails to respect duality. The algorithm of this problem is an alternative that does respect duality. The existence of the example above, however, shows that this ranking system violates a certain kind of monotonicity and suggests that it is an unreasonable system for actual use.

$-x^{-\alpha}$

SOLUTIONS

A Sufficient Condition for Generalized Commuting

12331 [2022, 588]. *Proposed by WeChat Group on Matrix Analysis, Nova Southeastern University, Fort Lauderdale, FL.* Let A and B be complex m-by-n matrices, and let C be a complex n-by-m matrix. Prove that if there are nonzero scalars x and y such that $ACB = xA + yB$, then $ACB = BCA$.

Solution by Li Zhou, Polk State College, Winter Haven, FL. Since $CACB = xCA + yCB$, we have $(CA - yI_n)(CB - xI_n) = xyI_n$. Thus $CB - xI_n$ has an inverse P that satisfies $CA - yI_n = xyP$. Hence

$$B(CA - yI_n) = xyBP = x(ACB - xA)P = xA(CB - xI_n)P = xA,$$

so $BCA = xA + yB = ACB$.

Also solved by C. P. Anil Kumar (India), M. Bataille (France), M. R. Elgersma, K. Gatesman, E. A. Herman, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), P. Oman & H. Wang, M. Reid, K. Sarma (India), K. Schilling, R. Stong, R. Tauraso (Italy), Southeast Missouri State University Math Club, UM6P Math Club (Morocco), and the proposer.

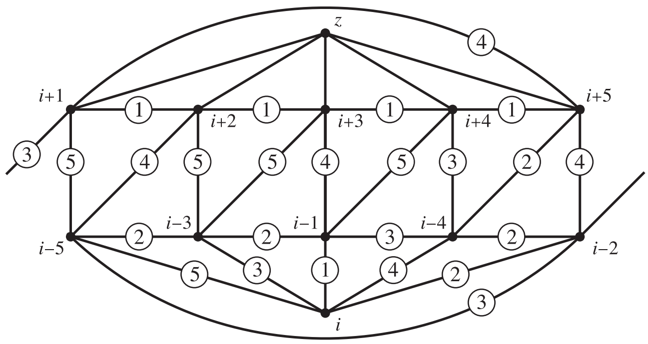
A Symmetric Decomposition into Icosahedra

12333 [2022, 588]. *Proposed by Moshe Rosenfeld, University of Washington, Seattle, WA, and Tacoma Institute of Technology, Tacoma, WA.* Let G be the multigraph obtained by replacing each edge of the complete graph K_{12} by five edges. Show that the 330 edges of G can be partitioned into 11 sets such that each set forms a graph isomorphic to the icosahedron.

Solution by Eagle Problem Solvers, Georgia Southern University, Statesboro and Savannah, GA. We assign the label z to one vertex of K_{12}, and label the other 11 vertices with the elements of $\mathbb{Z}_{11}$. Notice that each edge of K_{12} can be written uniquely as either (z, v), for some $v \in \mathbb{Z}_{11}$, or $(v, v + j)$, for some $v \in \mathbb{Z}_{11}$ and $j \in \{1, 2, 3, 4, 5\}$. In the latter case, we refer to j as the *difference* of the edge.

The icosahedron H is a 5-regular graph with 12 vertices and 30 edges. For each $i \in \mathbb{Z}_{11}$, we define a copy H_i of H in K_{12} and then show that these copies together use each edge of K_{12} five times. These copies of H can then be used to define the required partition of the edges of G.

We draw H_i with z at the top and vertex i as the vertex at distance 3 from z. The five vertices adjacent to z in H_i are $i+1$, $i+2$, $i+3$, $i+4$, and $i+5$, and the five vertices adjacent to i are $i-1, i-2, i-3, i-4$, and $i-5$. There is a 10-cycle alternating between the vertices adjacent to z and those adjacent to i. (The "wraparound" edge from $i-2$ to $i+1$ has been left broken to emphasize the symmetry.) In the drawing of H_i below, we have labeled each edge not incident to z with its difference. For example, the edge from $i+1$ to $i-5$ has difference 5 because $(i+1)+5 = i+6 = i-5$ in $\mathbb{Z}_{11}$.



It remains to show that each edge of K_{12} occurs in H_i for exactly five values of i in $\mathbb{Z}_{11}$. An edge of the form (z, v) for some $v \in \mathbb{Z}_{11}$ occurs in H_i if and only if

$$v \in \{i+1, i+2, i+3, i+4, i+5\},$$

in other words, if and only if

$$i \in \{v-1, v-2, v-3, v-4, v-5\}.$$

Now consider an edge $(v, v+j)$ with difference j. If $j = 1$, then the edge occurs in H_i if and only if $v \in \{i+1, i+2, i+3, i+4, i-1\}$. These values correspond to the five edges in the figure that are labeled 1, and they determine five values of i for which the edge occurs in H_i. Similarly, there are five edges in the figure labeled with each of the other differences 2, 3, 4, and 5, and therefore for each of these values of j, an edge of the form $(v, v+j)$ occurs in H_i for five values of i.

Editorial comment. Several solvers described a decomposition using a rotation modulo 11. Rob Pratt obtained a decomposition via integer linear programming.

The proposer used the fact that the graph H' whose edges are the pairs of vertices separated by distance 2 in a copy H of the icosahedron is isomorphic to the icosahedron. Since each vertex has exactly one antipodal vertex at distance 3, the union of H and H' omits exactly a perfect matching in K_{12}. Said another way, the complement of a perfect matching in K_{12} decomposes into two copies of the icosahedron. The icosahedron is 5-edge-colorable, meaning that it decomposes into five perfect matchings. This can be seen by drawing h with a central vertex and 5-fold rotational symmetry and forming a perfect matching using one edge from each of the six orbits of five edges. For each of these five matchings, the remaining edges of K_{12} decompose into two copies of the icosahedron. The resulting ten copies of the icosahedron together cover each edge outside H five times and cover each edge in H four times. Together with H itself, we obtain 11 copies of H covering each edge in K_{12} five times.

The decomposition argument generalizes in a straightforward way to yield the following result: If H is an n-vertex k-regular graph that decomposes into k perfect matchings, and K_n decomposes into t copies of H plus a leftover perfect matching, then the multigraph kK_n with k copies of each vertex pair as edges decomposes into $kt+1$ copies of H. Other applications besides the problem here include decomposing $4K_6$ into five octahedra and decomposing $3K_8$ into seven cubes, which were proved earlier in the literature.

Also solved by K. Gatesman, O. P. Lossers (Netherlands), R. Pratt, A. J. Schwenk, R. Stong, and the proposer.

A Recursively Defined Sequence

12334 [2022, 588]. *Proposed by Florin Stanescu, Şerban Cioculescu School, Găeşti, Romania.* Let f be a real-valued function on $[0,1]$ with a continuous second derivative. Assume that $f(0)=0$, $f'(0)=1$, $f''(0)\neq 0$, and $0<f'(x)<1$ for all $x\in(0,1]$. Let $x_1, x_2, \dots$ be a sequence with $0<x_1\leq 1$ and with

$$x_n = f\left(\frac{x_1+x_2+\cdots+x_{n-1}}{n-1}\right)$$

for $n\geq 2$. Prove $\lim_{n\to\infty} x_n \ln n = -2/f''(0)$.

Solution by Jinhai Yan, Fudan University, Shanghai, China. Let

$$s_n = \frac{x_1+x_2+\cdots+x_n}{n}.$$

Since $0<f'(x)<1$ for all $x\in(0,1]$, we find that f is increasing, $0<f(x)<x$ on $(0,1]$, and $s_n\in(0,1]$. Moreover,

$$ns_n-(n-1)s_{n-1}=x_n=f(s_{n-1})<s_{n-1}.$$

It follows that $0<s_n<s_{n-1}$. Hence s_n is monotone decreasing, so it converges. Let $\lim_{n\to\infty} s_n = A$. By the Stolz–Cesàro theorem and the continuity of f,

$$f(A)=\lim_{n\to\infty} f(s_n)=\lim_{n\to\infty} x_{n+1}=\lim_{n\to\infty}\frac{(n+1)s_{n+1}-ns_n}{n+1-n}=\lim_{n\to\infty} s_n = A,$$

which implies that $A=0$. By assumption,

$$f(x)=x+\frac{f''(0)}{2}x^2+o(x^2) \qquad (x\to 0^+).$$

Hence

$$s_{n+1}=\frac{ns_n+f(s_n)}{n+1}=\frac{ns_n+s_n+o(s_n)}{n+1}\sim s_n, \qquad \text{and}$$

$$x_n=f(s_{n-1})=s_{n-1}+o(s_{n-1})\sim s_n.$$

Notice that $1/s_n\to\infty$ monotonically. Applying the Stolz–Cesàro theorem again, together with $\ln(1+1/n)\sim 1/n\sim 1/(n+1)$, we find

$$\begin{aligned}\lim_{n\to\infty} x_n\ln n &= \lim_{n\to\infty}\frac{x_n}{s_n}\cdot\frac{\ln n}{1/s_n}=\lim_{n\to\infty}\frac{\ln(n+1)-\ln n}{1/s_{n+1}-1/s_n}\\ &=\lim_{n\to\infty}\frac{\ln(1+1/n)s_{n+1}s_n}{s_n-s_{n+1}}=\lim_{n\to\infty}\frac{s_n^2/(n+1)}{s_n-(ns_n+f(s_n))/(n+1)}\\ &=\lim_{n\to\infty}\frac{s_n^2}{s_n-f(s_n)}=\lim_{n\to\infty}\frac{s_n^2}{-f''(0)s_n^2/2+o(s_n^2)}=-\frac{2}{f''(0)},\end{aligned}$$

as claimed.

Editorial comment. This problem is a generalization of problem 12079 [2018, 944; 2020, 568] from this MONTHLY.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), H. Chen (US), C. Chiser (Romania), H. von Eitzen (Germany), K. Gatesman, M. Goldenberg & M. Kaplan, E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), N. D. Phuoc (Vietnam), K. Sarma (India), K. Schilling, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), and the proposer.

Gaussian Integers

12335 [2022, 685]. *Proposed by Tom Karzes, Sunnyvale, CA, Stephen Lucas, James Madison University, Harrisonburg, VA, and James Propp, University of Massachusetts, Lowell, MA.* A *Gaussian integer* is a complex number z such that $z = a + bi$ for integers a and b. Show that every Gaussian integer can be written in at most one way as a sum of distinct powers of $1 + i$, and that the Gaussian integer z can be expressed as such a sum if and only if $i - z$ cannot.

Solution by William J. Cowieson, Fullerton College, Fullerton, CA. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and nonnegative integers, respectively. Suppose first that $\sum_{s\in S}(1+i)^s = \sum_{t\in T}(1+i)^t$ for distinct sets $S, T \subset \mathbb{N}_0$. Subtract one side from the other, divide by the lowest remaining power of $1 + i$, and isolate 1 to obtain $1 = (1 + i)w$, where

$$w = \sum_{s\in S'}(1+i)^{s-1} - \sum_{t\in T'}(1+i)^{t-1}$$

for some disjoint sets $S', T' \subset \mathbb{N}$. Letting $N(z) = z\overline{z} = a^2 + b^2$ when $z = a + bi$, we conclude

$$1 = N(1) = N\big((1+i)w\big) = N(1+i)N(w) = 2N(w),$$

which is impossible. Thus equality requires $S = T$, so there is at most one way to write any Gaussian integer as a sum of distinct powers of $1 + i$.

Let $\mathbb{G}$ denote the set $\mathbb{Z}[i]$ of Gaussian integers. For $z = a + ib \in \mathbb{G}$, we have

$$\frac{z}{1+i} = \frac{1}{2}(1-i)z = \frac{a+b}{2} + i\frac{b-a}{2}.$$

Thus $z/(1 + i)$ is also in $\mathbb{G}$ if and only if $a + b$ is even, which holds if and only if $a^2 + b^2$ is even. On the other hand, if $a + b$ is odd, then $(a - 1) + b$ is even, so $(z - 1)/(1 + i) \in \mathbb{G}$. Writing these conditions in terms of N, we have a mapping $F\colon\ \mathbb{G} \to \mathbb{G}$ defined by

$$F(z) = \begin{cases} z/(1+i) & \text{if} \quad N(z) \text{ is even} \\ (z-1)/(1+i) & \text{if} \quad N(z) \text{ is odd} \end{cases}.$$

We now establish various properties of the mapping F.

Claim 1: *For all $z \in \mathbb{G}$ and $n \in \mathbb{N}_0$, we have $F^n(i - z) = i - F^n(z)$.*

Proof. If $N(z)$ is even, then $N(i - z)$ is odd, and

$$F(i-z) = (i-z-1)/(1+i) = i - z/(1+i) = i - F(z).$$

The computation when $N(z)$ is odd is similar, yielding $F(i - z) = i - F(z)$ for all $z \in \mathbb{G}$. This is the base case for a proof by induction on n. For the induction step, assuming the result for $n = k - 1$, we obtain

$$F^k(i-z) = F(F^{k-1}(i-z)) = F(i - F^{k-1}(z)) = i - F(F^{k-1}(z)) = i - F^k(z),$$

which is the result for $n = k$. ■

Claim 2: *The Gaussian integer z is a sum of distinct powers of $1+i$ if and only if $F(z)$ is also.*

Proof. If $F(z)$ is such a sum, then so are $(1+i)F(z)$ and $(1+i)F(z)+1$, one of which is z. Conversely, if z is such a sum, then either all powers are positive and $F(z)=z/(1+i)$ is such a sum, or 1 is a summand and $F(z)=(z-1)/(1+i)$ is such a sum. ■

Claim 3: *For all $z \in \mathbb{G}$, there exists $n \in \mathbb{N}_0$ such that either $F^n(z)=0$ or $F^n(z)=i$.*

Proof. If $N(z)$ is even, then $N(F(z)) = N\big(z/(1+i)\big) = N(z)/2$. If $N(z)$ is odd, then

$$N(F(z)) = N\big((z-1)/(1+i)\big) = N(z-1)/2 = \big((a-1)^2+b^2\big)/2,$$

which is at least a^2+b^2 if and only if $(a+1)^2+b^2 \le 2$. Thus $N(F(z)) < N(z)$ for all $z \in \mathbb{G}$ except $z \in \{0, i, -i, -1, -2+i, -2-i\}$, so for every $z \in \mathbb{G}$ there exists $m \in \mathbb{N}$ with $F^m(z)$ equal to a member of this set. Furthermore, $F^3(-i) = F^2(-1) = i = F(i)$ and $F^6(-2-i) = F^5(-2+i) = 0 = F(0)$, so always $F^n(z) \in \{0, i\}$ for some $n \in \mathbb{N}$. ■

Finally, observe that 0 is (vacuously) a sum of distinct powers of $1+i$, while i is not such a sum: If $i = \sum_{s\in S}(1+i)^s$ for some $S \subset \mathbb{N}_0$, then

$$\sum_{s\in S}(1+i)^s = i = 1+(1+i)i = 1+(1+i)\sum_{s\in S}(1+i)^s = \sum_{s\in(S+1)\cup\{0\}}(1+i)^s.$$

By uniqueness, $|S| = |(S+1)\cup\{0\}| = |S|+1$, which is impossible for finite S. Since no such infinite sum converges, i is not such a sum.

It follows from this and Claims 1–3 that z is a sum of distinct powers of $1+i$ if and only if $F^n(z)=0$ for some n. This is further equivalent to $F^n(i-z) = i - F^n(z) = i$ for some n, which holds if and only if $i-z$ is not a sum of distinct powers of $1+i$.

Editorial comment. Gagola, Ionaşcu, Meyerson, Tauraso, Wildon, and the Eagle Problem Solvers all mentioned the fractal nature of the Gaussian integers shaded in one of two colors depending on whether the Gaussian integer can or cannot be expressed as a sum of distinct powers of $1+i$, and they attached graphics showing this property. See W. J. Gilbert (1982), Fractal geometry derived from complex bases, *Math. Intelligencer* 4(2): 78–86.

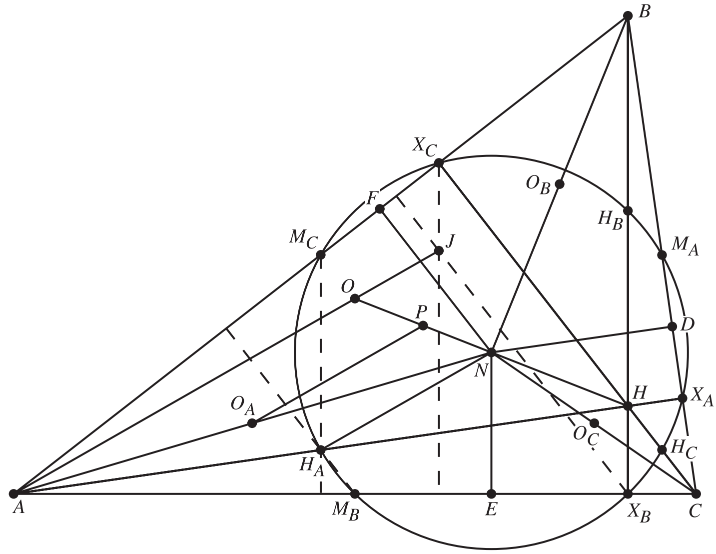
Also solved by J. Boswell & C. Curtis, T. Eisenkölbl (Austria), H. von Eitzen (Germany), S. M. Gagola Jr., K. Gatesman, F. Gesmundo (Germany) & T. M. Mazzoli (Austria), N. Hodges (UK), E. J. Ionaşcu, Y. J. Ionin, S. Lee, O. P. Lossers (Netherlands), M. D. Meyerson, K. Schilling, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), E. I. Verriest, M. Wildon (UK), Eagle Problem Solvers, Fejéntaláltuka Szeged Problem Solving Group (Hungary), and the proposers.

Four Concurrent Euler Lines

12336 [2022, 685]. *Proposed by Szilárd András, Babeş-Bolyai University, Cluj-Napoca, Romania.* Let N be the center of the nine-point circle of triangle ABC, and let D, E, and F be the orthogonal projections of N onto the sides BC, CA, and AB, respectively. Prove that the Euler lines of triangles ABC, AEF, BFD, and CDE are concurrent. Prove also that the point of concurrency is equidistant from the circumcenters of AEF, BFD, and CDE.

Solution by Kyle Gatesman, MITRE Corporation, Fairfax, VA. Let H be the orthocenter of triangle ABC, and let O be its circumcenter. Let the midpoints of the sides opposite A, B, and C be M_A, M_B, and M_C, respectively. Let the feet of the altitudes from A, B, and C (the *orthic points*) be X_A, X_B, and X_C, respectively. Let the midpoints of AH, BH, and CH (the *halfway points*) be H_A, H_B, and H_C, respectively. The nine-point circle passes through all three midpoints, all three orthic points, and all three halfway points. It is the image of the circumcircle of triangle ABC under a dilation centered at H with scaling factor 1/2, so its center N is the midpoint of the segment OH.

Let O_A, O_B, and O_C be the circumcenters of triangles AEF, BFD, and CDE, respectively. By definition of the points D, E, and F, quadrilaterals $AENF$, $BFND$, and $CDNE$ are cyclic, and their circumcircles have diameters AN, BN, and CN, respectively. Therefore O_A, O_B, and O_C are the midpoints of these diameters. It follows that the circumcircle of triangle $O_AO_BO_C$ is the image of the circumcircle of triangle ABC under a dilation centered at N with scaling factor 1/2, and its center P is the midpoint of NO. The point P is equidistant from O_A, O_B, and O_C, and it lies on the line HO, which is the Euler line of triangle ABC, so it suffices now to show that the Euler lines of triangles AEF, BFD, and CDE are concurrent at P. We show that the Euler line of triangle AEF passes through P; the claims for the other two triangles follow by symmetry of the construction.



Note that H_A is the orthocenter of triangle AM_BM_C, because $AM_BH_AM_C$ is the image of $ACHB$ under a dilation centered at A with scaling factor 1/2. Let J be the orthocenter of triangle AX_BX_C. Letting $\alpha = \angle BAC$, we have $AX_C = AC\cos\alpha$ and $AX_B = AB\cos\alpha$. Therefore triangle AX_BX_C is the image of triangle ABC under first a dilation centered at A with scale factor $\cos\alpha$ and then a reflection across the line m that bisects $\angle BAC$. It follows that J lies on the reflection of AH across m. Since

$$\angle OAB = \frac{\pi}{2} - \frac{1}{2}\angle AOB = \frac{\pi}{2} - \angle ACB = \frac{\pi}{2} - \angle ACX_A = \angle X_AAC = \angle HAC,$$

O also lies on the reflection of AH across m. Thus A, O, and J are collinear.

For any pair of parallel lines ℓ_1 and ℓ_2, we say that the line that is parallel to both ℓ_1 and ℓ_2 and halfway between them is their *midline*. This midline contains the midpoint of every line segment with an endpoint on each of ℓ_1 and ℓ_2. The nine-point circle of triangle ABC passes through M_B, X_B, M_C, and X_C, so E and F are the midpoints of M_BX_B and M_CX_C, respectively. The midline of the altitudes from M_B and X_B to AB is the altitude from E to AB, and the midline of the altitudes from M_C and X_C to AC is the altitude from F to AC. Since H_A lies on the altitudes from both M_B and M_C, and J lies on the altitudes from both X_B and X_C, the midpoint of the segment H_AJ lies on the altitudes from both E and F, and therefore it is the orthocenter of triangle AEF.

Since H_A is the midpoint of HA and N is the midpoint of HO, the line AO is parallel to H_AN. Since O_A is the midpoint of NA and P is the midpoint of NO, the line O_AP is parallel to AO, and it is the midline of AO and H_AN. The orthocenter of triangle AEF is the midpoint of the segment H_AJ, whose endpoints lie on the lines H_AN and AO, so this orthocenter lies on O_AP. This shows that O_AP is the Euler line of triangle AEF, and it passes through P, as required.

Editorial comment. N. S. Dasireddy pointed out that this problem was posed as part of the 2015 IMO preparation program in Vietnam. Several solutions can be found at artofproblemsolving.com/community/c6h1087710.

Also solved by M. Bataille (France), S. Bhadra (India), N. S. Dasireddy (India), G. Fera (Italy), O. Geupel (Germany), N. Hodges (UK), W. Janous (Austria), K.-W. Lau (China), G.-H. Liu (Taiwan), O. P. Lossers (Netherlands), F. Masroor, C. R. Pranesachar (India), C. Schindler (Germany), R. Stong, B. D. Suceavă, and the proposer.

Beta Integrals and Partial Fractions

12337 [2022, 685]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $k \in \{0, 1, 2\}$, let

$$S_k = \sum \frac{(-4)^n}{2n+1}\binom{2n}{n}^{-1},$$

where the sum is taken over all nonnegative integers n that are congruent to k modulo 3. Prove

(a) $S_0 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} + \dfrac{\pi}{6}$;

(b) $S_1 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} - \dfrac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} - \dfrac{\pi}{12}$; and

(c) $S_2 = \dfrac{\ln\left(1+\sqrt{2}\right)}{3\sqrt{2}} + \dfrac{\ln\left(2+\sqrt{3}\right)}{2\sqrt{3}} - \dfrac{\pi}{12}$.

Solution by Roberto Tauraso, Tor Vergata University of Rome, Rome, Italy. The presence of $\binom{2n}{n}^{-1}$ suggests expressing the sums using beta function integrals. We have

$$\begin{aligned}\frac{(-4)^n}{2n+1}\binom{2n}{n}^{-1} &= (-4)^n B(n+1, n+1) = (-4)^n \int_0^1 t^n(1-t)^n\,dt \\ &= \frac{1}{2}\int_{-1}^1 (s^2-1)^n\,ds = \int_0^1 (s^2-1)^n\,ds\end{aligned}$$

under the change of variable $s = 2t - 1$. Evaluating a geometric series and using partial fractions yields

$$S_0 = \int_0^1 \sum_{r=0}^{\infty} (s^2-1)^{3r}\,ds = \int_0^1 \frac{ds}{1-(s^2-1)^3} = \frac{1}{3}\int_0^1 \left(\frac{1}{2-s^2} + \frac{s^2+1}{s^4-s^2+1}\right) ds. \qquad (*)$$

To study this and similar expressions, let

$$I_0 = \int_0^1 \frac{ds}{2-s^2}, \qquad I_1 = \int_0^1 \frac{s^2-1}{s^4-s^2+1}\,ds, \qquad \text{and} \qquad I_2 = \int_0^1 \frac{s^2+1}{s^4-s^2+1}\,ds.$$

Using partial fractions and

$$s^4 - s^2 + 1 = (s^2 + \sqrt{3}s + 1)(s^2 - \sqrt{3}s + 1),$$

we obtain

$$I_0 = \frac{1}{2\sqrt{2}} \int_0^1 \left(\frac{1}{\sqrt{2}+s} + \frac{1}{\sqrt{2}-s} \right) ds = \frac{\ln(1+\sqrt{2})}{\sqrt{2}},$$

$$I_1 = -\frac{1}{2\sqrt{3}} \int_0^1 \left(\frac{2s+\sqrt{3}}{s^2+\sqrt{3}s+1} + \frac{2s-\sqrt{3}}{s^2-\sqrt{3}s+1} \right) ds = -\frac{\ln(2+\sqrt{3})}{\sqrt{3}},$$

and

$$\begin{aligned} I_2 &= \frac{1}{2} \int_0^1 \left(\frac{1}{s^2+\sqrt{3}s+1} + \frac{1}{s^2-\sqrt{3}s+1} \right) ds \\ &= \Big[\arctan(2s+\sqrt{3}) + \arctan(2s-\sqrt{3}) \Big]_0^1 = \frac{\pi}{2}. \end{aligned}$$

From $(*)$, we have $S_0 = (I_0 + I_2)/3$, which completes **(a)**. For the other sums, we compute

$$\begin{aligned} S_1 - S_2 &= \int_0^1 \sum_{k=0}^{\infty} \left((s^2-1)^{2k+1} - (s^2-1)^{3k+2} \right) ds \\ &= \int_0^1 \frac{(s^2-1)(2-s^2)}{1-(s^2-1)^3}\, ds = \int_0^1 \frac{s^2-1}{s^4-s^2+1}\, ds = I_1 \end{aligned}$$

and

$$S_0 + S_1 + S_2 = \int_0^1 \sum_{n=0}^{\infty} (s^2-1)^n \, ds = I_0.$$

This yields $S_1 = (2I_0 + 3I_1 - I_2)/6$ for **(b)** and $S_2 = (2I_0 - 3I_1 - I_2)/6$ for **(c)**.

Editorial comment. Michel Bataille based his solution on the general formula

$$\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}} = \frac{2x \arcsin(x)}{\sqrt{1-x^2}},$$

proved in D. H. Lehmer (1985), Interesting series involving the central binomial coefficient, this MONTHLY 92(7): 449–457.

Also solved by T. Amdeberhan & V. H. Moll, M. Bataille (France), A. Berkane (Algeria), P. Bracken, B. Bradie, H. Chen (US), W. J. Cowieson, K. Gatesman, M. L. Glasser, N. Hodges (UK), W. Janous (Austria), O. Kouba (Syria), C. Krattenthaler (Austria), P. Lalonde (Canada), G. Lavau (France), O. P. Lossers (Netherlands), R. Molinari, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. Terr, M. Vowe (Switzerland), T. Wiandt, M. Wildon (UK), Y. Zhang (China), and the proposer.

A Trigonometric Exponential Integral by the Leibniz Integral Rule

12338 [2022, 686]. *Proposed by István Mező, Nanjing, China.* Prove

$$\int_0^{\infty} \frac{\cos(x)-1}{x\ (e^x-1)}\, dx = \frac{1}{2} \ln\big(\pi \operatorname{csch}(\pi)\big).$$

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA. We start by writing the integral in the form

$$\int_0^{\infty} \frac{\cos x - 1}{x(e^x-1)}\, dx = \int_0^{\infty} \frac{\cos x - 1}{xe^x(1-e^{-x})}\, dx = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{(\cos x - 1)e^{-(n+1)x}}{x}\, dx.$$

For $a > 0$, let

$$I(a) = \int_0^\infty \frac{(\cos x - 1)e^{-ax}}{x}\, dx.$$

Differentiation under the integral sign (an application of what is known as the *Leibniz integral rule*) yields

$$I'(a) = \int_0^\infty (1 - \cos x)e^{-ax}\, dx = e^{-ax}\left(\frac{a\cos x - \sin x}{1+a^2} - \frac{1}{a}\right)\Bigg|_0^\infty = \frac{1}{a(1+a^2)},$$

and therefore

$$I(a) = \int \frac{da}{a(1+a^2)} = \frac{1}{2}\ln\left(\frac{a^2}{1+a^2}\right) + C.$$

Since $\lim_{a\to\infty} I(a) = 0$, we have $C = 0$, so the requested integral is given by

$$\int_0^\infty \frac{\cos x - 1}{x(e^x - 1)}\, dx = \frac{1}{2}\sum_{n=0}^{\infty} \ln\left(\frac{(n+1)^2}{1+(n+1)^2}\right) = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty} \frac{k^2}{1+k^2}\right).$$

To evaluate the infinite product, we use the known formula

$$\frac{\sinh z}{z} = \prod_{k=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 k^2}\right)$$

(see I. S. Gradshteyn, I. M. Ryzhik (2007), *Table of Integrals, Series, and Products, 7th ed.*, Burlington, MA: Academic Press, equation 1.431.2, p. 45). Applying this formula with $z = \pi$, we obtain

$$\int_0^\infty \frac{\cos x - 1}{x(e^x - 1)}\, dx = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty}\left(1 + \frac{1}{k^2}\right)^{-1}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh \pi}\right).$$

Also solved by U. Abel & V. Kushnirevych (Germany), A. Berkane (Algeria), S. Bhadra (India), R. Bittencourt (Brazil), K. N. Boyadzhiev, P. Bracken, B. Bradie, C. Burnette, H. Chen (US), W. J. Cowieson, B. E. Davis, M.-C. Fan (China), G. Fera (Italy), P. Fülöp (Hungary), M. L. Glasser, H. Grandmontagne (France), N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), S. Kaczkowski, O. Kouba (Syria), K.-W. Lau (China), G. Lavau (France), O. P. Lossers (Netherlands), J. Magliano, M. Maniquiz, F. Masroor, R. Mortini (Luxembourg) & R. Rupp (Germany), K. Nelson, M. Omarjee (France), D. Pascuas (Spain), P. Perfetti (Italy), S. Sharma (India), A. Stadler (Switzerland), S. M. Stewart (Saudi Arabia), R. Stong, R. Tauraso (Italy), E. I. Verriest, M. Vowe (Switzerland), T. Wiandt, Y. Zhang (China), L. Zhou, and the proposer.

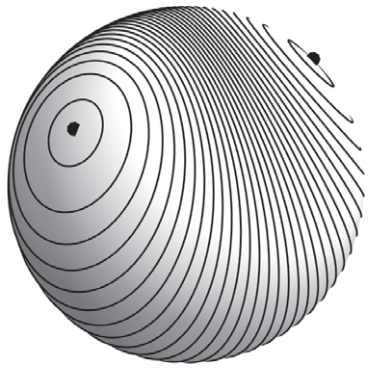
CLASSICS

C24. *Due to Colin L. Mallows.* Over the history of a certain tennis club, every player has played at least one match against every other player. Matches are played one at a time, and after each match a ranking of the players in the club is computed as follows. Starting with the most recent match and working backwards through time, use the match results to build up a partial order. Ignore any match that is inconsistent with more recent results. The final result is guaranteed to be a linear order, since any incomparability between a pair of players is resolved when a match between them is encountered. This linear order becomes the new club ranking. Prove or disprove: A player cannot rise in the club ranking by intentionally losing a match.

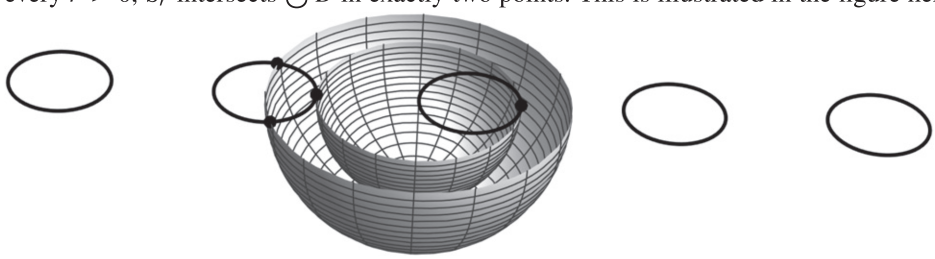
Decomposing Space into Disjoint Circles

C23. *Due to John H. Conway and Hallard T. Croft.* Determine whether it is possible to partition $\mathbb{R}^3$ into circles.

Solution. We first show that any sphere with two points deleted can be partitioned into circles. This is clear when the two deleted points are antipodal, witnessed by the circular latitude lines that cover the surface of the earth apart from the two poles. When the two deleted points are not antipodal, let L be the line in space common to the two planes that are tangent to the sphere at the deleted points. The cross-sections of the sphere with the planes through L decompose the sphere with two deleted points into circles. This is illustrated at right.



Let B be the family of unit circles in the xy-plane centered at $(4k + 1, 0, 0)$ for some integer k. These circles are disjoint, and one of them contains the origin. For $r > 0$, let S_r be the sphere of radius r centered at the origin. The key observation is that, for every $r > 0$, S_r intersects $\bigcup B$ in exactly two points. This is illustrated in the figure here:



Let T_r be a set of disjoint circles whose union is S_r with those two points deleted. The union of B and T_r for all $r > 0$ gives the desired decomposition of $\mathbb{R}^3$.

Editorial comment. The problem first appeared in J. H. Conway and H. T. Croft (1964), Covering a sphere with congruent great-circle arcs, in *Math. Proc. Cambridge Phil. Soc.*, 60: 787–800, where it was solved using the axiom of choice. While that solution gives the stronger result that all the circles can be of unit radius and no two circles are linked, it does not give an explicit construction. The solution here is due to Andrzej Szulkin (1983), $\mathbb{R}^3$ is the union of disjoint circles, this MONTHLY, 90: 640–641. For a more detailed treatment, see J. B. Wilker (1989), Tiling $\mathbb{R}^3$ with circles and disks, *Geom. Dedicata* 32: 203–209.

It is not possible to partition $\mathbb{R}^2$ into circles. In fact, if S is a family of disjoint circles in the plane, then in the interior of every circle in S is a point not contained in any circle in S. To see this, let C be such a circle. If the center of C is not part of any circle in S, then we are done. Otherwise, let C' be a circle in S containing the center of C. Note that the radius of C' is less than half the radius of C. If the center of C' is not part of any circle in S, then we are again done. Otherwise, in the same way, let C'' be a circle in S containing the center of C', and continue in this way to form an infinite family of nested circles in S whose radii converge to 0. The point in the interior of all these circles cannot be part of any circle in S.

SOLUTIONS

Two Tangent Circles

12325 [2022, 487]. *Proposed by Dong Luu, Hanoi University of Education, Hanoi, Vietnam.* Let $ABCD$ be a quadrilateral with a circumscribed circle ω and an inscribed circle γ. Prove that there are two circles α and β with the following property: For any triangle $\triangle MEF$ with (1) M on ω, (2) E and F on the line AB, and (3) the lines ME and MF tangent to γ, the circumcircle of $\triangle MEF$ is tangent to α and β.

Solution by Faraz Masroor, New York, NY. We first address a more general situation. Let ω and γ be circles, with γ inside ω, and let ℓ be a line tangent to γ. We show that there are circles α and β as in the statement of the problem, with ℓ playing the role of the line AB, except that one of the circles may degenerate to a point. At the end, we show that if ω and γ are the circumscribed and inscribed circles of a quadrilateral, then the degeneracy can be ruled out.

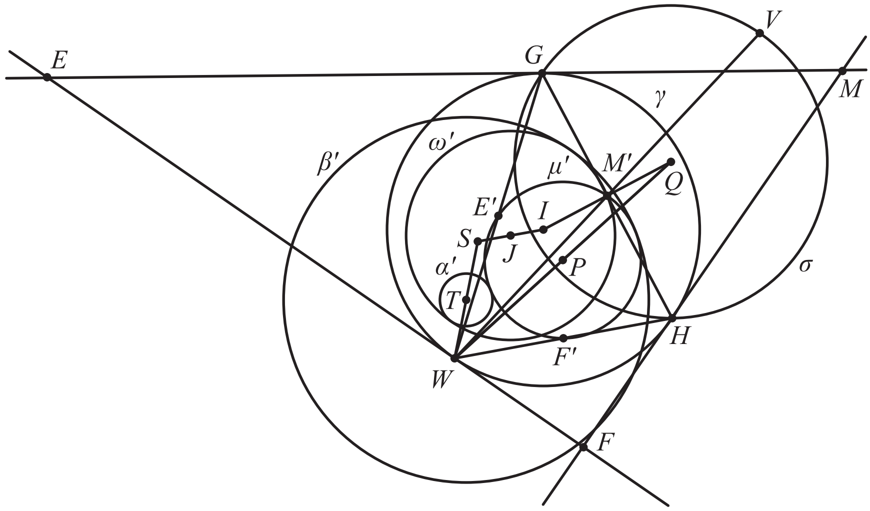
Let W be the point where ℓ is tangent to γ. Let I be the center of γ, and let R be its radius. Let ω' be the image of ω under inversion in γ. The circle ω' is inside γ and contains I in its interior. Let J be the center of ω', and let r be its radius. Let S be the image of I under reflection in J, and let T be the midpoint of WS.

Consider any triangle $\triangle MEF$ as in the statement of the problem, and let μ be its circumcircle. Let M', E', and F' be the images of M, E, and F under inversion in γ, and let μ' be the image of μ. The circle μ' circumscribes $\triangle M'E'F'$; let P be its center. We now make two claims about μ' and P:

Claim 1: The radius of μ' is $R/2$.

Claim 2: $TP = r$.

Before proving these claims, we show that they imply the desired conclusion. Let α' be a circle centered at T with radius $|R/2 - r|$, and let β' be a circle centered at T with radius $R/2 + r$. (It is possible that $r = R/2$, in which case α degenerates to a point.) The claims above imply that the circle μ' is tangent to both α' and β'. The point of tangency with α' is the point on μ' that is closest to T, and the point of tangency with β' is the point on μ' furthest from T. It follows that, as long as neither α' nor β' passes through I, the images of α' and β' under inversion in γ are circles α and β that are tangent to μ, as required.



To confirm that neither α' nor β' passes through I, note first that since J is the midpoint of IS and T is the midpoint of WS, we have $JT = (1/2)IW = R/2$. Also, since I is inside ω', which is inside γ, we must have $IJ < r$ and $IJ < R - r$. It follows that $TI > R/2 - r$ and $TI > R/2 - (R - r) = r - R/2$, so $TI > |R/2 - r|$, and therefore I lies outside of α'. Also, $TI \le JT + IJ < R/2 + r$, so I lies inside of β'. Thus neither α' nor β' passes through I.

To prove the claims, let ME and MF be tangent to γ at G and H, respectively. We use the fact that if the tangent lines to γ at two points X and Y intersect at Z, then the image of Z under inversion in γ is the midpoint of XY. Applying this fact three times, we see that E' is the midpoint of WG, F' is the midpoint of WH, and M' is the midpoint of GH. Let V and Q, respectively, be the images of W and I under reflection in M', and let σ be the image of γ under this reflection. The circle σ has radius R passing through G, H, and V, and its center is Q. Also, M' is the midpoint of both IQ and WV.

Since the midpoints of WV, WG, and WH are M', E', and F', respectively, the image of σ under a dilation with ratio 1/2 centered at W is μ', the circumcircle of $\triangle M'E'F'$. This proves the Claim 1. Also, the image of Q under this dilation is P, so P is the midpoint of WQ. Since T and P are the midpoints of WS and WQ, respectively, and J and M' are the midpoints of IS and IQ, respectively, we have $TP = (1/2)SQ = JM'$. But M is on ω, so M' is on ω', which is the circle of radius r centered at J. Therefore $TP = JM' = r$, which proves Claim 2.

Finally, we show that if ω and γ are the circumscribed and inscribed circles of a quadrilateral $ABCD$, then the case $r = R/2$ can be ruled out, thus eliminating the possibility that one of the circles is degenerate. In this case, by Poncelet's porism, there is another quadrilateral $A'B'C'D'$ such that ω and γ are the circumscribed and inscribed circles of $A'B'C'D'$ and, in addition, $A'C'$ is a diameter of ω and contains a diameter of γ. A straightforward calculation now shows that $r = R(\cos\theta + \sin\theta)/2 > R/2$, where $\theta = \angle A'C'B'$.

Editorial comment. Since ω' is a circle centered at J with radius r and $JT = R/2$, the circles α' and β' are both tangent to ω'. It follows that α and β are both tangent to ω. Also, the line through the centers of α and β passes through I.

Also solved by O. Kouba (Syria), O. P. Lossers (Netherlands), C. R. Pranesachar (India), R. Stong, and the proposer.

A Function with Polynomial Differences is a Polynomial

12326 [2022, 487]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every fixed $y \in \mathbb{R}$, $f(x+y) - f(x)$ is a polynomial in x. Prove that f is a polynomial function.

Solution by Jinhai Yan, Fudan University, Shanghai, China. We first note that, for every nonzero real number c, all polynomials in x can be expressed in the form $p(x+c) - p(x)$ for some other polynomial p. To see this, note that $(x+c)^n - x^n$ has degree $n-1$, and hence $\{(x+c)^n - x^n : n \geq 1\}$ forms a basis for the vector space of all polynomials in x.

Taking $y = 1$ in the hypothesis, we see that $f(x+1) - f(x)$ is a polynomial, so we can find a polynomial p such that $p(x+1) - p(x) = f(x+1) - f(x)$. Therefore $f(x+1) - p(x+1) = f(x) - p(x)$. If we let $T(x) = f(x) - p(x)$, then T is periodic with period 1.

Now let n be any positive integer. It is easy to check that T satisfies the same hypothesis as f, and taking $y = 1/n$ in that hypothesis we conclude that $T(x + 1/n) - T(x)$ is a polynomial. Therefore we can find a polynomial q such that $q(x + 1/n) - q(x) = T(x + 1/n) - T(x)$. It follows that

$$q(x+1) - q(x) = T(x+1) - T(x) = 0,$$

so q is periodic. But q is a polynomial, and therefore it is a constant function. Thus

$$T(x + 1/n) - T(x) = q(x + 1/n) - q(x) = 0,$$

so T is periodic with period $1/n$.

Since n was arbitrary, T has period $1/n$ for every positive integer n, so it is constant on the rationals. It is also continuous, so it must be a constant function. Finally, since $f(x) = p(x) + T(x)$, we conclude that f is a polynomial.

Editorial comment. The problem appears as Lemma 2.5 in F. Kühn and R. L. Schilling (2021), For which functions are $f(X_t) - \mathbb{E}f(X_t)$ and $g(X_t)/\mathbb{E}f(X_t)$ martingales?, *Theor. Prob. and Math. Statist.* 105: 79–91. The problem was submitted to the MONTHLY without mention of this reference. We regret publishing the problem without proper attribution to the source.

Omran Kouba pointed out that it is sufficient to assume that $f(x+y) - f(x)$ is a polynomial for two values of y that are independent over the rationals.

The hypothesis that f is continuous is necessary. We can prove this by imitating the reasoning used in the solution to classic problem C7 [2022, 694; 2022, 794]. Let B be a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$, and let $f : B \to \mathbb{R}$ be any function that takes the value 0 infinitely many times but is not identically 0. Now extend f to a function from $\mathbb{R}$ to $\mathbb{R}$ as follows: for any real number x, write x (uniquely) as a finite sum $\sum_{i=1}^{n} q_i b_i$, where $q_i \in \mathbb{Q} \setminus \{0\}$ and $b_i \in B$, and define $f(x)$ to be $\sum_{i=1}^{n} q_i f(b_i)$. It is easy to verify that for any fixed y, $f(x+y) - f(x)$ is equal to $f(y)$, which is a constant function and therefore a polynomial. Since f takes the value 0 infinitely many times but is not identically 0, it cannot be a polynomial. As explained in the editorial comment of problem C7, the reasoning here requires the axiom of choice, and indeed the result cannot be proved without using the axiom of choice.

Also solved by J. Boswell & C. Curtis, N. Caro-Montoya (Brazil), J.-P. Grivaux (France), D. A. Hejhal, N. Hodges (UK), Y. J. Ionin, O. Kouba (Syria), O. P. Lossers (Netherlands & ELTE (Hungary), F. Masroor, R. Mortini (France) & P. Pflug (Germany) & A. Sasane (UK), M. Omarjee (France), K. Sarma (India), K. Schilling, A. Stenger, R. Stong, R. Tuaraso (Italy), D. J. Velleman, UM6P Math Club (Morocco), and the proposer.

A Gaussian Binomial Identity

12327 [2022, 487]. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.* Let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \prod_{i=0}^{k-1} \dfrac{1-q^{n-i}}{1-q^{k-i}} & \text{if } 1 \le k \le n; \\ 1 & \text{if } k = 0. \end{cases}$$

Prove

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^{k(k-1)+(n-k)^2-n(n-1)/2}$$

for $n \ge 0$.

Solution by Doyle Henderson, Omaha, NE. Let S_n and T_n, respectively, denote the sums on the left and on the right, respectively. We show that both equal $\prod_{k=1}^{n}(1+q^k)$. Using $(1-q^{n+1})\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = (1-q^{n+1-k})\left[\begin{smallmatrix} n+1 \\ k \end{smallmatrix}\right]_q$ with q^2 replacing q, we obtain

$$\begin{aligned}(1-q^{2n+2})S_n &= \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} (1-q^{2n+2-2k})q^k \\ &= \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} q^k - \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} q^{2n+2-k}.\end{aligned}$$

Using the well-known identity $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right]_q$ and reversing the index of summation yields $S_{n+1} = \sum_{k=0}^{n+1} \left[\begin{smallmatrix} n+1 \\ k \end{smallmatrix}\right]_{q^2} q^{n+1-k}$, so

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} q^{2n+2-k} = q^{n+1}(S_{n+1}-1).$$

We have now proved

$$(1-q^{2n+2})S_n = S_{n+1} - q^{n+1} - q^{n+1}(S_{n+1}-1) = (1-q^{n+1})S_{n+1},$$

so $S_{n+1} = (1+q^{n+1})S_n$. This and $S_0 = 1$ yield $S_n = \prod_{k=1}^{n}(1+q^k)$.

To evaluate T_n, let $f(n,k) = k(k-1) + (n-k)^2 - n(n-1)/2$. Proceeding as for S_n yields

$$(1-q^{2n+2})T_n = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} q^{f(n,k)} - \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} q^{f(n,k)} q^{2n+2-2k}.$$

Since $f(n+1, n+1-k) = f(n,k)$, using $\left[\begin{smallmatrix} n \\ n-k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ and reversing the index of summation in T_{n+1} yields $T_{n+1} = \sum_{k=0}^{n+1} \left[\begin{smallmatrix} n+1 \\ k \end{smallmatrix}\right]_{q^2} q^{f(n,k)}$. Computing also that $f(n,k) + n + 1 - 2k = f(n+1,k)$, we obtain

$$\begin{aligned}\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} q^{f(n,k)} q^{2n+2-2k} &= q^{n+1} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} q^{f(n+1,k)} \\ &= q^{n+1}\left(T_{n+1} - q^{f(n+1,n+1)}\right).\end{aligned}$$

Since $f(n, n+1) = n + 1 + f(n+1, n+1)$, we have now proved

$$(1 - q^{2n+2})T_n = T_{n+1} - q^{f(n,n+1)} - q^{n+1}\left(T_{n+1} - q^{f(n+1,n+1)}\right) = (1 - q^{n+1})T_{n+1}.$$

Hence $T_{n+1} = (1 + q^{n+1})T_n$. This and $T_0 = 1$ yield $T_n = \prod_{k=1}^{n}(1 + q^k)$, finishing the proof.

Also solved by T. Amdeberhan & S. B. Ekhad, N. Hodges (UK), W. P. Johnson, P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Stong, R. Tauraso (Italy), and the proposer.

The Value Set of an Integer Quadratic Form

12328 [2022, 587]. *Proposed by Peter Koymans and Jeffrey Lagarias, University of Michigan, Ann Arbor, MI.* An integer binary quadratic form is a function $f : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $f(m, n) = am^2 + bmn + cn^2$ for some $a, b, c \in \mathbb{Z}$. The *value set V(f)* of such a form is defined to be $\left\{f(m, n) : (m, n) \in \mathbb{Z}^2\right\}$.
(a) Prove that if $f_1(m, n) = m^2 - mn - 3n^2$ and $f_2(m, n) = m^2 - 13n^2$, then $V(f_1) = V(f_2)$.
(b) Prove that if $f_1(m, n) = m^2 - mn - 4n^2$ and $f_2(m, n) = m^2 - 17n^2$, then $V(f_2) \subseteq V(f_1)$ but $V(f_1) \neq V(f_2)$.

Solution by Jacob Boswell and Charles Curtis, Missouri Southern State University, Joplin, MO. In both **(a)** and **(b)**, $f_2(m, n) = f_1(m + n, 2n)$, so $V(f_2) \subseteq V(f_1)$.

(a) It suffices to show $V(f_1) \subseteq V(f_2)$. We have

$$f_1(m, n) = f_2\left(\frac{11m - 25n}{2}, \frac{-3m + 7n}{2}\right),$$

so $f_1(m, n) \in V(f_2)$ when m and n have the same parity. When m is even and n is odd, we use $f_1(m, n) = f_1(m - n, -n)$ to see that $f_1(m, n) \in V(f_2)$, since in this case $m - n$ and $-n$ have the same parity. Finally, when m is odd and n is even, we have $f_1(m, n) = f_2(m - n/2, n/2) \in V(f_2)$.

(b) It suffices to show $V(f_1) \not\subseteq V(f_2)$. We have $f_2(m, n) = m^2 - 17n^2 \not\equiv 2 \pmod 4$. However $f_1(3, 1) = 2 \equiv 2 \pmod 4$.

Editorial comment. The identity used in **(a)** arises from the multiplicative property of the norm N in $\mathbb{Z}[\sqrt{13}]$ defined by $N(a + b\sqrt{13}) = f_2(a, b)$. Thus

$$f_1(m, n) = f_2\left(\frac{11}{2}, -\frac{3}{2}\right) f_2\left(m - \frac{n}{2}, \frac{n}{2}\right) = f_2\left(\frac{11m - 25n}{2}, \frac{7n - 3m}{2}\right).$$

Several solvers considered more generally $f_1(m, n) = m^2 - mn - an^2$ and $f_2(m, n) = m^2 - (4a + 1)n^2$, for integers a. The argument given for **(a)** applies when $f_2(m, n) = 4$ has a solution in odd integers. For example, when $a = 7$ we have $27^2 - 29 \cdot 5^2 = 4$, so $f_1(m, n) = f_2\big((27m + 59n)/2, (5m + 11n)/2\big)$.

Also solved by U. Abel & V. Kushnirevych (Germany), A. J. Bevelacqua, P. Corn, T. Eisenkölbl (Austria), G. Fera (Italy), K. Gatesman, Y. J. Ionin, O. P. Lossers (Netherlands), B. Phillabaum, C. R. Pranesachar (India), M. Reid, J. P. Robertson, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), D. Terr, L. Zhou, and the proposer.

Equally Spaced Unit Vectors

12329 [2022, 587]. *Proposed by Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let n be a positive integer with $n \geq 3$. For each positive integer m with $m \geq 2$, find all real values λ_m such that there are m distinct unit vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$ satisfying $v_i \cdot v_j = \lambda_m$ for all i, j with $1 \leq i < j \leq m$.

Solution by Kuldeep Sarma, Tezpur University, Tezpur, India. If $m \le n$, then the allowed values for λ_m are all numbers in the interval $[-1/(m-1), 1)$. If $m = n+1$, then the only allowed value for λ_m is $-1/n$. If $m > n+1$, then no such λ_m exists.

Assume that such vectors exist, and let them be the columns of a real $n \times m$ matrix V. The Gram matrix $V^T V$ has ones on the diagonal and λ_m in every off-diagonal position. Thus $V^T V = (1-\lambda_m)I_m + \lambda_m J_m$, where J_m denotes the m-by-m matrix of all 1s. The vector $(1, \dots, 1)^T$ is an eigenvector for this matrix with eigenvalue $1 + (m-1)\lambda_m$, and there are $m-1$ linearly independent vectors whose coordinates sum to zero, all of which are eigenvectors with eigenvalue $1 - \lambda_m$. Thus the eigenvalues of this matrix are $1 - \lambda_m$ with multiplicity $m-1$ and $1 + (m-1)\lambda_m$ with multiplicity 1. Since $V^T V$ is positive semidefinite, these eigenvalues must be nonnegative, and hence $-1/(m-1) \le \lambda_m \le 1$. However, the case $\lambda_m = 1$ is excluded, since it would force all of the v_i to be equal. Also note that $V^T V$ has rank at most n. Hence if $m = n+1$, then $V^T V$ must have 0 as an eigenvalue, which implies $\lambda_m = -1/n$. If $m > n+1$, then $V^T V$ must have 0 as an eigenvalue with multiplicity at least 2, which is impossible since $\lambda_m = 1$ has been excluded.

Conversely, suppose that the conditions on the eigenvalues λ_m are satisfied. Let $A = (1-\lambda_m)I_m + \lambda_m J_m$. The matrix A is a symmetric positive semidefinite m-by-m matrix with rank at most n. Let $r = \operatorname{rank}(A)$. Using either the Cholesky decomposition or the orthogonal diagonalizability of real symmetric matrices, we can write $A = X^T X$ for some r-by-m matrix X, and padding with $n-r$ extra rows of 0s, we can write $A = V^T V$ for some n-by-m matrix V. The columns of V are then the desired unit vectors v_i. They are distinct since $v_i \cdot v_j = \lambda_m < 1$ for $1 \le i < j \le m$.

Also solved by C. P. Anil Kumar (India), N. Caro-Montoya (Brazil), M. Elgersma, K. Gatesman, Y. J. Ionin, O. P. Lossers (Netherlands), R. Stong, L. Zhou, Eagle Problem Solvers, and the proposer.

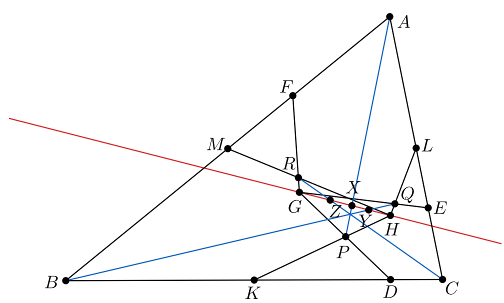
One Concurrency Leads To Another

12330 [2022, 587]. *Proposed by Oleh Faynshteyn, Leipzig, Germany.* In the acute and scalene triangle ABC, let G be the centroid, H be the orthocenter, D, E, and F be the feet of the altitudes from A, B, and C, respectively, and K, L, and M be the midpoints of BC, CA, and AB, respectively. Let P be the intersection of DG and KH, let Q be the intersection of EG and LH, and let R be the intersection of FG and MH.

(a) Prove that AP, BQ, and CR are concurrent.

(b) Let X, Y, and Z be the points where GH intersects AP, BQ, and CR. Prove

$$\frac{HX}{XG} + \frac{HY}{YG} + \frac{HZ}{ZG} = 3.$$

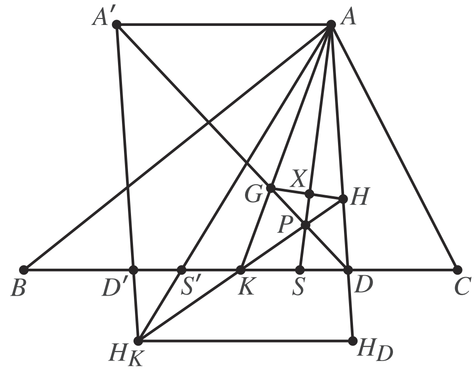


Solution by Faraz Masroor, New York, NY. Neither statement depends on the triangle being acute or H being the orthocenter of ABC. We can let H be any point in the interior of $\triangle ABC$ not lying on any of the lines AG, BG, or CG, as long as we redefine D, E, and F to be the intersections of AH, BH, and CH with BC, CA, and AB, respectively.

(**a**) Let S, T, and U be the intersections of AP, BQ, and CR with BC, CA, and AB, respectively. We must prove that AS, BT, and CU are concurrent. By Ceva's theorem it suffices to show

$$\frac{BS}{SC} \cdot \frac{CT}{TA} \cdot \frac{AU}{UB} = 1. \tag{1}$$

Let H_D and H_K be the reflections of H across D and K, and let A' be the point such that AH_DH_KA' is a parallelogram. Note that $A'A \parallel H_KH_D \parallel KD$, so $\angle A'AG = \angle DKG$. Also, $A'A = H_KH_D = 2KD$, and since G is the centroid of $\triangle ABC$, $AG = 2KG$. Therefore triangles $A'AG$ and DKG are similar. It follows that $\angle A'GA = \angle DGK$, so A', G, P, and D are collinear.



Since $\triangle SDP \sim \triangle AA'P$ and $\triangle DHP \sim \triangle A'H_KP$,

$$\frac{SD}{AA'} = \frac{PD}{PA'} = \frac{DH}{A'H_K} = \frac{DH_D}{AH_D}. \tag{2}$$

Let S' be the intersection of AH_K with BC. Since $\triangle S'D'H_K \sim \triangle AA'H_K$,

$$\frac{S'D'}{AA'} = \frac{D'H_K}{A'H_K} = \frac{DH_D}{AH_D}. \tag{3}$$

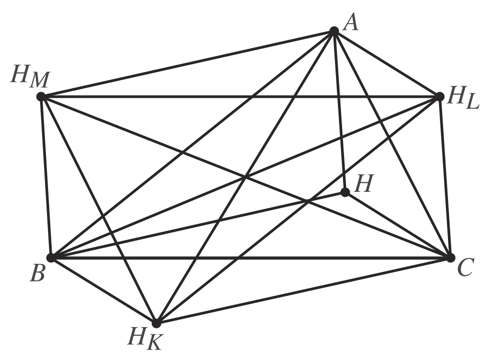
Combining (2) and (3), we see that $SD = S'D'$. Also, $D'D = H_KH_D = 2KD$, so K is the midpoint of DD', and therefore $KS = KS'$. We conclude $BS = CS'$ and $SC = S'B$.

Let H_L and H_M be the reflections of H across L and M, respectively, let T' be the intersection of BH_L with CA, and let U' be the intersection of CH_M with AB. Imitating the reasoning above, we can show $CT = AT'$, $TA = T'C$, $AU = BU'$, and $UB = U'A$. Therefore (1) is equivalent to

$$\frac{CS'}{S'B} \cdot \frac{AT'}{T'C} \cdot \frac{BU'}{U'A} = 1.$$

By another application of Ceva's theorem, to prove this it suffices to show that AH_K, BH_L, and CH_M are concurrent.

Since K is the midpoint of both HH_K and BC, $BHCH_K$ is a parallelogram. Similarly, $CHAH_L$ and $AHBH_M$ are parallelograms. We have $AH_L \parallel HC \parallel BH_K$ and $AH_L = HC = BH_K$, so AH_LH_KB is also a parallelogram. Therefore the midpoints of the diagonals AH_K and BH_L coincide. Similarly, this common midpoint coincides with the midpoint of CH_M, so the three segments are concurrent, as required.



(**b**) The law of sines implies

$$\frac{DS}{SK} = \frac{AD}{AK} \cdot \frac{\sin\angle DAP}{\sin\angle KAP}.$$

This is sometimes known as *the ratio lemma*. A second application of the ratio lemma yields

$$\frac{HX}{XG} = \frac{AH}{AG} \cdot \frac{\sin\angle DAP}{\sin\angle KAP}.$$

Combining these, and applying the fact that $AK = (3/2)AG$, we obtain

$$\frac{HX}{XG} = \frac{AH}{AG} \cdot \frac{AK}{AD} \cdot \frac{DS}{SK} = \frac{3}{2} \cdot \frac{AH}{AD} \cdot \frac{DS}{SK}. \tag{4}$$

By Ceva's theorem applied to $\triangle ADK$,

$$\frac{DS}{SK} \cdot \frac{KG}{GA} \cdot \frac{AH}{HD} = 1,$$

and therefore

$$\frac{DS}{SK} = \frac{GA}{KG} \cdot \frac{HD}{AH} = 2 \cdot \frac{HD}{AH}.$$

Substituting into (4), we obtain

$$\frac{HX}{XG} = \frac{3}{2} \cdot \frac{AH}{AD} \cdot 2 \cdot \frac{HD}{AH} = 3 \cdot \frac{HD}{AD} = 3 \cdot \frac{[HBC]}{[ABC]},$$

where for any points α, β, and γ, $[\alpha\beta\gamma]$ denotes the area of $\triangle\alpha\beta\gamma$. Similarly, $HY/YG = 3[HCA]/[ABC]$ and $HZ/ZG = 3[HAB]/[ABC]$, so

$$\frac{HX}{XG} + \frac{HY}{YG} + \frac{HZ}{ZG} = 3 \cdot \frac{[HBC] + [HCA] + [HAB]}{[ABC]} = 3 \cdot \frac{[ABC]}{[ABC]} = 3.$$

Editorial comment. Let O be the point on GH such that G is between O and H and $GH = 2OG$. It can be shown that O is the common midpoint of AH_K, BH_L, and CH_M.

Also solved by M. Bataille (France), H. Chen (China), C. Chiser (Romania), I. Dimitrić, G. Fera (Italy), M. Goldenberg & M. Kaplan, K. Gatesman, J.-P. Grivaux (France), K.-W. Lau (China), C. R. Pranesachar (India), V. Schindler (Germany), R. Stong, D. E. Türköz (Turkey), L. Zhou, and the proposer.

A Hyperbolic Integral

12332 [2022, 588]. *Proposed by Finbarr Holland, University College, Cork, Ireland.* Prove

$$\int_0^\infty \frac{\tanh^2 x}{x^2}\, dx = \frac{14\,\zeta(3)}{\pi^2},$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^\infty 1/k^3$.

Solution by Kuldeep Sarma, Tezpur University, Tezpur, India. We begin with the Weierstrass product formula for the hyperbolic cosine,

$$\cosh x = \prod_{n=0}^\infty \left(1 + \frac{x^2}{(n+1/2)^2\pi^2}\right).$$

Applying logarithmic differentiation, we obtain

$$\frac{\tanh x}{x} = 2\sum_{n=0}^\infty \frac{1}{(n+1/2)^2\pi^2 + x^2},$$

and therefore

$$\frac{\tanh^2 x}{x^2} = 4\sum_{n,m=0}^\infty \frac{1}{\big((n+1/2)^2\pi^2 + x^2\big)\big((m+1/2)^2\pi^2 + x^2\big)}. \tag{1}$$

Next we claim that for all positive a and b,

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}. \tag{2}$$

For distinct a and b, this follows from the calculation

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{b^2-a^2}\int_0^\infty \left(\frac{1}{a^2+x^2} - \frac{1}{b^2+x^2}\right) dx$$

$$= \frac{1}{b^2-a^2}\left(\frac{\pi}{2a} - \frac{\pi}{2b}\right) = \frac{\pi}{2ab(a+b)},$$

but it is easily verified that (2) also holds when $a = b$.

Combining (1) and (2), we have

$$\int_0^\infty \frac{\tanh^2 x}{x^2}\,dx = 4\sum_{n,m=0}^{\infty}\int_0^\infty \frac{1}{\big((n+1/2)^2\pi^2+x^2\big)\big((m+1/2)^2\pi^2+x^2\big)}\,dx$$

$$= \frac{16}{\pi^2}\sum_{n,m=0}^{\infty}\frac{1}{(2n+1)(2m+1)(2n+2m+2)}$$

$$= \frac{16}{\pi^2}\sum_{n,m=0}^{\infty}\frac{1}{(2n+1)(2m+1)}\int_0^1 x^{(2n+2m+2)}\,\frac{dx}{x}$$

$$= \frac{16}{\pi^2}\int_0^1 \left(\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}\cdot\sum_{m=0}^{\infty}\frac{x^{2m+1}}{2m+1}\right)\frac{dx}{x}$$

$$= \frac{4}{\pi^2}\int_0^1 \ln^2\left(\frac{1-x}{1+x}\right)\frac{dx}{x}.$$

Finally, to evaluate the last integral, we substitute $u = (1-x)/(1+x)$ and obtain

$$\int_0^\infty \frac{\tanh^2 x}{x^2}\,dx = \frac{8}{\pi^2}\int_0^1 \frac{\ln^2 u}{1-u^2}\,du = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\int_0^1 u^{2n}\ln^2 u\,du$$

$$= \frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = \frac{16}{\pi^2}\cdot\frac{7\zeta(3)}{8} = \frac{14\zeta(3)}{\pi^2}.$$

Editorial comment. Several solvers noted a relationship between this problem and problem 12317, which asked for a proof of

$$\int_0^{\pi/2}\frac{\sin(4x)}{\ln(\tan x)}\,dx = -14\frac{\zeta(3)}{\pi^2}.$$

Let I denote the integral in problem 12317, and J the integral in this problem. Using the substitution $u = \tan x$, we have

$$I = \int_0^\infty \frac{4u(1-u^2)}{(1+u^2)^3\ln u}\,du.$$

We can reexpress J by applying integration by parts, recognizing that the resulting integrand is odd, and expressing the hyperbolic functions in terms of exponentials:

$$J = 2\int_0^\infty \frac{\sinh x}{x\cosh^3 x}\,dx = \int_{-\infty}^\infty \frac{4e^{2x}(e^{2x}-1)}{x(e^{2x}+1)^3}\,dx.$$

Finally, using the substitution $u = e^x$, we obtain

$$J = \int_0^\infty \frac{4u(u^2-1)}{(1+u^2)^3\ln u}\,du = -I.$$

Also solved by U. Abel & V. Kushnirevych (Germany), T. Amdeberhan & V. Moll, K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), P. Bracken, B. Bradie, H. Chen, B. E. Davis, M.-C. Fan (China), G. Fera (Italy), D. Fleischman, K. Gatesman, M. L. Glasser, H. Grandmontagne (France), G. C. Greubel, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), S. Kaczkowski, L. Kempeneers & J. Van Casteren (Belgium), O. Kouba (Syria), O. P. Lossers (Netherlands), C. Maniquiz, K. Nelson, M. Omarjee (France), P. Perfetti (Italy), S. Sharma (India), A. Stadler (Switzerland), A. Stenger, S. M. Stewart (Saudi Arabia), R. Stong, R. Tauraso (Italy), T. Wiandt, H. Widmer (Germany), M. Wildon (UK), Y. Zhang (China), Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

CLASSICS

C23. *Due to John H. Conway and Hallard T. Croft.* Determine whether it is possible to partition $\mathbb{R}^3$ into circles.

The Erdős–Mordell Inequality

C22. *Due to Paul Erdős.* Prove that from any point in any triangle, the sum of the distances to the vertices of the triangle is at least twice as large as the sum of the distances to the sides of the triangle.

Solution. Let P be a point in $\triangle ABC$, let $a = BC$, $b = CA$, $c = AB$, $x = PA$, $y = PB$, and $z = PC$, and let d, e, and f be the distances from P to the sides BC, CA, and AB, respectively. The area of $\triangle ABC$ is given by $(ad + be + cf)/2$. Since $f + z$ is no less than the altitude of $\triangle ABC$ dropped to base AB, the area of $\triangle ABC$ is no greater than $c(f + z)/2$. It follows that $c(f + z) \geq ad + be + cf$, or $cz \geq ad + be$.

We apply this inequality not to the original point P, but to the point in $\triangle ABC$ that is the reflection of P across the angle bisector from C. We obtain the same inequality but with the roles of d and e reversed: $cz \geq ae + bd$.

The analogous inequalities $ax \geq bf + ce$ and $by \geq cd + af$ are obtained in the same way. We obtain

$$x + y + z \geq \frac{b}{a}f + \frac{c}{a}e + \frac{c}{b}d + \frac{a}{b}f + \frac{a}{c}e + \frac{b}{c}d = \left(\frac{c}{b} + \frac{b}{c}\right)d + \left(\frac{c}{a} + \frac{a}{c}\right)e + \left(\frac{b}{a} + \frac{a}{b}\right)f.$$

By the AM-GM inequality, the three quantities in parentheses are all at least 2, and the desired inequality $x + y + z \geq 2(d + e + f)$ follows.

Editorial Comment: Equality holds if and only if $\triangle ABC$ is equilateral and P is its center. The problem appeared as problem 3740 [1935, 396; 1937, 252] in this MONTHLY, proposed by Paul Erdős and solved by Louis Mordell and independently by David F. Barrow. It has come to be known as the Erdős–Mordell inequality. Barrow's solution proved the stronger claim that the inequality holds even if d, e, and f are the distances from P to the points where the angle bisectors meet the sides of the triangle.

Numerous alternative proofs and generalizations have appeared over the decades. For example, a proof that is more elementary than those of Mordell and Barrow appears in V. Komornik (1997), A short proof of the Erdős–Mordell theorem, this MONTHLY, 104(1): 57–60. Our proof here follows roughly that of C. Alsina and R. Nelsen (2007), A visual proof of the Erdős–Mordell inequality, *Forum Geom.* 7: 99–102. Peter Walker (2016), The Erdős–Mordell theorem in the exterior domain, *Internat. J. Geom.* 5(1): 31–38 examines the extent to which the result generalizes to points outside a triangle.

SOLUTIONS

A Logarithmic Trigonometric Integral

12317 [2022, 385]. *Proposed by Seán Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.* Prove

$$\int_0^{\pi/2} \frac{\sin(4x)}{\log(\tan x)}dx = -14\frac{\zeta(3)}{\pi^2},$$

where $\zeta(3)$ is Apéry's constant $\sum_{n=1}^{\infty} 1/n^3$.

Composite solution by Hongwei Chen, Christopher Newport University, Newport News, VA, and Thomas Dickens, Houston, TX. Let I denote the requested integral. Using the identity $\sin(4x) = 2\sin(2x)\cos(2x) = 4\sin x\cos x(\cos^2 x - \sin^2 x)$ and the substitutions $u = \tan x$ and $t = u^2$, we obtain

$$I = \int_0^{\pi/2} \frac{\sin(4x)}{\log(\tan x)}\,dx = \int_0^\infty \frac{4u(1-u^2)}{(\log u)(1+u^2)^3}\,du$$
$$= -4\int_0^\infty \frac{t-1}{(\log t)(1+t)^3}\,dt.$$

We now use parametric integration to compute this integral. First observe that for $t > 0$,

$$\int_0^1 t^p\,dp = \frac{t-1}{\log t}.$$

Thus

$$I = -4\int_0^\infty \int_0^1 \frac{t^p}{(1+t)^3}\,dp\,dt = -4\int_0^1 \left(\int_0^\infty \frac{t^p}{(1+t)^3}\,dt\right) dp.$$

We evaluate the inner integral using the substitution $s = 1/(1+t)$ and then recognizing the beta function. Using the identities $B(z_1, z_2) = \Gamma(z_1)\Gamma(z_2)/\Gamma(z_1+z_2)$, $\Gamma(z+1) = z\Gamma(z)$, and $\Gamma(z)\Gamma(1-z) = \pi/\sin(z\pi)$ yields

$$\int_0^\infty \frac{t^p}{(1+t)^3}\,dt = \int_0^1 s^{1-p}(1-s)^p\,ds = B(2-p, p+1)$$
$$= \frac{\Gamma(2-p)\Gamma(p+1)}{\Gamma(3)} = \frac{p(1-p)\Gamma(1-p)\Gamma(p)}{2} = \frac{p(1-p)\pi}{2\sin(p\pi)},$$

for $p \in (0, 1)$. Thus

$$I = -2\pi\int_0^1 \frac{p(1-p)}{\sin(p\pi)}\,dp = -\frac{2}{\pi^2}\int_0^\pi \frac{x(\pi-x)}{\sin x}\,dx, \qquad (*)$$

where we have used the substitution $x = p\pi$ in the second step. Applying the Fourier sine series of $x(\pi - x)$ on $[0, \pi]$ yields

$$x(\pi - x) = \frac{8}{\pi}\sum_{n=0}^\infty \frac{1}{(2n+1)^3}\sin\big((2n+1)x\big),$$

and so

$$I = -\frac{16}{\pi^3}\sum_{n=0}^\infty \frac{1}{(2n+1)^3}\int_0^\pi \frac{\sin\big((2n+1)x\big)}{\sin x}\,dx.$$

We now use the formula

$$\int_0^\pi \frac{\sin(nx)}{\sin x}\,dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \pi, & \text{if } n \text{ is odd,} \end{cases}$$

for $n \in \mathbb{N}$, which follows from the fact that, for $n \geq 2$,

$$\int_0^\pi \frac{\sin(nx)}{\sin x}\,dx - \int_0^\pi \frac{\sin\big((n-2)x\big)}{\sin x}\,dx = \int_0^\pi \frac{\sin(nx) - \sin\big((n-2)x\big)}{\sin x}\,dx$$
$$= 2\int_0^\pi \cos\big((n-1)x\big)\,dx = 0.$$

This yields

$$I = -\frac{16}{\pi^3}\sum_{n=0}^{\infty}\frac{\pi}{(2n+1)^3} = -\frac{16}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{\infty}\frac{1}{(2n)^3}\right) = -\frac{16}{\pi^2}\cdot\frac{7}{8}\zeta(3) = -14\frac{\zeta(3)}{\pi^2}.$$

Editorial comment. Most solvers converted the proposed integral into $(*)$ and then used either the power series of $\csc x$ or a contour integral.

Also solved by A. Berkane (Algeria), N. Bhandari (Nepal), P. Bracken, B. Bradie, B. S. Burdick, B. E. Davis, M.-C. Fan (China), G. Fera (Italy), M. L. Glasser (Spain), H. Grandmontagne (France), E. A. Herman, N. Hodges (UK), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), M. Maniquiz, K. Nelson, M. Omarjee (France), P. Perfetti (Italy), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), T. Wiandt, H. Widmer (Switzerland), Y. Zhang (China), Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

Two Operator Norms

12318 [2022, 386]. *Proposed by Mohammadhossein Mehrabi, University of Gothenburg, Gothenburg, Sweden.* Let a be a positive real number, and let S_a be the set of functions $f : [-a, a] \to \mathbb{R}$ such that $\int_{-a}^{a}(f(x))^2\,dx = 1$. Let $A(f) = \int_{-a}^{a} f(x)\,dx$, $B(f) = \int_{-a}^{a} x\,f(x)\,dx$, and $C(f) = \int_{-a}^{a} x^2 f(x)\,dx$.

(a) What is $\sup\left\{A(f)^2 + B(f)^2 : f \in S_a\right\}$?

(b) What is $\sup\left\{A(f)^2 + B(f)^2 + C(f)^2 : f \in S_a\right\}$?

Solution by Kenneth Andersen, University of Alberta, Edmonton, AB, Canada. Applying the Gram–Schmidt process to the basis $\{1, x, x^2, \ldots\}$ in $L^2[-a, a]$, with inner product $\langle f, g\rangle = \int_{-a}^{a} f(x)\overline{g(x)}\,dx$, produces an orthonormal basis $\{e_1, e_2, \ldots\}$ with

$$e_1 = \frac{1}{\sqrt{2a}}, \quad e_2 = \sqrt{\frac{3}{2}}a^{-3/2}x, \quad \text{and} \quad e_3 = \sqrt{\frac{5}{8}}a^{-5/2}(3x^2 - a^2).$$

For $f \in S_a$ we have $f = \sum_{n=1}^{\infty} f_n e_n$ and $\sum_{n=1}^{\infty} f_n^2 = \langle f, f\rangle = \int_{-a}^{a}(f(x))^2\,dx = 1$, where $f_n = \langle e_n, f\rangle$. Note that

$$1 = \sqrt{2a}e_1, \quad x = \sqrt{\frac{2}{3}}a^{3/2}e_2, \quad \text{and} \quad x^2 = \frac{\sqrt{2}}{3}a^{5/2}\left(\frac{2}{\sqrt{5}}e_3 + e_1\right),$$

so

$$A(f) = \int_{-a}^{a} f(x)\,dx = \langle 1, f\rangle = \sqrt{2a}\,f_1,$$

$$B(f) = \int_{-a}^{a} xf(x)\,dx = \langle x, f\rangle = \sqrt{\frac{2}{3}}a^{3/2}f_2,$$

$$C(f) = \int_{-a}^{a} x^2 f(x)\,dx = \langle x^2, f\rangle = \frac{\sqrt{2}}{3}a^{5/2}\left(\frac{2}{\sqrt{5}}f_3 + f_1\right).$$

(a) For any $f \in S_a$ we have

$$A(f)^2 + B(f)^2 = 2af_1^2 + \frac{2a^3}{3}f_2^2 = [f_1\ f_2]\,S\,[f_1\ f_2]^T,$$

where

$$S = \begin{pmatrix} 2a & 0 \\ 0 & 2a^3/3 \end{pmatrix}.$$

When computing the requested supremum, we may restrict attention to functions f with $f_1^2 + f_2^2 = 1$ and $f_n = 0$ for $n \geq 3$. The supremum is the largest eigenvalue of S, so

$$\sup\left\{A(f)^2 + B(f)^2 : f \in S_a\right\} = \begin{cases} 2a, & \text{if } a \leq \sqrt{3}, \\ 2a^3/3, & \text{if } a > \sqrt{3}. \end{cases}$$

(b) For any $f \in S_a$, we have

$$\begin{aligned} A(f)^2 + B(f)^2 + C(f)^2 &= \left(2a + \frac{2a^5}{9}\right) f_1^2 + \frac{2a^3}{3} f_2^2 + \frac{8a^5}{45} f_3^2 + \frac{8\sqrt{5}a^5}{45} f_1 f_3 \\ &= [f_1\ f_2\ f_3]\, U\, [f_1\ f_2\ f_3]^T, \end{aligned}$$

where

$$U = \begin{pmatrix} 2a + 2a^5/9 & 0 & 4\sqrt{5}a^5/45 \\ 0 & 2a^3/3 & 0 \\ 4\sqrt{5}a^5/45 & 0 & 8a^5/45 \end{pmatrix}.$$

As in part **(a)**, the requested supremum is the largest eigenvalue of U. The eigenvalues of U are

$$\frac{2a^3}{3} \quad \text{and} \quad a + \frac{a^5}{5} \pm \frac{a}{15}\sqrt{9a^8 + 10a^4 + 225}.$$

Since $\sqrt{5}/a^2 + a^2/\sqrt{5} \geq 2$, we have $a + a^5/5 \geq 2a^3/\sqrt{5} > 2a^3/3$. Therefore

$$\sup\left\{A(f)^2 + B(f)^2 + C(f)^2 : f \in S_a\right\} = a + \frac{a^5}{5} + \frac{a}{15}\sqrt{9a^8 + 10a^4 + 225}.$$

Also solved by A. Berkane (Algeria), H. Chen (US), O. Kouba (Syria), B. Lai (China), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), M. Omarjee (France), K. Schilling, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Yan (China), and the proposer. Part **(a)** also solved by E. A. Herman.

Counting Rectangles with Prime Area

12320 [2022, 386]. *Proposed by Enrique Treviño, Lake Forest College, Lake Forest, IL.* Consider the grid of n^2 lattice points $\{1, \ldots, n\}^2$. Let $S_1(n)$ be the number of rectangles with corners in the grid (though not necessarily with horizontal and vertical sides) that have area equal to a prime integer congruent to 1 (mod 4). Define $S_3(n)$ similarly using primes congruent to 3 (mod 4). Prove that there is a value n_0 such that $S_1(n) > S_3(n)$ for $n \geq n_0$.

Solution by Nigel Hodges, Cheltenham, UK. We say that a rectangle is *aligned* if it has horizontal and vertical sides; otherwise it is *unaligned*. First consider aligned rectangles. When the area is a prime p, the sidelengths must be 1 and p. For $p \leq n - 1$, there are $2(n-1)(n-p)$ aligned rectangles with area p. Therefore the total number of aligned rectangles with prime area is

$$\sum_{\substack{p=1 \\ p \text{ prime}}}^{n-1} 2(n-1)(n-p),$$

which is less than $2n^3$.

Now consider an unaligned rectangle with area p. Because each side has length $\sqrt{a}$ for some integer a that is at least 2, unaligned rectangles must be squares of side-length $\sqrt{p}$.

Since an integer congruent to 3 modulo 4 cannot be the sum of two squares, no unaligned rectangles will contribute to S_3.

On the other hand, every prime p congruent to 1 mod 4 can be written as the sum of two squares (uniquely up to order). Let $p = k^2 + m^2$ be one such representation. The smallest aligned square that contains an unaligned square of area p has side-length $k + m$, and such an aligned square contains two unaligned squares of area p. There are $(n - (k + m))^2$ aligned squares with side-length $k + m$ within the grid, so there are at least $2(n - (k + m))^2$ unaligned rectangles of area p.

Restricting to $p \leq n^2/8$, we have

$$k + m = \sqrt{2p - (k - m)^2} < \sqrt{2p} \leq n/2.$$

Thus there are at least $n^2/2$ unaligned squares of area p in the grid. Therefore

$$S_1(n) \geq \sum \pi(n^2/8; 4, 1) \cdot \frac{n^2}{2},$$

where $\pi(x; q, r)$ denotes the number of primes up to x that are congruent to r mod q. By the prime number theorem for arithmetic progressions (see, for instance, H. Davenport (1980), *Multiplicative Number Theory*, 2nd ed., Springer-Verlag, Berlin, Ch. 20),

$$\pi(x; 4, 1) \sim \frac{x}{\phi(4) \ln x} = \frac{x}{2 \ln x}.$$

Therefore, for $\varepsilon \in (0, 1)$ and sufficiently large n,

$$S_1(n) \geq \frac{\varepsilon n^2/8}{2 \ln(n^2/8)} \cdot \frac{n^2}{2} = \frac{\varepsilon n^4}{32 \ln(n^2/8)} > 2n^3 > S_3(n).$$

Also solved by N. Caro-Montoya (Brazil), N. Fellini (Canada), D. Fleischman, K. Gatesman, A. Stadler (Switzerland), R. Tauraso (Italy), and the proposer.

Primes are Rarely Squares Modulo Squares

12321 [2022, 486]. *Proposed by Mohammadamin Sharifi, Sharif University of Technology, Tehran, Iran.* Let p be a prime number. Prove that the number of perfect squares m such that the least nonnegative remainder of p (mod m) is a perfect square is less than $2p^{1/3}$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Writing $m = b^2$ and letting a^2 be the least nonnegative residue of $p \pmod{m}$, we can write $p = kb^2 + a^2$ for integers a, b, k with $b > a \geq 0$ and $k > 0$ (since p is not a square). Conversely, any such expression gives a solution $m = b^2$. We prove the following:

Claim. For each positive integer k, there is at most one pair (a, b) with $b > a \geq 0$ such that $p = kb^2 + a^2$.

Proof: Suppose that (a, b) and (c, d) are two such pairs. Since b^2 determines a, we must have $d \neq b$. We write

$$0 \neq (b^2 - d^2)p = b^2(kd^2 + c^2) - d^2(kb^2 + a^2) = b^2c^2 - a^2d^2 = (bc + ad)(bc - ad).$$

Hence $bc + ad$ or $bc - ad$ is a nonzero multiple of p. Since $b/a > 1 > c/d$, the vectors (b, a) and (c, d) are not parallel. By the Cauchy–Schwarz inequality (with strict inequality for nonparallel vectors),

$$0 < |bc - ad| \leq bc + ad < (b^2 + a^2)^{1/2}(c^2 + d^2)^{1/2} \leq p,$$

so neither factor is a multiple of p. This contradiction yields the claim. ■

Given $p = kb^2 + a^2$, either $k \le p^{1/3}$ or $b^2 \le p^{2/3}$, since otherwise $kb^2 > p$. Since k determines the solution (by the claim), the first case gives at most $\lfloor p^{1/3} \rfloor$ solutions. In the second case, since b determines a and hence determines the solution, we also have at most $\lfloor p^{1/3} \rfloor$ solutions. Therefore the total number of solutions is at most $2\lfloor p^{1/3} \rfloor$.

Also solved by N. Hodges (UK), O. P. Lossers (Netherlands), R. Tauraso (Italy), and the proposer.

A Skew-Symmetric Determinant

12322 [2022, 486]. *Proposed by Askar Dzhumadil'daev, Kazakh-British Technical University, Almaty, Kazakhstan.* Given real numbers $x_1, \dots, x_{2n}$, let A be the skew-symmetric $2n$-by-$2n$ matrix with entries $a_{i,j} = (x_i - x_j)^2$ for $1 \le i < j \le 2n$. Prove

$$\det(A) = 4^{n-1}\Big((x_1 - x_2)(x_2 - x_3)\cdots(x_{2n-1} - x_{2n})(x_{2n} - x_1)\Big)^2.$$

Solution by Kuldeep Sarma, Tezpur University, Tezpur, India. Let $p(x_1, \dots, x_{2n})$ be the desired determinant as a polynomial in $x_1, \dots, x_{2n}$. It is a homogeneous polynomial of degree $4n$. We claim $(x_k - x_{k+1})^2 \mid p$ for $1 \le k < 2n$ and $(x_{2n} - x_1)^2 \mid p$, which implies

$$(x_1 - x_2)^2(x_2 - x_3)^2 \cdots (x_{2n-1} - x_{2n})^2(x_{2n} - x_1)^2 \mid p.$$

Noting the degree of p, we conclude that p is a scalar multiple of the desired polynomial.

To prove the claim for $k = 1$, fix arbitrary real numbers $x_2, \dots, x_{2n}$ and let $x_1 = x_2 + \epsilon$, where ϵ may vary. It suffices to show that $p(x_2 + \epsilon, x_2, x_3, \dots, x_{2n}) = O(\epsilon^2)$. With this expression for x_1, the matrix A becomes

$$\begin{bmatrix} 0 & \epsilon^2 & (x_2+\epsilon-x_3)^2 & (x_2+\epsilon-x_4)^2 & \cdots \\ -\epsilon^2 & 0 & (x_2-x_3)^2 & (x_2-x_4)^2 & \cdots \\ -(x_2+\epsilon-x_3)^2 & -(x_2-x_3)^2 & \cdot & \cdot & \cdots \\ -(x_2+\epsilon-x_4)^2 & -(x_2-x_4)^2 & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Subtracting the second row from the first and then the second column from the first (which does not change the determinant) gives

$$\begin{bmatrix} 0 & \epsilon^2 & 2(x_2-x_3)\epsilon+\epsilon^2 & 2(x_2-x_4)\epsilon+\epsilon^2 & \cdots \\ -\epsilon^2 & 0 & (x_2-x_3)^2 & (x_2-x_4)^2 & \cdots \\ -2(x_2-x_3)\epsilon-\epsilon^2 & -(x_2-x_3)^2 & \cdot & \cdot & \cdots \\ -2(x_2-x_4)\epsilon-\epsilon^2 & -(x_2-x_4)^2 & \cdot & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Factors of ϵ can now be taken out from the first row and the first column. This yields $p(x_2 + \epsilon, x_2, \dots, x_{2n}) = O(\epsilon^2)$ and thus $(x_1 - x_2)^2 \mid p$.

The same argument works when $1 < k < 2n$: set $x_k = x_{k+1} + \epsilon$ and perform the operations on the kth and $(k+1)$th rows and columns instead of the first and second. For $(x_{2n} - x_1)^2 \mid p$, set $x_{2n} = x_1 + \epsilon$ and add the $2n$th row to the first and the $2n$th column to the first; the rest of the argument is identical.

It remains only to find the scalar coefficient. We do this by evaluating $p(x_1, \dots, x_{2n})$ when $x_k = (-1)^{k-1}/2$. In this case,

$$(x_1 - x_2)^2(x_2 - x_3)^2 \cdots (x_{2n-1} - x_{2n})^2(x_{2n} - x_1)^2 = 1,$$

so we need to show that the determinant of A is 4^{n-1}. Changing the order of the rows and columns to put the odd-indexed rows and columns in order before the even-indexed rows and columns does not change the sign of the determinant and yields

$$\det A = \begin{vmatrix} 0 & B_n \\ -B_n^T & 0 \end{vmatrix},$$

where B_n is the n-by-n matrix with all entries 1 on and above the main diagonal and all entries -1 below the main diagonal. Thus $\det A = (\det B_n)^2$.

We prove by induction on n that $\det B_n = 2^{n-1}$. Note $\det B_1 = 1$. For $n > 1$, subtracting the second row of B_n from the first gives the block matrix

$$\begin{pmatrix} 2 & \mathbf{0} \\ -\mathbf{1} & B_{n-1} \end{pmatrix},$$

where the second block of rows or columns has length $n-1$. Thus $\det B_n = 2 \det B_{n-1} = 2^{n-1}$. Hence $\det A = 4^{n-1}$, and

$$p(x_1, \dots, x_{2n}) = 4^{n-1}(x_1 - x_2)^2(x_2 - x_3)^2 \cdots (x_{2n-1} - x_{2n})^2(x_{2n} - x_1)^2,$$

as desired.

Also solved by N. Caro-Montoya (Brazil), H. Chen (US), D. Fleischman, J.-P. Grivaux (France), N. Hodges (UK), O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), M. Omarjee (France), C. R. Pranesachar (India), R. Stong, R. Tauraso (Italy), Fejéntaláltuka Szeged Problem Solving Group (Hungary), and the proposer.

Beyond Bell Numbers

12323 [2022, 486]. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden, and Andreas Dieckmann, Physikalisches Institut der Universität Bonn, Bonn, Germany.*

(a) Find integers c_0, c_1, and c_2 such that

$$\sum_{k=0}^{\infty} \frac{k^{11}}{(k!)^3} = \sum_{k=0}^{\infty} \frac{c_0 + c_1 k + c_2 k^2}{(k!)^3}.$$

(b) Prove that for any integers n and b with $1 \le b \le n$, there are integers c_m for $0 \le m \le b-1$ such that

$$\sum_{k=0}^{\infty} \frac{k^n}{(k!)^b} = \sum_{k=0}^{\infty} \left(\frac{1}{(k!)^b} \sum_{m=0}^{b-1} c_m k^m \right).$$

(c) Prove that the integers c_m from part **(b)** are unique.

Solution by Kenneth Schilling, University of Michigan, Flint, MI. We first prove the claim in **(b)**. For $1 \le b \le n$, we construct a sequence $\{p_i\}$ of polynomials as follows: let $p_0(x) = x^n$, and if $p_i(x) = \sum_{m=0}^{d} a_m x^m$, then let

$$p_{i+1}(x) = \sum_{m=0}^{b-1} a_m x^m + \sum_{m=b}^{d} a_m (x+1)^{m-b}.$$

An easy induction shows that the polynomials $p_i(x)$ have integer coefficients. Since $\text{degree}(p_{i+1}) \le \max\{b-1, \text{degree}(p_i) - b\}$, the sequence of degrees of p_i decreases until we reach the first polynomial p_r with degree less than b (at which point the sequence repeats p_r indefinitely). For $m \ge b$ we have

$$\sum_{k=0}^{\infty} \frac{k^m}{(k!)^b} = \sum_{k=1}^{\infty} \frac{k^{m-b}}{((k-1)!)^b} = \sum_{k=0}^{\infty} \frac{(k+1)^{m-b}}{(k!)^b},$$

and it follows that for all i,

$$\sum_{k=0}^{\infty} \frac{p_{i+1}(k)}{(k!)^b} = \sum_{k=0}^{\infty} \frac{p_i(k)}{(k!)^b}.$$

Hence

$$\sum_{k=0}^{\infty} \frac{k^n}{(k!)^b} = \sum_{k=0}^{\infty} \frac{p_r(k)}{(k!)^b}$$

and the required integers $c_0, \ldots, c_{b-1}$ are the coefficients of $p_r(x)$.

For part **(a)**, following the procedure above, we have

$$\begin{aligned}
p_0(x) &= x^{11},\\
p_1(x) &= (x+1)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8,\\
p_2(x) &= 1 + 8x + 28x^2 + 56 + 70(x+1) + 56(x+1)^2 + 28(x+1)^3 +\\
&\qquad 8(x+1)^4 + (x+1)^5 = 220 + 311x + 226x^2 + 70x^3 + 13x^4 + x^5,\\
p_3(x) &= 220 + 311x + 226x^2 + 70 + 13(x+1) + (x+1)^2 = 304 + 326x + 227x^2.
\end{aligned}$$

Hence the required integers are given by $c_0 = 304$, $c_1 = 326$, and $c_2 = 227$.

For part **(c)**, suppose the contrary. Taking the difference of two distinct solutions, we get a nonzero polynomial $p(x)$ with integer coefficients and degree at most $b-1$ such that $\sum_{k=0}^{\infty} p(k)/(k!)^b = 0$. Assume without loss of generality that the leading coefficient of p is positive. For sufficiently large N we have $p(k) > 0$ for all $k > N$, and it follows that in the equation

$$(N!)^b \sum_{k=N+1}^{\infty} \frac{p(k)}{(k!)^b} = -(N!)^b \sum_{k=0}^{N} \frac{p(k)}{(k!)^b},$$

the left side is positive and the right side is an integer. Hence this quantity is a positive integer and in particular it is at least 1.

Let C be the sum of the absolute values of all the coefficients of p. For $x \geq 1$ we have $p(x) \leq Cx^{b-1}$ and hence for every positive integer k,

$$p(N+k) \leq C(N+k)^{b-1} \leq C(kN+k)^{b-1} = Ck^{b-1}(N+1)^{b-1}.$$

We also have

$$\frac{N!}{(N+k)!} = \frac{1}{(N+1)\cdots(N+k)} \leq \frac{1}{(N+1)k!}.$$

Thus

$$\begin{aligned}
1 \leq (N!)^b \sum_{k=N+1}^{\infty} \frac{p(k)}{(k!)^b} &= \sum_{k=1}^{\infty} \frac{p(N+k)(N!)^b}{((N+k)!)^b}\\
&\leq \sum_{k=1}^{\infty} \frac{Ck^{b-1}(N+1)^{b-1}}{(N+1)^b (k!)^b} \leq \frac{C}{N+1} \sum_{k=1}^{\infty} \frac{1}{k!} \to 0 \quad \text{as } N \to \infty,
\end{aligned}$$

so we have a contradiction.

Editorial comment. The problem can be viewed as saying that for all $n \geq 0$ and $b \geq 1$, there is a unique polynomial $P_{n,b}(x)$ with integer coefficients and of degree at most $b-1$ such that

$$\sum_{k=0}^{\infty} \frac{k^n}{(k!)^b} = \sum_{k=0}^{\infty} \frac{P_{n,b}(k)}{(k!)^b}.$$

The polynomial $P_{n,1}$ is constant and equal to the Bell number B_n, the number of partitions of the set $\{1, \dots, n\}$. Hence the polynomials $P_{n,k}$ can be viewed as a generalization of the Bell numbers. The solution above gives a recurrence for these polynomials, and the proposers have shown that a similar recurrence holds when the factorials are replaced by multifactorials (see www-elsa.physik.uni-bonn.de/ dieckman/InfProd/InfProd.html#Sumsxinvolvingxreciprocalxmultifactorialsxorxfactorials).

One can also give a summation formula for $P_{n,b}$ analogous to the formula for the Bell numbers as a sum of Stirling numbers of the second kind. Specifically, let $h_k(x_0, \dots, x_m)$ denote the complete homogeneous symmetric polynomial of degree k and define generalized Stirling numbers $S_b(n, m)$ to be h_{n-m} evaluated at $x_i = \lfloor i/b \rfloor$ for $0 \le i \le m$. For $b = 1$ these are the Stirling numbers of the second kind. One can prove the polynomial identity

$$X^n = \sum_{m=0}^{n} S_b(n, m) \prod_{i=0}^{m-1} (X - \lfloor i/b \rfloor),$$

from which it follows that

$$P_{n,b}(x) = \sum_{m=0}^{n} S_b(n, m) x^{(m \bmod b)}.$$

Also solved by N. Hodges (UK), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), M. Omarjee (France), C. R. Pranesachar (India), R. Tauraso (Italy), Eagle Problem Solvers, and the proposer. Parts **(a)** and **(b)** were solved by P. Bracken, H. Chen (US), N. Grivaux (France), E. A. Herman, W. Janous (Austria), and D. Terr.

A Symmetrical Integral

12324 [2022, 486]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let a and b be positive real numbers. Prove

$$\int_0^\infty \frac{1}{\sqrt{ax^4 + 2(2b - a)x^2 + a}}\, dx = \int_0^\infty \frac{1}{\sqrt{bx^4 + 2(2a - b)x^2 + b}}\, dx.$$

Solution by Giuseppe Fera, Vicenza, Italy. Let

$$f(a, b) = \int_0^\infty \frac{1}{\sqrt{ax^4 + 2(2b - a)x^2 + a}}\, dx.$$

Splitting the integral at 1 and then making the change of variables $y = 1/x$ in the second integral, we get

$$\begin{aligned} f(a, b) &= \int_0^1 \frac{1}{\sqrt{ax^4 + 2(2b - a)x^2 + a}}\, dx + \int_1^\infty \frac{1}{\sqrt{ax^4 + 2(2b - a)x^2 + a}}\, dx \\ &= \int_0^1 \frac{1}{\sqrt{ax^4 + 2(2b - a)x^2 + a}}\, dx + \int_0^1 \frac{1}{\sqrt{ay^4 + 2(2b - a)y^2 + a}}\, dy \\ &= 2\int_0^1 \frac{1}{\sqrt{ax^4 + 2(2b - a)x^2 + a}}\, dx. \qquad (*) \end{aligned}$$

Substituting $x = (1 - u)/(1 + u)$ and $dx = -2\, du/(1 + u)^2$, we obtain

$$f(a, b) = 2\int_0^1 \frac{1}{\sqrt{bu^4 + 2(2a - b)u^2 + b}}\, du = f(b, a),$$

which completes the proof.

Editorial comment. Many solvers used the substitution $x = \tan(\theta/2)$ in $(*)$ to get

$$f(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a\cos^2\theta + b\sin^2\theta}},$$

which also easily yields the desired symmetric property with respect to a and b.

Also solved by K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), P. Bracken, H. Chen (US), K. Gatesman, M. L. Glasser, D. Henderson, O. Kouba (Syria), B. Lai (China) & R. Wang (China), S. Lee, O. P. Lossers (Netherlands), J. Magliano, F. Masroor, M. Omarjee (France), C. R. Pranesachar (India), V. Schindler (Germany), S. M. Stewart (Saudi Arabia), R. Stong, R. Tauraso (Italy), T. Wiandt, H. Widmer (Switzerland), L. Zhou, UM6P Math Club (Morocco), and the proposer.

CLASSICS

C22. *Due to Paul Erdős.* Prove that from any point in any triangle, the sum of the distances to the vertices of the triangle is at least twice as large as the sum of the distances to the sides of the triangle.

The Infamous Pentagon Problem

C21. *From the 1986 International Mathematical Olympiad.* An integer is assigned to each vertex of a regular pentagon in such a way that the sum of the five integers is positive. If three consecutive vertices are assigned the numbers x, y, z in order, and y is negative, then one may replace x, y, and z by $x + y$, $-y$, and $z + y$, respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution. The procedure must end. To see this, let

$$F(a, b, c, d, e) = (a - c)^2 + (b - d)^2 + (c - e)^2 + (d - a)^2 + (e - b)^2.$$

When the integers a, b, c, d, and e are assigned to the pentagon, we call $F(a, b, c, d, e)$ the *score* of that assignment. The score of any assignment is nonnegative, but the effect of the replacement move is to lower the score, since, with $c < 0$, we have

$$\begin{aligned} &F(a,b, c, d, e) - F(a, b + c, -c, d + c, e) \\ &\quad = (a - c)^2 + (b - d)^2 + (c - e)^2 + (d - a)^2 + (e - b)^2 \\ &\qquad - \left((a + c)^2 + (b - d)^2 + (-c - e)^2 + (d + c - a)^2 + (e - b - c)^2\right) \\ &\quad = -2c(a + b + c + d + e), \end{aligned}$$

which is positive. Since there is no infinite strictly decreasing sequence of nonnegative integers, no infinite sequence of replacement moves is possible.

Editorial Comment: Although the solution above appears simple, this was the hardest problem on the 1986 International Mathematical Olympiad, because the function F is difficult to find. The problem has seen many incarnations and generalizations, and it has spawned many papers in research journals. The solution in P. Winkler (2003) *Mathematical Puzzles: A Connoisseur's Collection*, A K Peters, is attributed to B. Chazelle. It allows for the generalization to an n-gon for any n and further yields the surprising conclusion that the number of replacement moves until no negative integers remain and the final assignment of these integers is independent of which sequence of moves is selected.

SOLUTIONS

Minimizing an Integral

12308 [2022, 285]. *Proposed by Cezar Lupu, Yanqi Lake BIMSA and Tsinghua University, Beijing, China.* What is the minimum value of $\int_0^1 \left(f'(x)\right)^2\,dx$ over all continuously differentiable functions $f\colon [0,1] \to \mathbb{R}$ such that $\int_0^1 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = 1$?

Solution by Raymond Mortini, University of Luxembourg, Esch-sur-Alzette, Luxembourg. The minimum value is 105/2.

By the Cauchy–Schwarz inequality,

$$\left(\int_0^1 f'(x)(x^3 - x)\,dx\right)^2 \le \int_0^1 (f'(x))^2\,dx \int_0^1 (x^3 - x)^2\,dx. \tag{$*$}$$

Integrating by parts in the integral on the left in ($*$), we obtain

$$\begin{aligned}\int_0^1 f'(x)(x^3 - x)\,dx &= f(x)(x^3 - x)\Big|_0^1 - \int_0^1 f(x)(3x^2 - 1)\,dx \\ &= -3\int_0^1 x^2 f(x)\,dx + \int_0^1 f(x)\,dx = -3 + 1 = -2.\end{aligned}$$

Moreover,

$$\int_0^1 (x^3 - x)^2\,dx = \int_0^1 (x^6 - 2x^4 + x^2)\,dx = \frac{8}{105}.$$

Therefore ($*$) implies

$$\int_0^1 (f'(x))^2\,dx \ge \frac{4}{8/105} = \frac{105}{2}.$$

Equality holds in ($*$) when $f'(x)$ is a scalar multiple of $x^3 - x$, or equivalently when

$$f(x) = \frac{ax^4}{4} - \frac{ax^2}{2} + c$$

for some real numbers a and c. The constraint $\int_0^1 f(x)\,dx = \int_0^1 x^2 f(x)\,dx = 1$ leads to the values $a = -105/4$ and $c = -33/16$. Thus the minimum value $105/2$ is attained when $f(x) = -(105/16)x^4 + (105/8)x^2 - 33/16$.

Also solved by R. A. Agnew, K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), P. Bracken, H. Chen (US), C. Chiser (Romania), P. J. Fitzsimmons, K. Gatesman, L. Han, K. T. L. Koo (China), O. Kouba (Syria), B. Lai & R. Wang (China), K.-W. Lau (China), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), P. Perfetti (Italy), A. D. Pîrvuceanu (Romania), K. Schilling, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, F. Visescu (Romania), J. Vukmirović (Serbia), T. Wiandt, J. Yan (China), L. Zhou, UM6P Math Club (Morocco), and the proposer.

Two Inequalities Involving Power Means

12311 [2022, 286]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let m and n be positive integers, and let $r, x_1, x_2, \dots, x_n$ be positive real numbers.

(a) Prove $\displaystyle\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r \geq \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}$ when $r \leq m/2$.

(b) Prove $\displaystyle\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r \leq \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}$ when $r \geq m$.

Solution by Faraz Masroor, New York, NY. For $t > 0$, let

$$S_t = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^t\right)^{1/t},$$

which is the power mean with exponent t of the numbers $x_1, \dots, x_n$. By the power mean inequality, $S_a \leq S_b$ when $0 < a \leq b$.

(a) Suppose $r \leq m/2$. By the Cauchy–Schwarz inequality, for any i and j in $\{0, \dots, m\}$,

$$\frac{1}{n}\sum_{k=1}^{n} x_k^i \cdot \frac{1}{n}\sum_{k=1}^{n} x_k^j \geq \left(\frac{1}{n}\sum_{k=1}^{n} x_k^{(i+j)/2}\right)^2.$$

Therefore

$$\begin{aligned}
\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r &= \prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^{r/2} \cdot \prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^{m-j}\right)^{r/2} \\
&= \prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j \cdot \frac{1}{n}\sum_{k=1}^{n} x_k^{m-j}\right)^{r/2} \geq \prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^{m/2}\right)^r \\
&= \prod_{j=0}^{m}(S_{m/2})^{mr/2} \geq (S_r)^{(m+1)mr/2} = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}.
\end{aligned}$$

(b) Suppose $r \geq m$. Since the $j = 0$ term in the product is 1, we have

$$\prod_{j=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n} x_k^j\right)^r = \prod_{j=1}^{m}(S_j)^{jr} \leq \prod_{j=1}^{m}(S_r)^{jr} = (S_r)^{r\sum_{j=1}^{m} j} = \left(\frac{1}{n}\sum_{k=1}^{n} x_k^r\right)^{\binom{m+1}{2}}.$$

Editorial comment. Some solvers observed that the inequality in **(a)** actually holds for $r \leq (m+1)/2$. This can be proven by letting j run from 1 to m in the product (as in the solution to **(b)**) and then changing x_k^{m-j} in the solution above to x_k^{m+1-j}, with appropriate modifications in the later steps.

Also solved by K. F. Andersen (Canada), M. Bataille (France), O. Geupel (Germany), O. Kouba (Syria), H. Kwong, O. P. Lossers (Netherlands), A. Mhanna (Lebanon), P. Perfetti (Italy), M. Reid, R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

An Above Average Function

12312 [2022, 286]. *Proposed by Martin Tchernookov, University of Wisconsin, Whitewater, WI.* Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ such that, for all positive x,

$$f(x)\left(f(x) - \frac{1}{x}\int_0^x f(t)\,dt\right) \geq \big(f(x)-1\big)^2.$$

Solution by Edward Schmeichel, San Jose State University, San Jose, CA. Clearly the constant function defined by $f(x) = 1$ satisfies the given inequality. We show that it is the only continuous function that does so.

Let f be a continuous function satisfying the inequality. For $x \geq 0$, let

$$A(x) = \begin{cases} \frac{1}{x}\int_0^x f(t)\,dt, & \text{if } x > 0, \\ f(0), & \text{if } x = 0. \end{cases}$$

Note that A is continuous from the right at 0. Letting $x \to 0^+$ in the given inequality, we obtain $0 = f(0)(f(0) - f(0)) \geq (f(0)-1)^2$, and therefore $f(0) = 1$.

Since $f(x) = 0$ for any $x > 0$ gives a contradiction, the intermediate value theorem implies $f(x) > 0$ for all $x \in [0, \infty)$. If follows that $f(x) - A(x) \geq (f(x)-1)^2/f(x) \geq 0$ and hence $f(x) \geq A(x)$. Thus $A'(x) = (f(x) - A(x))/x \geq 0$ for all $x > 0$, so $A(x)$ is nondecreasing, and we obtain $f(x) \geq A(x) \geq A(0) = 1$ for all $x \geq 0$.

The given inequality can be rearranged to read $f(x)(2 - A(x)) \geq 1$, so $A(x) < 2$. Thus $A(x)$ is both nondecreasing and bounded above, so as x tends to infinity, $A(x)$ approaches a limit L from below, where $1 \leq L \leq 2$. If $L = 1$, then $A(x) = 1$ and hence $f(x) = 1$ for all x, and we are done. Thus we may assume $L > 1$.

Say $L = 1 + \epsilon$, where $0 < \epsilon \leq 1$. Let a be any number with $1/(1+\epsilon) < a < 1$, and choose b large enough that $A(x) \geq 1 + a\epsilon$ for $x \geq b$. For $x \geq b$,

$$f(x) \geq \frac{1}{2 - A(x)} \geq \frac{1}{1 - a\epsilon},$$

and therefore

$$A(x) = \frac{1}{x}\int_0^x f(t)\,dt \geq \frac{1}{x}\int_b^x \frac{1}{1-a\epsilon}\,dt = \frac{x-b}{x(1-a\epsilon)}.$$

It follows that

$$L = \lim_{x\to\infty} A(x) \geq \lim_{x\to\infty} \frac{x-b}{x(1-a\epsilon)} = \frac{1}{1-a\epsilon} > \frac{1}{1-\epsilon/(1+\epsilon)} = 1 + \epsilon = L,$$

a contradiction.

Also solved by K. F. Andersen (Canada), P. Bracken, J. Boswell & C. Curtis, H. Chen (China), C. Chiser (Romania), P. J. Fitzsimmons, L. Han, D. A. Hejhal, D. Henderson, E. A. Herman, G. Herzog (Germany) & R. Mortini (France), N. Hodges (UK), K.-W. Lau (China), O. P. Lossers (Netherlands), M. Omarjee (France), L. J. Peterson, A. Sinha (India), R. Stong, R. Tauraso (Italy), J. Vukmirović (Serbia), J. Yan (China), and the proposer.

Trees with Pairwise Isomorphic Subtrees

12313 [2022, 286]. *Proposed by Douglas B. West, University of Illinois, Urbana IL.* For all $n \in \mathbb{N}$, determine all n-vertex trees having the property that the connected $(n-2)$-vertex subgraphs that can be obtained by deleting two vertices are pairwise isomorphic.

Solution by O. P. Lossers, Eindhoven University of Technology, Netherlands and Eötvös Loránd University, Hungary. For all n, the trees with this property are the star and the path, plus when $n = 5$ the one tree that is not a star or path. In each of these cases, the subtrees all are stars or all are paths.

For $n \geq 6$, let T be an n-vertex tree, and let ℓ be the number of leaves of T. If $\ell = 2$, then T is a path. If $\ell = 3$, then T consists of three paths with a common endpoint, and any tree with three leaves is determined by the multiset of lengths of those paths. Let a, b, and c be their lengths in T, with $a \geq b \geq c \geq 1$. Note that $a + b + c = n - 1 \geq 5$. If $c \leq 2$, then T has both paths and non-paths as connected $(n-2)$-vertex subgraphs. If $c \geq 3$, then the sets $\{a, b, c-2\}$ and $\{a-1, b-1, c\}$ are different and determine nonisomorphic subtrees.

Hence we may assume $\ell \geq 4$. The diameter of a tree is the maximum length of its paths, and any longest path connects two leaves. Let d be the diameter of T. Deleting two leaves outside a longest path produces a subtree having diameter d. If some two leaves together cover all longest paths, then deleting them produces a subtree with smaller diameter. Hence we may assume that T has no such pair of leaves, which implies that every connected subgraph of T with $n-2$ vertices has diameter d.

If d is odd, then T has a central edge e belonging to all paths of length d. Since all connected subgraphs with $n-2$ vertices have diameter d, both components of $T - e$ contain at least two leaves of T. The tree obtained by deleting any two leaves still has diameter d and the same central edge e. Hence a subtree obtained by deleting two of the leaves of T from the smallest component of $T - e$ is not isomorphic to a subtree obtained by deleting one leaf of T from each component of $T - e$; in these two subtrees the sizes of the components obtained by deleting the central edge e are different.

The argument for even d with $d \geq 4$ is similar. In this case the tree has a unique central vertex z at the middle of every longest path. Let k be the degree of z. Note that $k \geq 2$, and k is the number of components of $T - z$. Let $n_1, \ldots, n_k$ be the numbers of vertices in the components of $T - z$, in nonincreasing order. Every connected subgraph of T with $n-2$ vertices has diameter d and the same central vertex z. If $n_k \leq 2$, then T has $(n-2)$-vertex subtrees whose central vertices have different degrees. If $n_k \geq 3$, then a subtree obtained by deleting two vertices from a smallest component of $T - z$ is not isomorphic to a subtree obtained by deleting one vertex each from two largest components of $T - z$.

Editorial comment. Motivated by this problem, Stan Wagon conjectured that among all graphs, the graphs whose connected subgraphs obtained by deleting two vertices are pairwise isomorphic are the stars, paths, cycles, complete graphs, and five other graphs with at most five vertices. Wagon's conjecture was proved by the proposer.

Also solved by K. Gatesman, O. Geupel (Germany), Y. Ionin, R. Stong, R. Tauraso (Italy), Texas State Problem Solvers, and the proposer.

Rotating Devices

12314 [2022, 385]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let n, m, and k be positive integers with $k \leq n-1$. Consider n devices each of which can be in any of m states denoted $0, 1, \ldots, m-1$. A move consists of selecting a set of k devices and adding $1 \pmod{m}$ to each of their states. Prove that for any n, m, k as specified and any initial states of the n devices, there exists a sequence of moves that leaves each device in the state 0 or 1.

Solution by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA. Fixing k, we prove the claim by induction on $n - k$. For the induction step, suppose $n > k + 1$. Let $(a_1, \dots, a_n)$ be the initial states. First, add 1 modulo m to the first k devices until the state of the first device reaches 0. Next apply the induction hypothesis to the last $n - 1$ devices, leaving a_1 unchanged, to bring all those devices to state 0 or 1.

Thus it suffices to prove the claim when $n = k + 1$. For this, we begin with a lemma.

Lemma. *Given any distinct indices i and j, moves can be made that result in replacing a_i with $a_i - s$, replacing a_j with $a_j + s$, and leaving all other states unchanged, for any s with* $1 \le s \le m - 1$.

Proof. By symmetry, we may assume $i = 1$ and $j = k + 1$. Add 1 to the last k devices s times, and add 1 to the first k devices $m - s$ times. ■

Given initial states $(a_1, \dots, a_{k+1})$, using the lemma k times with $(i, j, s) = (i, k + 1, a_i)$ for $1 \le i \le k$ brings the first k devices to 0 while accumulating $\sum_{i=1}^{k+1} a_i$ in position $k + 1$. That is, we obtain $(0, \dots, 0, \ell)$, where $\ell \equiv \sum_{i=1}^{k+1} a_i \pmod m$ and $0 \le \ell \le m - 1$.

If $\ell < k$, then we apply the lemma ℓ times with $(i, j, s) = (k + 1, j, 1)$ for $1 \le j \le \ell$ to obtain $(1, \dots, 1, 0, \dots, 0)$, with 1 in the first ℓ devices and 0 in the others.

If $\ell \ge k$, then write $\ell = qk + r$ for integers q and r with $0 \le r < k$. Using the lemma k times with $(i, j, s) = (k + 1, j, 1)$ for $1 \le j \le k$ yields $(1, \dots, 1, \ell - k)$. Now adding 1 to the first k devices $m - 1$ times produces $(0, \dots, 0, \ell - k)$.

Repeating this process q times brings the states to $(0, \dots, 0, r)$, where $0 \le r < k$. The argument in the case $\ell < k$ then allows us to reach $(1, \dots, 1, 0, \dots, 0)$, with 1 in the first r devices and 0 in the others.

Also solved by J. Boswell & C. Curtis, B. S. Burdick, N. Caro-Montoya (Brazil), H. Chen (China), P. Corn, K. Gatesman, A. Goel, E. A. Herman, N. Hodges (UK), O. P. Lossers (Netherlands), A. Mandal (India), F. Masroor, G. Raduns, T. Song, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), O. Zhang, and the proposer.

Determinants and Rooted Trees

12315 [2022, 385]. *Proposed by Mikael P. Sundqvist and Victor Ufnarovski, Lund University, Sölvegatan, Sweden.* Suppose $a_{i,j} \in [0, 1]$ for $1 \le i \le n$ and $1 \le j \le n$. Let B be the n-by-n matrix with i, j-entry $b_{i,j}$ defined by $b_{i,j} = a_{i,j}$ when $j \ne i - 1$ and $b_{i,j} = -\sum_{k=1}^{n} a_{i,k}$ when $j = i - 1$.
(a) Evaluate $\det(B)$ in the case where $a_{i,j} = 1$ for all i and j.
(b) Show that the value in part **(a)** is the maximum possible value of $\det(B)$.
(c) Show that $\det(B) \ge 0$ in all cases.

Solution I by Richard Stong, Center for Communications Research, San Diego, CA.
(a) We denote by A the matrix with i, j-entry $a_{i,j}$. The matrix B in this case has a 1 as every entry except just below the main diagonal, where the entries are $-n$. Subtracting the top row from each other row yields a matrix whose only nonzero entry in row i for $i > 1$ is $-n - 1$ in column $i - 1$. The cofactor expansion of the determinant along column n yields $\det(B) = (-1)^{n+1}(-n - 1)^{n-1} = (n + 1)^{n-1}$.

(b) and **(c)** Clearly, $\det(B)$ is a homogeneous polynomial of degree n in the values $a_{i,j}$, where each monomial has the form $\prod_{i=1}^{n} a_{i,f(i)}$ for some function $f\colon [n] \to [n]$ (generally not a permutation). We prove a stronger version of **(c)** that implies **(b)**, namely that the coefficient of each such monomial is nonnegative. Increasing all entries of A to 1 therefore maximizes each monomial and the value of $\det(B)$. The coefficient of a particular monomial is the value of $\det(B)$ in the case where the corresponding n entries of A equal 1 and all other entries are 0. These n entries are one from each row of A. Thus it suffices to prove **(c)** in the special case where A has a single 1 in each row and the rest of A is 0. This is the statement proved in Lemma 2, since the corresponding matrix B always has -1 in each subdiagonal entry.

Lemma 1. *If M is an n-by-n matrix with all entries 0 except for diagonal entries 1 and at most one entry in each row equal to -1, then $\det(M) \geq 0$.*

Proof. We proceed by induction on n, with trivial base case $n = 1$. If any row of M lacks a -1, then the expansion of $\det(M)$ along that row reduces to the case for $n-1$ and the induction hypothesis suffices. If every row of M has a -1, then the row sums of M are all zero, so $\det(M) = 0$. ∎

Lemma 2. *If B is an n-by-n matrix with all entries 0 except for subdiagonal entries -1 and at most one entry equal to 1 in each row, then $\det(B) \geq 0$.*

Proof. We again proceed by induction on n, with trivial base case $n = 1$. If B has no 1 in the top row, then already $\det(B) = 0$.

If $b_{1,j} = 1$ with $j < n$, then obtain B' from B by deleting row 1 and column j. Expanding along row 1 yields $\det(B) = (-1)^{j+1}\det(B')$. Obtain B'' from B' by moving row j of B' to the top, so $\det(B'') = (-1)^{j-1}\det(B') = \det(B)$. Since the -1 in row $j+1$ of B was deleted before moving the row, B'' satisfies the conditions of Lemma 2, and hence $\det(B'') \geq 0$ by the induction hypothesis.

If $b_{1,n} = 1$, then move row 1 to the bottom, introducing a factor of $(-1)^{n-1}$, and negate the top $n-1$ rows, introducing a second factor of $(-1)^{n-1}$. The resulting matrix M has the form described in Lemma 1, so $\det(B) = \det(M) \geq 0$, as desired. ∎

Solution II by Pierre Lalonde, Plessisville, QC, Canada. We express $\det(B)$ as the sum of the weights of certain rooted spanning trees in a directed graph. To do this, we locate $-B$ as a submatrix in a particular matrix obtained from a weighted digraph with $n+1$ vertices. Let G be the directed graph on vertices $v_1, \dots, v_{n+1}$ in which every ordered pair of distinct vertices is an edge. Define the weight of edge v_iv_j with $i \neq j$ as follows: let $w(v_iv_1) = a_{i,i-1}$ for $2 \leq i \leq n$, let $w(v_iv_j) = a_{i,j-1}$ for $1 \leq i \leq n$ and $2 \leq j \leq n+1$, and let $w(v_{n+1}v_j) = 0$ for $1 \leq j \leq n$.

The *Laplacian matrix* $L(G)$ of G is defined by letting the entry in position (i, j) be $-w(v_iv_j)$ when $i \neq j$ and in position (i, i) be $\sum_{j \neq i} w(v_iv_j)$. Thus

$$L(G) = \begin{bmatrix} \sum_{j=1}^{n} a_{1,j} & -a_{1,1} & -a_{1,2} & \cdots & -a_{1,n-1} & -a_{1,n} \\ -a_{2,1} & \sum_{j=1}^{n} a_{2,j} & -a_{2,2} & \cdots & -a_{2,n-1} & -a_{2,n} \\ -a_{3,2} & -a_{3,1} & \sum_{j=1}^{n} a_{3,j} & \cdots & -a_{3,n-1} & -a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n,n-1} & -a_{n,1} & -a_{n,2} & \cdots & \sum_{j=1}^{n} a_{n,j} & -a_{n,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The n-by-n submatrix obtained by deleting row $n+1$ and column 1 is $-B$.

A *tree* is an acyclic connected graph. We consider orientations of trees, in which each edge is given a direction. A *rooted tree* is an orientation of a tree in which all edges are oriented along paths to a distinguished vertex called the *root*. The *weight* of a rooted tree is the product of the weights of its edges. A special case of a deep generalization of Tutte's directed matrix tree theorem (S. Chaiken and D. J. Kleitman (1978), Matrix tree theorems, *J. Combinatorial Theory (A)* 24:3, 377–381) states that the sum of the weights of the rooted trees in a directed graph G that are rooted at a particular vertex v_j is given by any cofactor in the row for that vertex in the Laplacian matrix $L(G)$ defined above. (Their proof is also presented in D. B. West (2021), *Combinatorial Mathematics*, Cambridge, p. 750.) Thus $(-1)^{n+2}\det(-B)$, which equals $\det(B)$, is the sum of the weights of all spanning trees of G rooted at v_{n+1}.

When $a_{i,j} = 1$ for all i and j, we are just counting spanning trees in a complete graph with $n+1$ vertices, which by Cayley's formula yields $\det(B) = (n+1)^{n-1}$, solving **(a)**. Because the weights are nonnegative, $\det(B)$ is always nonnegative, solving **(c)**, and $\det(B)$ is maximized when all weights are maximized at 1, solving **(b)**.

Also solved by L. Han & J. Xu, O. P. Lossers (Netherlands), R. Tauraso (Italy), and the proposer.

Organizing a Row of Coins

12316 [2022, 385]. *Proposed by H. A. ShahAli, Tehran, Iran, and Manija Shahali, Bakersfield, CA.* For each i in $\{1, 2, \ldots, C\}$, we have $2i$ coins with color i. Place these $C(C+1)$ coins in a line. A *move* consists of the transposition of two adjacent coins. Let m be the minimum number of moves required to reach a configuration where all coins of the same color are together in a run of consecutive coins. Show that the maximum value of m over all initial configurations is $(C-1)C(C+1)(3C+2)/12$.

Solution by Nigel Hodges, Cheltenham, UK. We show first that, from any configuration, $(C-1)C(C+1)(3C+2)/12$ moves suffice to complete the task.

We proceed by induction on C. When $C = 1$ there is only one configuration, and the number of moves needed is 0, which is the value of the specified formula at $C = 1$.

Now consider $C > 1$. Index the positions from 1 at the left end of the row to $C(C+1)$ at the right end. We first move all coins of color C to one end, whichever end requires fewer moves. Consider the leftmost coin of color C. Since an optimal set of moves will never swap two coins of the same color, we may assume that this coin ends at position 1 or position $C^2 - C + 1$. The sum of the number of moves involving it if it goes left plus the number if it goes right is $C(C-1)$.

This sum is independent of which coin of color C we consider, so the total number of moves spent on all color C coins if moved left, plus the number of moves spent if moved right, equals $2C^2(C-1)$. Therefore at most $C^2(C-1)$ moves are needed to gather the coins of color C at one end.

With all coins of color C coins at one end, what remains is an instance with $C-1$ colors. By the induction hypothesis, the number of moves remaining to complete the task now is at most $(C-2)(C-1)C(3C-1)/12$, and adding this to $2C^2(C-1)$ yields the desired formula.

Consider the initial configuration whose right half consists of i coins of color i together for each i, from color 1 in the middle of the row through color C at the right end. The left half is the reflection of this, with half the coins of color C on the left end. We now show that this configuration requires the full $(C-1)C(C+1)(3C+2)/12$ moves. Again we use induction on C, and again the case $C = 1$ is trivial.

With $C > 1$, the two blocks of color C have all other $C(C-1)$ coins between them, so forming a single block of color C will require each of at least C coins to move at least $C(C-1)$ times, requiring a total of at least $C^2(C-1)$ moves involving coins of color C. Moves that involve a coin of color C do not change the order among the coins with earlier colors. Hence the number of moves that must be made not involving a coin with color C must be at least the number required to solve the problem with $C-1$ colors that is obtained by ignoring the coins with color C. That ordering is the instance of the specified configuration when the number of colors is $C-1$. By the induction hypothesis, there must be at least $(C-2)(C-1)C(3C-1)/12$ moves not involving a coin with color C. Again the sum of the two contributions is the desired formula.

Also solved by H. Chen (China), A. De la Fuente, L. Guala & S. Leucci & R. Tauraso (Italy), Y. J. Ionin, Y. Kim (Korea), O. P. Lossers (Netherlands), K. Schilling, R. Stong, and the proposer.

A Fermat Point Inequality

12319 [2022, 386]. *Proposed by Mihály Bencze, Braşov, Romania.* Let ABC be a triangle with all angles less than 120°, and let F be the Fermat point of ABC (the point in the interior that minimizes the sum of the distances to A, B, and C). Prove

$$\frac{FA^4}{AB^2}+\frac{FB^4}{BC^2}+\frac{FC^4}{CA^2}\geq\frac{FA^3+FB^3+FC^3}{FA+FB+FC}.$$

Solution by Nandan Sai Dasireddy, Hyderabad, India. Write $FA=x$, $FB=y$, and $FC=z$. It is well known that all of the angles $\angle AFB$, $\angle BFC$, and $\angle CFA$ are equal to 120°. Therefore, by the law of cosines, $AB^2=x^2+xy+y^2$, and similarly for BC^2 and CA^2. Thus the desired inequality is

$$\sum_{\text{cyc}}\frac{x^4}{x^2+xy+y^2}\geq\frac{x^3+y^3+z^3}{x+y+z},$$

where we use $\sum_{\text{cyc}} f(x,y,z)$ as a shorthand for $f(x,y,z)+f(y,z,x)+f(z,x,y)$. Since

$$\frac{x^3+y^3+z^3}{x+y+z}=\frac{3xyz}{x+y+z}+\sum_{\text{cyc}}(x^2-xy),$$

it suffices to show

$$\frac{3xyz}{x+y+z}\leq\sum_{\text{cyc}}\left(\frac{x^4}{x^2+xy+y^2}-(x^2-xy)\right)=\sum_{\text{cyc}}\frac{xy^3}{x^2+xy+y^2}.$$

By the Cauchy–Schwarz inequality,

$$\sum_{\text{cyc}}\frac{xy^3}{x^2+xy+y^2}=\sum_{\text{cyc}}\frac{y^2}{x/y+y/x+1}\geq\frac{(x+y+z)^2}{\sum_{\text{cyc}}(x/y+y/x+1)}=\frac{xyz(x+y+z)}{xy+yz+zx}.$$

Therefore it suffices to prove

$$\frac{xyz(x+y+z)}{xy+yz+zx}\geq\frac{3xyz}{x+y+z}.$$

But this is equivalent to $(x+y+z)^2\geq 3(xy+yz+zx)$, which is true because

$$(x+y+z)^2-3(xy+yz+zx)=\sum_{\text{cyc}}\frac{(x-y)^2}{2}\geq 0.$$

Also solved by O. Geupel (Germany), K. T. L. Koo (China), C. G. Petalas (Greece), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), A. Tzavellas (Greece), L. Zhou, UM6P Math Club (Morocco), and the proposer.

CLASSICS

C21. *From the 1986 International Mathematical Olympiad, suggested by the editors.* An integer is assigned to each vertex of a regular pentagon in such a way that the sum of the five integers is positive. If three consecutive vertices are assigned the numbers x, y, z in order, and y is negative, then one may replace x, y, and z by $x+y$, $-y$, and $z+y$, respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Period of the Fibonacci Sequence Modulo m

C20. *Due to Peter Freyd, suggested by the editors.* Given a positive integer m, let $f(m)$ be the period of the Fibonacci sequence taken modulo m. Prove $f(m) \le 6m$ and that equality holds for infinitely many m.

Solution. Let A denote the Fibonacci matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, and let I denote the 2-by-2 identity matrix. The values $F_0 = 0$ and $F_1 = 1$ and the relation $F_{n+1} = F_n + F_{n-1}$ imply that $A^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$. Hence $f(m)$ is the multiplicative order of A modulo m. From the first few Fibonacci numbers we learn that $A^3 \equiv I \pmod 2$, so $f(2) = 3$. Similarly, $A^4 \equiv -I \pmod 3$, so $f(3)$ divides 8; it does not divide 4, so $f(3) = 8$. To compute $f(5)$, note that $A^5 \equiv 3I \pmod 5$; hence $A^{20} \equiv I \pmod 5$, and so $f(5)$ divides 20. Because neither A^4 nor A^{10} equals I modulo 5, we infer $f(5) = 20$.

We next note that $f(m_1m_2) = \operatorname{lcm}(f(m_1), f(m_2))$ when $\gcd(m_1, m_2) = 1$. This is a consequence of the fact that a and b are congruent modulo m_1m_2 if and only if they are congruent modulo both m_1 and m_2. This reduces the problem of bounding $f(m)$ for general m to the problem of finding such bounds when m is a prime power. For example, $f(10) = \operatorname{lcm}(f(2), f(5)) = 60$, so $f(m) = 6m$ when $m = 10$.

When p is prime and $a \ge 2$, the calculation $A^{pf\left(p^{a-1}\right)} = \left(I + p^{a-1}M\right)^p \equiv I \pmod{p^a}$ for some matrix M shows that $f(p^a)$ divides $pf\left(p^{a-1}\right)$. Applying this $a-1$ times yields that $f(p^a)$ divides $p^{a-1}f(p)$.

We need some special information about the case $p = 5$.

Claim 1. $f(5^a) = 4 \cdot 5^a$.

Proof. We have $A^{20} = I + 5K$ for some matrix K, and it is easily checked that K is nonzero modulo 5. This is the base case for an induction proof of the stronger claim that $A^{4\cdot 5^a} \equiv I + 5^aK \pmod{5^{a+1}}$. If this holds for some a, then

$$A^{4\cdot 5^{a+1}} = (I + 5^aK + 5^{a+1}M)^5 \equiv I + 5^{a+1}K \pmod{5^{a+2}},$$

for some matrix M. This completes the induction. ■

For odd primes p not equal to 5, we say that p is *type 1* if $p \equiv \pm 1 \pmod 5$ and *type 2* if $p \equiv \pm 2 \pmod 5$. Let $\left(\frac{a}{p}\right)$ be the Legendre symbol, equal to 0 if $a \equiv 0 \pmod p$, equal to 1 if a is a quadratic residue modulo p, and equal to -1 otherwise. Thus $\left(\frac{p}{5}\right)$ is 1 if p is type 1 and is -1 if p is type 2.

Claim 2. If p is type 1, then $f(p)$ divides $p-1$. If p is type 2, then $f(p)$ divides $2(p+1)$.

Proof. All congruences here are modulo p. Expand the Binet formula for the Fibonacci numbers to obtain

$$F_m = \frac{(1+\sqrt{5})^m - (1-\sqrt{5})^m}{2^m\sqrt{5}} = \frac{1}{2^{m-1}} \sum_{i=0}^{\lceil m/2 \rceil - 1} 5^i \binom{m}{2i+1}.$$

We examine this formula for $m = p - 1$ and $m = p$, reducing modulo the prime p and using $2^{p-1} \equiv 1 \pmod{p}$. When $m = p$, all terms in the sum except the last one are divisible by p, and so F_p is congruent modulo p to $5^{(p-1)/2}$, which is $\left(\frac{5}{p}\right)$. By the law of quadratic reciprocity, $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$, and so $F_p \equiv 1$ in the type 1 case and $F_p \equiv -1$ in the type 2 case.

When $m = p - 1$, note that $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$. Hence

$$F_{p-1} \equiv 2^{p-1} F_{p-1} \equiv 2 \sum_{i=0}^{(p-3)/2} 5^i (-1)^{2i+1} \equiv -2 \left(\frac{5^{(p-1)/2} - 1}{5 - 1} \right),$$

and so $F_{p-1} \equiv 0$ in the type 1 case and $F_{p-1} \equiv 1$ in the type 2 case. In the type 1 case, we have $(F_{p-1}, F_p) \equiv (0, 1)$ and hence $F_{p-2} \equiv 1$ and $A^{p-1} = I$. In the type 1 case, we have $(F_{p-1}, F_p) \equiv (1, -1)$ and hence $F_{p+1} \equiv 0$, $F_{p+2} = -1$, and $A^{p+1} = -I$. ∎

We now combine the preceding facts to get the desired bound on $f(m)$. Consider an arbitrary modulus m with prime factorization $m = 2^a 5^b \prod p_i^{c_i} \prod q_j^{d_j}$, where $a, b \geq 0$ and p_i and q_j range, respectively, over all type 1 and type 2 primes that divide m. We have

$$f(m) \leq \mathrm{lcm}_{i,j} \left\{ \left[3 \cdot 2^{a-1}\right]_2, \left[4 \cdot 5^b\right]_5, (p_i - 1)\, p_i^{c_i - 1}, 2\left(q_j + 1\right) q_j^{d_j - 1} \right\},$$

where $[x]_p$ is x if p divides m and 1 otherwise. It follows that

$$\begin{aligned} \frac{f(m)}{m} &\leq \frac{\left[3 \cdot 2^{a-1}\right]_2}{2^a} \cdot \prod_i \frac{p_i - 1}{p_i} \cdot \frac{\mathrm{lcm}_j \left\{4 \cdot 5^b, 4\left(q_j + 1\right)/2 \cdot q_j^{d_j - 1}\right\}}{5^b \prod_j q_j^{d_j}} \\ &\leq \frac{3}{2} \cdot 4 \cdot \prod_i \frac{p_i - 1}{p_i} \cdot \prod_j \frac{q_j + 1}{2q_j} \leq 6. \end{aligned} \qquad (*)$$

This proves that $f(m) \leq 6m$.

The inequality in $(*)$ is strict if either 2 or 5 does not divide m or if m has any type 1 divisors p_i or type 2 divisors q_j. Thus $f(m)$ can equal $6m$ only if m has the form $2^a 5^b$ for $a, b \geq 1$. In that case, $f(m) \leq \mathrm{lcm}\left(3 \cdot 2^{a-1}, 4 \cdot 5^b\right) = 3 \cdot 5^b \cdot \mathrm{lcm}\left(2^{a-1}, 4\right)$. If $a \geq 2$, then $\mathrm{lcm}\left(2^{a-1}, 4\right)/2^a \leq 1$, so $6m$ cannot be reached. If $a = 1$, then $f(m) = \mathrm{lcm}\left(3, 4 \cdot 5^b\right) = 3 \cdot 4 \cdot 5^b = 6m$, so $f(m) = 6m$ exactly when m is one of the infinitely many values $2 \cdot 5^b$ with $b \geq 1$.

Editorial Comments: The present problem appeared as part of Problem E3410 [1990, 916; 1992, 278] in this MONTHLY, with solution by Kevin Brown.

The first detailed investigation into Fibonacci periodicity was D. D. Wall (1960), Fibonacci series modulo m, this MONTHLY 67, 525–532. Included there is an alternative proof of Claim 2 that uses linear algebra, along the following lines: For type 1 primes p, the characteristic polynomial of A factors into distinct factors as $x^2 - x - 1 = (x - \alpha)(x - \beta)$ in the field with p elements. Hence A is similar modulo p to the diagonal matrix $\mathrm{diag}(\alpha, \beta)$, and so $A^{p-1} \equiv I \pmod{p}$, because both α and β have multiplicative order dividing $p - 1$. A similar argument exists for type 2 primes, though then α and β reside in the field with p^2 elements.

It is easy to see that the Fibonacci sequence modulo m is in fact periodic (and not merely eventually periodic), which explains the implicit assumption in the problem statement.

When p is prime, $f\left(p^2\right)$ equals either $pf(p)$ or $f(p)$. Primes p with $f(p^2) = f(p)$ are known as Wall–Sun–Sun primes. Surprisingly, it is unknown if any such primes exist. There are none less than 10^{14} (A.-S. Elsenhans and J. Jahnel, The Fibonacci sequence modulo p^2—an investigation by computer for $p < 10^{14}$, arxiv.org/pdf/1006.0824.pdf).

The editors thank Joe Buhler for his contribution in producing the solution here.

SOLUTIONS

An Exponential Field Homomorphism

12301 [2022, 186]. *Proposed by Jan Mycielski, University of Colorado, Boulder, Colorado.* Suppose that $\alpha : \mathbb{C} \to \mathbb{C}$ respects addition and exponentiation, in the sense that $\alpha(x + y) = \alpha(x) + \alpha(y)$ and $\alpha(e^x) = e^{\alpha(x)}$ for all complex numbers x and y. (An example is complex conjugation: $\alpha(z) = \bar{z}$.)
(a) Prove $\alpha(\sqrt{2}) = \sqrt{2}$.
(b)* Must it be the case that $\alpha(2^{1/3}) = 2^{1/3}$? What about $\alpha(2^{1/4}) = 2^{1/4}$ or $\alpha(\ln 2) = \ln 2$?

Solution to **(a)** *by Jayanta Manoharmayum, University of Sheffield, Sheffield, UK.* We first show that α is a field homomorphism. To see this, note first that $\alpha(0) = \alpha(0 + 0) = \alpha(0) + \alpha(0)$, so $\alpha(0) = 0$. Hence $\alpha(1) = \alpha(e^0) = e^{\alpha(0)} = e^0 = 1$. To check that α respects multiplication, let $x, y \in \mathbb{C}$. If either $x = 0$ or $y = 0$, then $\alpha(xy) = 0 = \alpha(x)\alpha(y)$. If not, then we can find complex numbers u and v such that $x = e^u$ and $y = e^v$, and therefore

$$\alpha(xy) = \alpha(e^{u+v}) = e^{\alpha(u+v)} = e^{\alpha(u)+\alpha(v)} = e^{\alpha(u)}e^{\alpha(v)} = \alpha(x)\alpha(y).$$

Being a field homomorphism, the map α is injective and fixes every rational number. From $-1 = \alpha(-1) = \alpha(e^{i\pi}) = e^{\alpha(i\pi)}$ we conclude that $\alpha(i\pi) = in\pi$ for some odd integer n. Since

$$\alpha(e^{i\pi/n}) = e^{\alpha(i\pi)\alpha(1/n)} = e^{in\pi/n} = -1 = \alpha(-1),$$

we must have $e^{i\pi/n} = -1$, so $n = \pm 1$, and hence $\alpha(i\pi) = \pm i\pi$. From $\sqrt{2} = e^{i\pi/4} + e^{-i\pi/4}$, it follows that

$$\alpha(\sqrt{2}) = \alpha(e^{i\pi/4} + e^{-i\pi/4}) = e^{\pm i\pi/4} + e^{\mp i\pi/4} = \sqrt{2}.$$

Editorial comment. Let $K \subseteq \mathbb{C}$ be the maximal cyclotomic extension of $\mathbb{Q}$ (that is, the field extension of $\mathbb{Q}$ generated by the roots of unity $e^{ir\pi}$ for $r \in \mathbb{Q}$). The argument above shows that $\alpha(e^{ir\pi}) = e^{\alpha(r)\alpha(i\pi)} = e^{\pm ir\pi}$ for all rational r. Hence either α acts on K trivially, or α acts on K as complex conjugation. The Kronecker–Weber theorem then implies that $\alpha(\theta) = \theta$ for any $\theta \in \mathbb{R}$ with $\mathbb{Q}(\theta)/\mathbb{Q}$ an abelian extension. In particular, $\alpha(\sqrt{r}) = \sqrt{r}$ for any positive rational r.

A sketch of a solution to part **(a)** of this problem appeared in J. Mycielski (1985), Remarks on infinite systems of equations, *Alg. Univ.* 21, 307–309.

No correct solutions to **(b)** were received.

Part **(a)** also solved by J. Boswell & C. Curtis, N. Caro-Montoya (Brazil), G. A. Edgar, N. Grivaux (France), E. A. Herman, Y. J. Ionin, O. P. Lossers (Netherlands), G. Plumpton & R. Su (Canada), M. Reid, K. Schilling, A. Stenger, R. Stong, Missouri State University Problem Solving Group, and the proposer.

Where Angle Bisectors Meet Opposite Sides

12303 [2022, 186]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E, and F be chosen on sides BC, CA, and AB so that AD, BE, and CF bisect the angles of ABC. Prove

$$\frac{FD}{AB+BC} + \frac{DE}{BC+CA} + \frac{EF}{CA+AB} \le \frac{3}{8}\left(1 + \frac{R}{2r}\right).$$

Composite solution by Richard Stong, Center for Communications Research, San Diego, CA, and Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. We prove

$$\frac{FD}{AB+BC} + \frac{DE}{BC+CA} + \frac{EF}{CA+AB} \le \frac{3}{4},$$

which, by Euler's inequality $R \ge 2r$, implies the stated inequality.

Let a, b, and c denote the lengths of sides BC, CA, and AB, respectively. By the angle bisector theorem, we have $BD = ac/(b+c)$ and $BF = ac/(a+b)$. Therefore, by the law of cosines (twice),

$$\begin{aligned} FD^2 &= \left(\frac{ac}{b+c}\right)^2 + \left(\frac{ac}{a+b}\right)^2 - 2\left(\frac{ac}{b+c}\right)\left(\frac{ac}{a+b}\right)\cos(\angle ABC) \\ &= \left(\frac{ac}{b+c}\right)^2 + \left(\frac{ac}{a+b}\right)^2 - 2\left(\frac{ac}{b+c}\right)\left(\frac{ac}{a+b}\right)\left(\frac{a^2+c^2-b^2}{2ac}\right) \\ &= \frac{abc(b^3+ab^2+cb^2+3abc-a^2b-c^2b+a^2c+c^2a-a^3-c^3)}{(a+b)^2(b+c)^2}. \end{aligned}$$

Combining this with similar formulas for DE^2 and EF^2, we obtain

$$\left(\frac{FD}{AB+BC}\right)^2+\left(\frac{DE}{BC+CA}\right)^2+\left(\frac{EF}{CA+AB}\right)^2= \frac{abc(9abc+a^2b+a^2c+b^2c+b^2a+c^2a+c^2b-a^3-b^3-c^3)}{(a+b)^2(b+c)^2(c+a)^2}.$$

By Schur's inequality, $a^2b+a^2c+b^2c+b^2a+c^2a+c^2b-a^3-b^3-c^3\le 3abc$, so

$$\left(\frac{FD}{AB+BC}\right)^2+\left(\frac{DE}{BC+CA}\right)^2+\left(\frac{EF}{CA+AB}\right)^2\le\frac{12(abc)^2}{(a+b)^2(b+c)^2(c+a)^2}$$
$$\le\frac{12(abc)^2}{(4ab)(4bc)(4ac)}=\frac{3}{16}.$$

The desired conclusion now follows by the Cauchy–Schwarz inequality.

Editorial comment. Several solvers noted that this problem is related to problem 12182 [2020, 461; 2022, 92] from this MONTHLY. Indeed, the inequality $EF\le(2a+b+c)/8$, derived in the published solution there, can be used as the basis for another solution to this problem.

Also solved by M. Bataille (France), N. S. Dasireddy (India), M. Drăgan & N. Stanciu (Romania), G. Fera (Italy), O. Geupel (Germany), N. Hodges (UK), W. Janous (Austria), C. G. Petalas (Greece), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Tauraso (Italy), M. Vowe (Switzerland), L. Zhou, and the proposer.

An Interpolation Identity

12304 [2022, 186]. *Proposed by Michel Bataille, Rouen, France.* Let m and n be positive integers with $m<n$. Prove

$$\left(\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n-k}\right)\left(\sum_{k=0}^{m}\binom{n}{k}\frac{(-1)^k}{k+1}\right)=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{(n-k)(k+1)}.$$

Solution by Pierre Lalonde, Plessisville, QC, Canada. For a nonnegative integer k, extend the binomial coefficient in the usual way as

$$\binom{x}{k}=\frac{x(x-1)\cdots(x-k+1)}{k!}.$$

We prove the polynomial identity

$$\sum_{k=0}^{m}\binom{x}{k}\frac{(-1)^k}{k+1}=\sum_{k=0}^{m}(-1)^{m-k}\binom{m+1}{k+1}\frac{\binom{x}{m+1}}{x-k}. \tag{1}$$

(The right side is a polynomial since $x-k$ divides $\binom{x}{m+1}$.) Evaluating the left side of (1) at $x=j$ for $0\le j\le m$ yields

$$\sum_{k=0}^{m}\binom{j}{k}\frac{(-1)^k}{k+1}=\frac{-1}{j+1}\sum_{k=0}^{j}\binom{j+1}{k+1}(-1)^{k+1}=\frac{-1}{j+1}\left((1-1)^{j+1}-1\right)=\frac{1}{j+1}.$$

When we evaluate the right side of (1) at $x=j$, the only term in the sum that is nonzero is the one with $k=j$. Thus

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m+1}{k+1}\frac{\binom{x}{m+1}}{x-k}=(-1)^{m-j}\binom{m+1}{j+1}\frac{j!(-1)^{m-j}(m-j)!}{(m+1)!}=\frac{1}{j+1}.$$

Since both sides of (1) are polynomials of degree m that agree on $m+1$ values, they are equal.

Dividing both sides of (1) by $(-1)^m \binom{x}{m+1}(m+1)$, we have

$$\frac{(-1)^m}{\binom{x}{m+1}(m+1)} \sum_{k=0}^{m} \binom{x}{k} \frac{(-1)^k}{k+1} = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(k+1)(x-k)}. \tag{2}$$

Expanding the coefficient on the left side of (2) by partial fractions yields

$$\frac{(-1)^m}{\binom{x}{m+1}(m+1)} = \frac{(-1)^m m!}{x(x-1)\cdots(x-m)} = \sum_{k=0}^{m} \frac{a_k}{x-k},$$

for some coefficients $a_0, \ldots, a_m$. To compute these coefficients, clear fractions and set $x = j$. Only the term for $k = j$ survives, and so we obtain

$$(-1)^m m! = \sum_{k=0}^{m} \frac{a_k j(j-1)\cdots(j-m)}{j-k} = a_j j!(-1)^{m-j}(m-j)!\,.$$

Thus $a_j = (-1)^j \binom{m}{j}$. Substituting this expansion into (2) gives

$$\left(\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{x-k}\right)\left(\sum_{k=0}^{m} \binom{x}{k} \frac{(-1)^k}{k+1}\right) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{(x-k)(k+1)}.$$

Evaluating at $x = n$ when $n > m$ yields the result.

Editorial comment. Lalonde notes that the left and right sides of (1) are the Newton and Lagrange interpolation polynomials, respectively, for the points $\{(j, \frac{1}{j+1}) : 0 \le j \le m\}$.

Most solvers evaluated the three sums individually, either in terms of Euler's beta integrals or by induction using binomial identities.

Also solved by U. Abel (Germany), A. Berkane (Algeria), P. Bracken, C. Curtis, G. Fera (Italy), O. Geupel (Germany), N. Hodges (UK), W. Janous (Austria), O. Kouba (Syria), O. P. Lossers (Netherlands), F. Masroor, E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

Laplace Simplifies an Integral

12305 [2022, 187]. *Proposed by Shivam Sharma, Delhi University, New Delhi, India.* Prove

$$\int_0^1 \frac{x-1-x\ln x}{x\ln x - x\ln^2 x}\,dx = \gamma,$$

where γ is Euler's constant $\lim_{n\to\infty}(-\ln n + \sum_{k=1}^{n} 1/k)$.

Solution by Seán Stewart, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia. Denote the integral to be calculated by I. The substitution $x = e^{-t}$ produces

$$I = \int_0^\infty \frac{1-e^{-t}(1+t)}{t(1+t)}\,dt.$$

To evaluate this integral, we use the Laplace transform $\mathcal{L}$, defined by $\mathcal{L}\{g\}(t) = \int_0^\infty g(s)e^{-st}\,ds$. In particular, we use the property that, for positive functions f and g,

$$\int_0^\infty f(t)\cdot\mathcal{L}\{g\}(t)\,dt = \int_0^\infty \mathcal{L}\{f\}(s)\cdot g(s)\,ds, \tag{$*$}$$

as long as both improper integrals are defined. This property is proved by expressing both sides of $(*)$ as double integrals and reversing the order of integration.

From elementary properties of the Laplace transform we have

$$\mathcal{L}\{1-e^{-t}(1+t)\}(s)=\mathcal{L}\{1\}(s)-\mathcal{L}\{e^{-t}\}(s)-\mathcal{L}\{te^{-t}\}(s)=\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}$$

and

$$\mathcal{L}\{1-e^{-s}\}(t)=\mathcal{L}\{1\}(t)-\mathcal{L}\{e^{-s}\}(t)=\frac{1}{t}-\frac{1}{t+1}=\frac{1}{t(t+1)}.$$

Applying $(*)$ and then integration by parts, we get

$$\begin{aligned}
I&=\int_0^\infty (1-e^{-t}(1+t))\cdot\mathcal{L}\{1-e^{-s}\}(t)\,dt\\
&=\int_0^\infty \mathcal{L}\{1-e^{-t}(1+t)\}(s)\cdot(1-e^{-s})\,ds\\
&=\int_0^\infty\left(\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right)\cdot(1-e^{-s})\,ds\\
&=\left(\ln s-\ln(s+1)+\frac{1}{s+1}\right)(1-e^{-s})\Big|_0^\infty-\int_0^\infty\left(\ln s-\ln(s+1)+\frac{1}{s+1}\right)e^{-s}\,ds\\
&=-\int_0^\infty e^{-s}\ln s\,ds-\int_0^\infty e^{-s}\left(\frac{1}{s+1}-\ln(s+1)\right)ds=\gamma-\left[e^{-s}\ln(s+1)\right]_0^\infty=\gamma,
\end{aligned}$$

where in the last line we use the well-known integral representation $\gamma=-\int_0^\infty e^{-x}\ln x\,dx$. (See, for example, F. W. J. Olver et al. (2010), *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, p. 140, Eq. 5.9.17.)

Also solved by E. Alan, T. Amdeberhan & V. H. Moll, M. Bataille (France), A. Berkane (Algeria), N. Bhandari (Nepal), P. Bracken, B. Bradie, B. S. Burdick, W. Chang, H. Chen, H. Chen & F. Zhuang (Canada), B. E. Davis, G. Fera (Italy), D. Fleischman, M. L. Glasser (Spain), R. Gordon, J.-P. Grivaux (France), L. Han, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), S. Kaczkowski, A. M. Karparvar (Iran), O. Kouba (Syria), O. P. Lossers (Netherlands), F. Masroor, M. Omarjee (France), H. Ricardo, V. Schindler (Germany), T. P. Sharma (India), A. Stadler (Switzerland), M. Štofka (Slovakia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), J. Vukmirović (Serbia), T. Wiandt, H. Widmer, J. Yan (China), L. Zhou, Fejéntaláltuka Szeged Problem Solving Group (Hungary), and the proposer.

A Sum of Euler and von Mangoldt Functions

12306 [2022, 187]. *Proposed by Amrit Awasthi, Amritsar, India.* For a positive integer n, evaluate

$$\sum_{a|n}\phi(a)\ln a+\sum_{a|n}\sum_{b|(n/a)}\phi(a)\Lambda(b),$$

where ϕ is the Euler phi function ($\phi(m)$ is the number of integers k with $1\le k\le m$ that are relatively prime to m) and Λ is the von Mangoldt function ($\Lambda(m)$ equals $\ln p$ when m is a power of the prime number p and equals 0 when m is not a prime power).

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The sum equals $n\ln n$. We use the well-known identities $\sum_{b|n}\Lambda(b)=\ln n$ and $\sum_{a|n}\phi(a)=n$. The first identity follows from the prime factorization $n=p_1^{m_1}\cdots p_r^{m_r}$: for each i, the m_i powers of p_i each contribute $\ln p_i$ to the sum, and all other terms are zero. The second follows because $\phi(a)$ counts the elements k of $\{1,2,\dots,n\}$ such that $\gcd(k,n)=n/a$ for each divisor a of n.

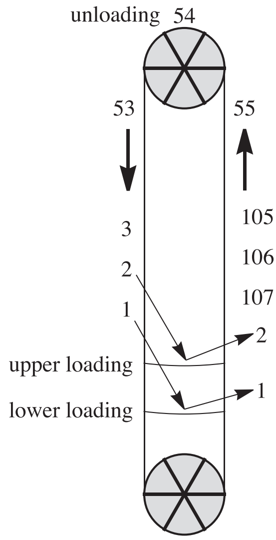
Using these identities, we obtain

$$\sum_{a|n}\phi(a)\ln a+\sum_{a|n}\sum_{b|(n/a)}\phi(a)\Lambda(b)=\sum_{a|n}\phi(a)\left(\ln a+\ln\frac{n}{a}\right)=\ln n\sum_{a|n}\phi(a)=n\ln n.$$

Also solved by B. S. Burdick, C. Burnette, N. Caro-Montoya (Brazil), W. Chang, C. Curtis, T. Dickens, O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin, W. Janous (Austria), A. M. Karparvar (Iran), K. T. L. Koo (China), O. Kouba (Syria), O. P. Lossers (Netherlands), R. Molinari, M. Reid, H. Ricardo, A. Stadler (Switzerland), R. Tauraso (Italy), L. Zhou, Missouri State University Problem Solving Group, and the proposer.

Double-Loading Six-Pack

12307 [2022, 285]. *Proposed by Stuart Boersma, Central Washington University, Ellensburg, WA, Kim Ruhland, Breckenridge, CO, and Bruce Torrence, Randolph-Macon College, Ashland, VA.* Consider a ski lift with n chairs attached to a cable loop. Let m be an integer such that $1 \le m \le n$. At each loading stage at the bottom, the lowest m descending chairs are detached from the cable in order, loaded with skiers, and then reattached in the reverse order but otherwise at the same locations around the cable from which they were removed (see figure for the case $n = 107$ and $m = 2$; a lift of this type is used at the Breckenridge ski area). At the next stage, the same steps are carried out with the next m descending chairs; the process continues indefinitely. After how many loading stages are the chairs returned to the same cyclic order they had at the beginning?



Solution by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA. Let $f(n, m)$ denote the (minimum) number of loading stages required before the chairs are returned to the same cylic order they had at the beginning. If $m = 1$, then there is no change in the order of the chairs after each loading stage, so $f(n, 1) = 1$ for each n. Henceforth we assume $m > 1$.

Let q and r be the unique integers such that $n = qm + r$ and $0 \le r < m$. We prove

$$f(n,m)=\begin{cases}2q & \text{if } r=0;\\ q(q+1) & \text{if } r=1 \text{ and } q \text{ is odd, or } r=m-1 \text{ and } q \text{ is even;}\\ 2q(q+1) & \text{otherwise.}\end{cases}$$

If $r = 0$, then after the first q loading stages each successive group of m chairs is internally reversed. Since $m > 1$, this differs from the original order. After q more stages, the original order within each group is restored, and the order of the groups is unchanged, so $f(n, m) = 2q$.

Now suppose $r > 0$. When k is a nonnegative integer, let $A_k = (mk + 1, \dots, mk + r)$ and $B_k = (mk + r + 1, \dots, mk + m)$. Let $\overline{A}_k$ and $\overline{B}_k$ denote the reversals of A_k and B_k, respectively, and let S_k denote the order of the chairs after k loading stages. We assume the initial order S_0 is $(1, \dots, n)$, so

$$S_0 = A_0B_0A_1B_1 \cdots A_{q-1}B_{q-1}A_q.$$

After each loading stage, the first (leftmost) m chairs are reversed and moved to the end of the cable, so

$$S_1 = A_1 B_1 A_2 B_2 \cdots A_q \overline{B}_0\ \overline{A}_0$$

and

$$S_{q+1} = \overline{A}_0\ \overline{B}_1\ \overline{A}_1 \cdots \overline{B}_{q-1}\ \overline{A}_{q-1} B_0 \overline{A}_q.$$

Notice that in each configuration, exactly one pair of A_k groups are adjacent (possibly with one or both being reversed). In the initial configuration, A_0 and A_q are adjacent. After one loading stage, A_1 and A_0 are adjacent (and A_0 has been reversed); after two loading stages, the adjacent pair is A_2 and A_1. After $q+1$ loading stages, the adjacent pair is again A_0 and A_q, and then the pattern repeats. It follows that $f(n,m)$ must be a multiple of $q+1$. After the first $q+1$ loading stages, the groups A_k are back in their original positions, but reversed, while the positions of the B_k have shifted by one position. The group B_0 has been reversed twice, so it is back in its original order, and every other B_k has been reversed once. Each set of $q+1$ loading stages has a similar effect, so the first time that the A_k and B_k groups are back in their original cyclic order is after q sets of $q+1$ loading stages, at which point each A_k has been reversed q times and each B_k has been reversed $q+1$ times. Therefore

$$S_{q(q+1)} = \begin{cases} \overline{A}_0 B_0 \overline{A}_1 B_1 \cdots \overline{A}_{q-1} B_{q-1} \overline{A}_q & \text{if } q \text{ is odd} \\ A_0 \overline{B}_0 A_1 \overline{B}_1 \cdots A_{q-1} \overline{B}_{q-1} A_q & \text{if } q \text{ is even.} \end{cases}$$

If $r=1$, then $\overline{A}_k = A_k$, while if $r = m-1$ then $\overline{B}_k = B_k$. Thus $S_{q(q+1)} = S_0$ when $r=1$ and q is odd, and when $r = m-1$ and q is even. Otherwise, we need another $q(q+1)$ loading stages for the A_k and B_k groups to once again return to their original cyclic order, and $S_{2q(q+1)} = S_0$. This establishes the desired result for $f(n,m)$.

Since $107 = 53 \cdot 2 + 1$, for the Breckenridge ski lift $r=1$ and q is odd, so the number of loading stages is $f(107,2) = q(q+1) = 53 \cdot 54 = 2862$.

Editorial comment. The Quicksilver Super6 ski lift at Breckenridge seats 6 skiers per chair and is the only ski lift in North America to have double loading. This arrangement is known to the locals as a “double-loading six-pack”.

The problem is a discrete version of classic problem C4 [2022, 394; 2022, 495] from this MONTHLY, and the methods of solution are similar.

Also solved by C. Farnsworth, K. Gatesman, O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin, O. P. Lossers (Netherlands), A. Mandal (India), R. Stong, L. Zhou, and the proposers.

Eliminating Tiles

12309 [2022, 286]. *Proposed by Joseph DeVincentis, Salem, MA, Thomas C. Occhipinti, Luther College, Decorah IA, and Daniel J. Velleman, Amherst College, Amherst, MA, and University of Vermont, Burlington, VT.* Consider a square grid that is infinite in all directions, with tiles placed on finitely many squares of the grid. Two grid squares are called adjacent if they share an edge. There are two types of legal moves:
(A) If two tiles are on adjacent squares, then they can both be removed.
(B) If a tile is on a square and all adjacent squares are unoccupied, then the tile can be removed with four new tiles then placed on the four adjacent squares.
For which initial configurations is it possible to eliminate all tiles from the grid?

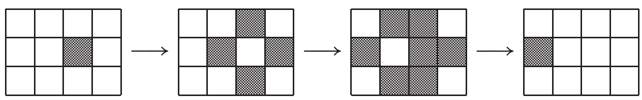
Solution by José Heber Nieto, University of Zulia, Maracaibo, Venezuela. Color the squares of the grid alternating black and white, as in an infinite chessboard. Given a distribution of tiles, let w and b denote the number of tiles placed on white and black squares, respectively. We prove that it is possible to eliminate all of the tiles if and only if $w \equiv b \pmod 5$.

The necessity of $w \equiv b \pmod 5$ holds because A-moves do not change $w - b$ and B-moves increase or decrease $w - b$ by 5. Hence $w - b \pmod 5$ is an invariant and, if it is possible to eliminate all of the tiles, then $w - b \equiv 0 \pmod 5$.

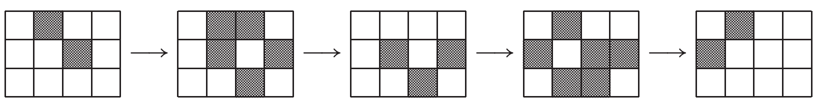
To prove sufficiency, we apply induction on $\max\{w, b\}$, beginning with the obvious case $w = b = 0$. If any tiles are adjacent, we can apply an A-move and then apply the induction hypothesis. If no tiles are adjacent and $w \neq b$, say $w > b$ without loss of generality, we apply a B-move to any tile on a white square, leaving $w - 1$ tiles on white squares and $b + 4$ tiles on black squares. Because $w \geq b + 5$, we may again apply the induction hypothesis. Thus we may assume $w = b > 0$ and that no tiles are adjacent.

Now choose a tile on a white square and one on a black square so that the Euclidean distance between the centers of the squares is minimal. We may assume that the squares are unit squares, assign integral coordinates to their centers, and name each square by its center. By symmetry, we may assume one tile is on $(0, 0)$ and the other on (a, b), where $a > b \geq 0$. Observe that $a \geq 2$ and that a and b have opposite parity. Squares $(a \pm 1, b)$ and $(a, b \pm 1)$ are adjacent to (a, b), so they are empty. Squares $(a - 2, b)$ and $(a - 1, b - 1)$ are also empty because they are closer to $(0, 0)$ than is (a, b).

Case 1: $(a - 1, b + 1)$ *is empty.* Apply a B-move to (a, b), a B-move to $(a - 1, b)$, and A-moves to $(a - 1, b + 1)$ and $(a, b + 1)$, to (a, b) and $(a + 1, b)$, and to $(a - 1, b - 1)$ and $(a, b - 1)$. The net effect is to move the tile on (a, b) to $(a - 2, b)$.



Case 2: $(a - 1, b + 1)$ *is occupied.* Again first apply a B-move to (a, b), this time followed by applying an A-move to $(a - 1, b + 1)$ and $(a, b + 1)$. Now apply a B-move to $(a - 1, b)$ and A-moves to (a, b) and $(a + 1, b)$ and to $(a - 1, b - 1)$ and $(a, b - 1)$. The net effect is again to move the tile on (a, b) to $(a - 2, b)$.



Case 1 and Case 2 both reduce the distance between $(0, 0)$ and the nearest square of the opposite color. Hence iterating the appropriate case brings a tile next to $(0, 0)$. At that point an A-move removes the pair, and the induction hypothesis applies.

Also solved by J. Boswell & C. Curtis, V. Chen & O. Zhang, K. Gatesman, O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin, O. P. Lossers (Netherlands), A. Martin & P. Martin & R. Martin (Germany), K. Schilling, R. Stong, R. Tauraso (Italy), and the proposer.

Parallel Segments and Concurrent Cevians

12310 [2022, 286]. *Proposed by Thanos Kalogerakis, Kiato, Greece, Dan-Stefan Marinescu, Hunedoara, Romania, and Mehmet Şahin, Ankara, Turkey.* Let P be a point inside triangle ABC, and let D, E, and F be points on BC, CA, and AB, respectively, such that PE, PF, and PD are parallel to AB, BC, and CA, respectively. Let A', B', and C' be points on BC, CA, and AB, respectively, such that AA', BB', and CC' are parallel to DE, EF, and FD, respectively.

(a) Prove $\text{Area}(ABC) \geq 3 \cdot \text{Area}(DEF)$, and determine conditions for equality.

(b) Prove that AA', BB', and CC' are concurrent.

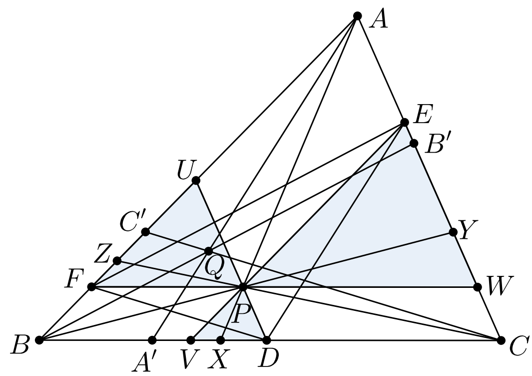
(c) Prove $PD \cdot PE \cdot PF \geq 8 \cdot A'D \cdot B'E \cdot C'F$, and determine conditions for equality.

Solution by Faraz Masroor, New York, NY.

(a) Suppose that AP, BP, and CP intersect BC, CA, and AB at X, Y, and Z, respectively. Let $x = PX/AX$, $y = PY/BY$, and $z = PZ/CZ$. For any polygon $IJKL$, denote its area by $[IJKL]$. We have $x = [PBC]/[ABC]$, $y = [PCA]/[ABC]$, and $z = [PAB]/[ABC]$. Thus $x + y + z = 1$. By the Cauchy–Schwarz inequality,

$$(1+1+1)\left(x^2+y^2+z^2\right) \geq (x+y+z)^2 = 1,$$

with equality if and only if $x = y = z = 1/3$. Suppose that PD, PE, and PF intersect AB, BC, and CA at U, V, and W, respectively. We have $[PVD]/[ABC] = x^2$, $[PWE]/[ABC] = y^2$, and $[PUF]/[ABC] = z^2$. Also, $[PDE] = [PDC] = [PDCW]/2$. Likewise, $[PEF] = [PEAU]/2$ and $[PFD] = [PFBV]/2$. Therefore



$$\begin{aligned}[DEF] &= [PDE] + [PEF] + [PFD] \\ &= \frac{[ABC] - [PVD] - [PWE] - [PUF]}{2} \\ &= \frac{[ABC]}{2}\left(1 - x^2 - y^2 - z^2\right) \leq \frac{[ABC]}{3},\end{aligned}$$

with equality if and only if P is the centroid of triangle ABC.

(b) Since $\angle BAA' = \angle PED$ and $\angle A'AC = \angle EDP$, applying the law of sines in $\triangle PDE$ we get $\sin\angle BAA'/\sin\angle A'AC = PD/PE$. Multiplying this by the other two analogous equations, we have

$$\frac{\sin\angle BAA'}{\sin\angle A'AC}\cdot\frac{\sin\angle CBB'}{\sin\angle B'BA}\cdot\frac{\sin\angle ACC'}{\sin\angle C'CB} = \frac{PD}{PE}\cdot\frac{PE}{PF}\cdot\frac{PF}{PD} = 1.$$

Therefore, by the trigonometric form of Ceva's theorem, AA', BB', and CC' concur at some point Q.

(c) Let $a = BC$, $b = CA$, and $c = AB$. Notice that $PD = bx$, $PE = cy$, and $PF = az$. Also, $AE = UP = bz$, $DC = PW = ay$, and $EC = EW + PD = by + bx$. From $A'D/AE = DC/EC$ we have $A'D = ayz/(x+y)$. Likewise, $B'E = bzx/(y+z)$ and $C'F = cxy/(z+x)$. The AM–GM inequality now yields

$$\frac{PD\cdot PE\cdot PF}{A'D\cdot B'E\cdot C'F} = \frac{(x+y)(y+z)(z+x)}{xyz} \geq 8.$$

Equality holds if and only if $x = y = z = 1/3$, that is, exactly when P is the centroid of triangle ABC.

Editorial comment. The problem statement here corrects a typographical error in the original statement of the problem.

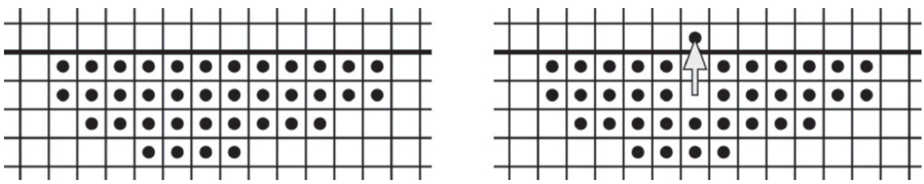
Also solved by M. Bataille (France), H. Chen (China), C. Curtis, G. Fera (Italy), K. Gatesman, O. Geupel (Germany), O. Kouba (Syria), O. P. Lossers (Netherlands), C. Petalas (Greece), C. R. Pranesachar (India), V. Schindler (Germany), R. Stong, L. Zhou, Davis Problem Solving Group, and the proposer. Parts **(a)** and **(b)** also solved by J. P. Grivaux.

CLASSICS

C20. *Due to Peter Freyd, suggested by the editors.* Given a positive integer m, let $f(m)$ be the period of the Fibonacci sequence taken modulo m. Prove $f(m) \le 6m$ and that equality holds for infinitely many m.

Conway's Solitaire Army

C19. *Due to John H. Conway, suggested by the editors.* A battlefield is modeled by an infinite grid of unit squares, whose centers are indexed by $\{(a, b): a, b \in \mathbb{Z}\}$. The soldiers are modeled by pegs, which are placed initially at a finite number of squares (a, b) with $b \le 0$. The soldiers advance by jumping in the style of peg solitaire: A jump is permitted when there are three squares in the grid forming a 1-by-3 rectangle with one end square of this rectangle empty while the other two squares are occupied by pegs. Where this configuration exists, the peg on the end may jump over the peg in the middle and move to the empty end, while the peg in the middle is removed. How many pegs are needed in an initial configuration to allow a peg to advance to the square $(0, 5)$?



A possible initial configuration   A possible initial jump

Solution. It is impossible to advance a peg to the square $(0, 5)$, no matter how many pegs are in the initial configuration. To see this, assign to the square (a, b) the weight $\lambda^{|a|-b}$, where $\lambda = (\sqrt{5}-1)/2$. This λ is chosen so that $0 < \lambda < 1$ and $\lambda + \lambda^2 = 1$. For any position of pegs on the grid, define the *weight* of the position to be the sum of the weights of the squares that are occupied by pegs. The weight of the entire halfplane with $b \le 0$ is

$$\sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{0} \lambda^{|a|-b} = \sum_{a=0}^{\infty} \lambda^a \sum_{b=0}^{\infty} \lambda^b + \sum_{a=1}^{\infty} \lambda^a \sum_{b=0}^{\infty} \lambda^b$$
$$= \frac{1}{1-\lambda}\frac{1}{1-\lambda} + \frac{\lambda}{1-\lambda}\frac{1}{1-\lambda} = \frac{1+\lambda}{(1-\lambda)^2} = \frac{1+\lambda}{\lambda^4} = \frac{1}{\lambda^5}.$$

Since there are only finitely many soldiers at the start, the weight of the original configuration is strictly less than $1/\lambda^5$.

Any jump involves eliminating pegs in squares of weight λ^n and λ^{n+1}, respectively, for some n, and adding a peg to an empty square of weight λ^{n-1} or λ^{n+1} or λ^{n+2}. Since $\lambda^n + \lambda^{n+1} = \lambda^{n-1}$, while λ^{n+1} and λ^{n+2} are smaller, we see that no jump can increase the weight of the position. Yet the weight of the target square $(0, 5)$ is $1/\lambda^5$, which exceeds the weight of the original configuration. Thus no finite number of initial pegs allows a peg to reach the square $(0, 5)$.

Editorial Comment: The number of pegs required in an initial configuration to advance a peg to the square $(0, n)$ for $n = 1, 2, 3$, and 4 is 2, 4, 8, and 20, respectively. The problem has seen many reincarnations and generalizations. It originated with John H. Conway in 1961 and appears in R. Honsberger (1976), A problem in checker jumping, in *Mathematical Gems II*, Mathematical Association of America, pp. 23–28.

SOLUTIONS

Two Cyclic Quadrilaterals

12294 [2022, 86]. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle with center O. Let $B_1B_2B_3B_4$ be the quadrilateral that contains $A_1A_2A_3A_4$ in its interior such that, for $1 \leq i \leq 4$ and with subscripts taken cyclically, B_iB_{i+1} is parallel to A_iA_{i+1} and at distance $|A_iA_{i+1}|$ from it. Because $B_1B_2B_3B_4$ has the same angles as $A_1A_2A_3A_4$, there is a circle in which it is inscribed. Let P be the center of that circle. Show that A_1A_3, A_2A_4, and OP are concurrent.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Lay down complex coordinates with the circumcircle of $A_1A_2A_3A_4$ as the unit circle and $A_1A_2A_3A_4$ oriented clockwise. The coordinate of the circumcenter O is 0. We use lowercase letters to denote the coordinates of the corresponding uppercase points. Hence the complex numbers a_i, for $1 \leq i \leq 4$, have modulus one, so their complex conjugates are their respective inverses.

Let X be the intersection point of the diagonals A_1A_3 and A_2A_4. Its coordinate can be found by treating the equation

$$x = ta_1 + (1-t)a_3 = ua_2 + (1-u)a_4$$

and its complex conjugate as two linear equations in the (real) unknowns t and u. The result is

$$t = \frac{a_1(a_2-a_3)(a_3-a_4)}{(a_1-a_3)(a_1a_3-a_2a_4)}, \quad u = \frac{a_2(a_3-a_4)(a_4-a_1)}{(a_2-a_4)(a_2a_4-a_1a_3)}.$$

Hence

$$x = \frac{a_1a_2a_3 + a_1a_3a_4 - a_1a_2a_4 - a_2a_3a_4}{a_1a_3 - a_2a_4}.$$

The line parallel to A_1A_2, at a distance $|a_2 - a_1|$ from A_1A_2 and outside $A_1A_2A_3A_4$, can be parametrized (by real t) as

$$\ell_{12}(t) = ta_1 + (1-t)a_2 + i(a_2 - a_1),$$

and symmetrically for the other sides. Treating the equation $b_1 = \ell_{12}(t) = \ell_{41}(u)$ and its complex conjugate as two linear equations in the unknowns t and u, we get

$$b_1 = \frac{a_1a_2 - a_1a_4 + 2a_1a_2i + 2a_1a_4i - 4a_2a_4i}{a_2 - a_4},$$

and symmetrically for the other points.

Since P is equidistant from B_1, B_2, and B_3, we have

$$(p - b_1)(\overline{p} - \overline{b_1}) = (p - b_2)(\overline{p} - \overline{b_2}) = (p - b_3)(\overline{p} - \overline{b_3}).$$

This gives two linear equations in the two unknowns p and $\overline{p}$, and solving these equations we find the coordinate of P to be

$$p = \frac{-4i(a_1a_2a_3 + a_1a_3a_4 - a_1a_2a_4 - a_2a_3a_4)}{(a_1 - a_3)(a_2 - a_4)} = \frac{-4i(a_1a_3 - a_2a_4)}{(a_1 - a_3)(a_2 - a_4)} \cdot x.$$

The coefficient of x in the last equation is easily checked to be its own complex conjugate, so it is real. Thus X and P lie on the same line through the origin O, as desired.

Editorial comment. Using a similar argument, Roberto Tauraso showed that the same conclusion holds if each B_iB_{i+1} is at a distance $r|A_iA_{i+1}|$ from A_iA_{i+1}, for any $r > 0$.

Also solved by J.-P. Grivaux (France), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Tauraso (Italy), and the proposer.

A Geometric Progression as a Sum of Two Squares

12295 [2022, 86]. *Proposed by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China.*
(a) Show that when n is an odd positive integer, $1 + 7^n + 7^{2n} + 7^{3n} + 7^{4n} + 7^{5n} + 7^{6n}$ is a sum of two squares.
(b)* Show that when n is even, the expression in part **(a)** is not a sum of two squares.

Solution by Michael Reid, University of Central Florida, Orlando, FL. (**a**) The expression equals $(7^n - 1)^6 + 7^{n+1}(7^{2n} - 7^n + 1)^2$. When n is odd, 7^{n+1} is a square, so the number is the sum of a sixth power and a square.
(**b**) When n is even, the expression is congruent to 7 modulo 8, so it is not even the sum of three squares.

Editorial comment. Several solvers used quadratic reciprocity to solve (**b**), which yields a generalization. For a prime p congruent to 3 modulo 4,

$$1 + p^n + p^{2n} + \cdots + p^{(p-1)n}$$

is the sum of two squares if and only if n is odd. As in the proof above, for even n the sum is congruent to p mod 8, so it is not the sum of two squares. Conversely, factor

$$p^{pn} - 1 = (p^n - 1)(1 + p^n + p^{2n} + \cdots + p^{(p-1)n}).$$

For a prime q dividing the first factor, the second factor is p modulo q, so the two factors are relatively prime. Thus for a prime q dividing the second factor, the multiplicative order of p modulo q is divisible by p. By Lagrange's theorem, p divides $q - 1$, so $q \equiv 1 \pmod{p}$ and q is a square modulo p. Since p has odd order modulo q, it further follows that p is a square modulo q. Therefore quadratic reciprocity yields $(-1)^{(p-1)(q-1)/4} = 1$, implying $q \equiv 1 \pmod 4$. Such primes q are the sum of two squares, as are the products of sums of two squares. See, for instance, G. H. Hardy and E. M. Wright (2008), *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, for an explanation of quadratic reciprocity and the needed facts about sums of two squares.

Other solvers used a theorem of Aurifeuille, Le Lasseur, and Lucas from the 1870s to give another existence proof for (**a**). The theorem implies that, for m odd and square-free, the cyclotomic polynomial $\Phi_m(x)$ may be expressed as $p^2(x) - (-1)^{(m-1)/2} m x q^2(x)$, where the polynomials p and q have integer coefficients. See A. Schinzel (1962), On primitive prime factors of $a^n - b^n$, *Proc. Cambridge Phil. Soc.*, 58(4), 555–562. When $m \equiv 3 \pmod 4$ and n is odd, setting $x = m^n$ generalizes (**a**).

Also solved by M. Chamberland, R. Dietmann & M. Widmer (UK), A. Dixit (India), S. Fan, N. Fellini (Canada), P. Lalonde (Canada), O. P. Lossers (Netherlands), J. Manoharmayum (UK), R. Martin (Germany), J. P. Robertson, J. Silverberg, A. Stenger, R. Stong, Eagle Problem Solvers, and the proposer. Part (**a**) also solved by C. Curtis and O. Geupel (Germany). Part (**b**) also solved by C. Degenkolb, B. Finkel, N. Hodges (UK), and B. Sury (India).

Coloring the Complement of a Matching

12296 [2022, 86]. *Proposed by David A. Kalarkop and R. Rangarajan, University of Mysore, Mysuru, India, and Douglas B. West, University of Illinois, Urbana, IL.* For $t \le n/2$, let $H(n,t)$ be the graph obtained from the complete graph on n vertices by deleting t pairwise disjoint edges. Determine the number of ways to assign each vertex of $H(n,t)$ a color from a set of k available colors so that vertices forming an edge receive distinct colors.

Solution by Oliver Geupel, Brühl, Germany. Such a coloring of vertices is known as a *proper coloring* of a graph. We show that the number of proper colorings of $H(n,t)$ from k available colors is

$$\sum_{r=0}^{t} \binom{t}{r} k_{(n-t+r)},$$

where $x_{(m)}$ denotes the *falling factorial* $\prod_{i=0}^{m-1}(x-i)$.

Let $e_1, \dots, e_t$ denote the deleted edges, which we view as pairs of vertices. The set of vertices of $H(n,t)$ is then the disjoint union of the sets $e_1, \dots, e_t$ and a set U of size $n-2t$. An induced subgraph consisting of all of U and exactly one vertex from each e_j is the complete graph on its $n-t$ vertices. Hence the vertices of any such subgraph require distinct colors. Given a proper coloring of $H(n,t)$, let $C_1, \dots, C_t$ and C denote the disjoint sets of colors assigned to vertices in $e_1, \dots, e_t$ and U, respectively. Note that $|C_j| \in \{1,2\}$ for each j and $|C| = |U| = n-2t$. For $0 \le r \le t$, there are $\binom{t}{r}$ ways to choose r among $C_1, \dots, C_t$ to have two elements. Each such choice defines a partition of the vertices into $n-(t-r)$ sets receiving distinct colors, and then there are $k_{(n-t+r)}$ ways to assign colors to these sets. Summing over r counts all the proper colorings using k available colors.

Also solved by N. Caro-Montoya (Brazil), K. Gatesman, O. Geupel (Germany), S. C. Locke, O. P. Lossers (Netherlands), R. Martin (Germany), J. H. Nieto (Venezuela), E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), Missouri State University Problem Solving Group, and the proposers.

A Hyperbolic Trigonometric Integral

12297 [2022, 86]. *Proposed by Narendra Bhandari, Bajura District, Nepal.* Prove

$$\int_0^{\pi/2} \frac{\left(\sinh^{-1}(\sin x)\right)^2}{\sin^2 x}\,dx = \frac{\pi}{2}\left(\frac{\pi}{2} - \ln 2\right).$$

Solution by John E. Kampmeyer III, Springfield, PA. We first use integration by parts and the identities

$$\sinh^{-1} x = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \quad \text{and} \quad \tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \int_0^1 \frac{z}{1-z^2y^2}dy$$

to compute

$$\int_0^{\pi/2} \frac{\left(\sinh^{-1}(\sin x)\right)^2}{\sin^2 x}\,dx = 2\int_0^{\pi/2} \frac{\cos^2 x \sinh^{-1}(\sin x)}{\sin x\sqrt{1+\sin^2 x}}\,dx - \left.\frac{\left(\sinh^{-1}(\sin x)\right)^2}{\tan x}\right|_0^{\pi/2}$$

$$= 2\int_0^{\pi/2} \frac{\cos^2 x}{\sin x\sqrt{1+\sin^2 x}} \tanh^{-1}\left(\frac{\sin x}{\sqrt{1+\sin^2 x}}\right)dx$$

$$= 2\int_0^{\pi/2}\int_0^1 \frac{\cos^2 x}{1+(1-y^2)\sin^2 x}\,dy\,dx = 2\int_0^1\int_0^{\pi/2} \frac{1}{1+(2-y^2)\tan^2 x}\,dx\,dy.$$

Using the substitutions $t = \tan x$ and then $u = t\sqrt{2-y^2}$ we obtain

$$\int_0^{\pi/2} \frac{\left(\sinh^{-1}(\sin x)\right)^2}{\sin^2 x}\,dx = 2\int_0^1\int_0^\infty \frac{1}{1+(2-y^2)t^2}\cdot\frac{1}{1+t^2}\,dt\,dy$$

$$= 2\int_0^1 \left(\left(1-\frac{1}{y^2-1}\right)\int_0^\infty \frac{dt}{1+(2-y^2)t^2} + \frac{1}{y^2-1}\int_0^\infty \frac{dt}{1+t^2}\right)dy$$

$$= 2\int_0^1 \left(\left(1-\frac{1}{y^2-1}\right)\frac{1}{\sqrt{2-y^2}}\int_0^\infty \frac{du}{1+u^2} + \frac{1}{y^2-1}\int_0^\infty \frac{dt}{1+t^2}\right)dy$$

$$= \pi\int_0^1 \left(\left(1-\frac{1}{y^2-1}\right)\frac{1}{\sqrt{2-y^2}} + \frac{1}{y^2-1}\right)dy$$

$$= \pi\left(\int_0^1 \frac{dy}{\sqrt{2-y^2}} + \int_0^1 \frac{1}{y^2-1}\left(1-\frac{1}{\sqrt{2-y^2}}\right)dy\right). \qquad (*)$$

The first integral $\int_0^1 dy/\sqrt{2-y^2}$ in $(*)$ is equal to $\sin^{-1}\left(y/\sqrt{2}\right)\Big|_0^1 = \pi/4$. For the second we use the substitution $u = y/\sqrt{2-y^2}$ to compute

$$\int_0^1 \frac{1}{y^2-1}\left(1-\frac{1}{\sqrt{2-y^2}}\right)dy = \lim_{t\to1^-}\left(\int_0^t \frac{dy}{y^2-1} - \int_0^t \frac{dy}{(y^2-1)\sqrt{2-y^2}}\right)$$

$$= \lim_{t\to1^-}\left(\int_0^t \frac{dy}{y^2-1} - \int_0^{t/\sqrt{2-t^2}} \frac{du}{u^2-1}\right)$$

$$= \frac{1}{2}\lim_{t\to1^-}\left(\ln(1-t) - \ln(1+t) - \ln\left(1-\frac{t}{\sqrt{2-t^2}}\right) + \ln\left(1+\frac{t}{\sqrt{2-t^2}}\right)\right)$$

$$= \frac{1}{2}\lim_{t\to1^-}\left(\ln\left(\frac{1-t}{1-t/\sqrt{2-t^2}}\right) - \ln\left(\frac{1+t}{1+t/\sqrt{2-t^2}}\right)\right) = -\frac{1}{2}\ln 2.$$

Substituting these values into $(*)$ yields the desired result.

Editorial comment. Many solvers used the Maclaurin series for $(\sinh^{-1} x)^2$ and then reversed the order of the integration and summation.

Also solved by A. Berkane (Algeria), P. Bracken, H. Chen (US), B. E. Davis, S. Fan, G. Fera (Italy), M. L. Glasser, H. Grandmontagne (France), F. Holland (Ireland), O. Kouba (Syria), J. Magliano, J. Manoharmayum (UK), M. Omarjee (France), K. Sarma (India), V. Schindler (Germany), A. Stadler (Switzerland), A. Stenger, S. M. Stewart (Saudi Arabia), M. Štofka (Slovakia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), M. Wildon (UK), UM6P Math Club (Morocco), and the proposer.

A Disappearing Root of Unity

12298 [2022, 87]. *Proposed by George Stoica, Saint John, NB, Canada.* Let n be a positive integer, S_n be the group of all permutations of $\{1, 2, \ldots, n\}$, and z be a primitive complex nth root of unity. Prove

$$\sum_{\tau \in S_n} \prod_{j=1}^{n} \left(1 - x_j z^{\tau(j)}\right) = n! \left(1 - \prod_{i=1}^{n} x_i\right)$$

for any $x_1, \ldots, x_n \in \mathbb{C}$.

Solution by José Heber Nieto, University of Zulia, Maracaibo, Venezuela. Since

$$x^n - 1 = \prod_{j=1}^{n}(x - z^j) = x^n + \sum_{k=1}^{n}(-1)^k x^{n-k} \sum_{1 \le i_1 < \cdots < i_k \le n} z^{i_1+\cdots+i_k},$$

we have

$$\sum_{1 \le i_1 \le n} z^{i_1} = \sum_{1 \le i_1 < i_2 \le n} z^{i_1+i_2} = \cdots = \sum_{1 \le i_1 < \cdots < i_{n-1} \le n} z^{i_1+\cdots+i_{n-1}} = 0$$

and

$$\sum_{1 \le i_1 < \cdots < i_n \le n} z^{i_1+\cdots+i_n} = z^{1+\cdots+n} = (-1)^{n-1}.$$

Hence for any nonempty set $J \subseteq \{1, \ldots, n\}$,

$$\sum_{\tau \in S_n} z^{\sum_{j \in J} \tau(j)} = \sum_{1 \le i_1 < \cdots < i_{|J|} \le n} \; \sum_{\tau(J) = \{i_1, \ldots, i_{|J|}\}} z^{i_1+\cdots+i_{|J|}}$$

$$= \sum_{1 \le i_1 < \cdots < i_{|J|} \le n} |J|!\,(n - |J|)!\, z^{i_1+\cdots+i_{|J|}} = \begin{cases} (-1)^{n-1} n!, & \text{if } J = \{1, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

We can now compute

$$\prod_{j=1}^{n} \left(1 - x_j z^{\tau(j)}\right) = 1 - \sum_{1 \le i_1 \le n} x_{i_1} z^{\tau(i_1)} + \sum_{1 \le i_1 < i_2 \le n} x_{i_1} x_{i_2} z^{\tau(i_1)+\tau(i_2)} - \cdots$$
$$+ (-1)^n \sum_{1 \le i_1 < i_2 < \cdots < i_n \le n} x_{i_1} \cdots x_{i_n} z^{\tau(i_1)+\tau(i_2)+\cdots+\tau(i_n)},$$

and therefore

$$\sum_{\tau \in S_n} \prod_{j=1}^{n} \left(1 - x_j z^{\tau(j)}\right) = n! - \sum_{1 \le i_1 \le n} x_{i_1} \sum_{\tau \in S_n} z^{\tau(i_1)} + \sum_{1 \le i_1 < i_2 \le n} x_{i_1} x_{i_2} \sum_{\tau \in S_n} z^{\tau(i_1)+\tau(i_2)} - \cdots$$
$$+ (-1)^n \sum_{1 \le i_1 < \cdots < i_n \le n} x_{i_1} \cdots x_{i_n} \sum_{\tau \in S_n} z^{\tau(i_1)+\cdots+\tau(i_n)}$$
$$= n! + (-1)^{2n-1} n!\, x_1 \cdots x_n = n! \left(1 - \prod_{i=1}^{n} x_i\right).$$

Also solved by N. Caro-Montoya (Brazil), R. Dietmann (UK) & M. Widmer (Switzerland), K. Gatesman, O. Geupel (Germany), E. A. Herman, N. Hodges (UK), Y. J. Ionin, O. Kouba (Syria), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), F. Gesmundo (Germany) & T. M. Mazzoli (Austria), T. Amdeberhan & V. H. Moll, M. Omarjee (France), D. Pinchon (France), G. Plumpton (Canada), M. Reid, K. Sarma (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), T. Wiandt, UM6P Math Club (Morocco), and the proposer.

Wait Till You See Them All Together

12299 [2022, 87]. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden.* For n a positive integer, let $x_{0,n} = x_{1,n} = 1$ and, for integers k with $2 \le k \le n-1$, let $x_{k,n} = \left(nx_{k-1,n} - \sum_{j=1}^{k-1} x_{j,n}\right)/k$. Let $T_n = n^2 x_{n-1,n} - n + 1$. The first few values of T_n are 1, 3, 7, 47/3, 427/12, 416/5. Prove that T_n is the expected number of throws of an n-sided die until the last n throws contain all possible face values. For example, if throws of a 6-sided die give the sequence 12345266426351, then it took 14 throws for the event to occur.

Solution by Haoran Chen, Xi'an Jiaotong–Liverpool University, Suzhou, China. We prove that T_n and the expected value both equal U_n, where

$$U_n = 1 + \sum_{j=1}^{n-1} \frac{n^j}{\prod_{i=1}^{j}(n-i)}.$$

For $n = 1$, the assertion is true, so we assume $n \ge 2$.

Say that the process is in state S_k when the maximum number of distinct throws at the end of the current list is k. For $0 \le k \le n$, let e_k be the expected number of additional throws when in state S_k until the event occurs. Note that $e_n = 0$ and e_0 is the desired expected number of throws. Also $e_0 = e_1 + 1$, since there must be a throw to reach S_1 from S_0.

We derive a recurrence for e_k. When in S_k with $1 \le k \le n-1$, if the next throw differs from the previous k, then the process moves to S_{k+1}. This event occurs with probability $(n-k)/n$. On the other hand, if the next throw equals one of the previous k numbers (each with probability $1/n$), then the process enters one of $S_1, \dots, S_k$. Hence for $1 \le k \le n-1$,

$$e_k = 1 + \frac{1}{n}\sum_{j=1}^{k} e_j + \frac{n-k}{n} e_{k+1}.$$

We solve this recurrence by considering differences:

$$e_{k-1} - e_k = \frac{n-k}{n}(e_k - e_{k+1}), \quad \text{for } 1 \le k \le n-1.$$

From $e_0 - e_1 = 1$, we derive $e_k - e_{k+1} = \prod_{j=1}^{k} n/(n-j)$. By summing these differences up to $k = n-1$, the telescoping sum yields the desired result $e_0 = U_n$.

We next obtain the same formula for T_n. Keeping n fixed, we simplify notation by writing x_j for $x_{j,n}$. Let $s_0 = 0$ and $s_k = \sum_{j=1}^{k} x_j$ for $k \ge 1$, so

$$s_k = s_{k-1} + x_k. \tag{1}$$

We rewrite the given recurrence for x_k as

$$kx_k = nx_{k-1} - s_{k-1}. \tag{2}$$

Multiplying (1) by k and combining that with (2) yields

$$ks_k = (k-1)s_{k-1} + nx_{k-1}. \tag{3}$$

Writing (3) with index j instead of k and then summing produces

$$\sum_{j=2}^{k} js_j = \sum_{j=2}^{k}(j-1)s_{j-1} + \sum_{j=2}^{k} nx_{j-1} = \left(\sum_{j=1}^{k-1} js_j\right) + ns_{k-1},$$

and hence

$$ks_k = s_1 + ns_{k-1} = 1 + ns_{k-1}. \tag{4}$$

We now prove

$$s_k = \frac{1}{n}\sum_{j=1}^{k} \frac{n^j}{k(k-1)\cdots(k+1-j)},$$

by induction on k. This formula holds for $k = 1$ since $s_1 = 1$. For $k \geq 2$, (4) yields

$$\begin{aligned} s_k &= \frac{1}{k} + \frac{1}{k}\sum_{j=1}^{k-1} \frac{n^j}{(k-1)\cdots(k-j)} = \frac{1}{k} + \frac{1}{n}\sum_{j=1}^{k-1} \frac{n^{j+1}}{k(k-1)\cdots(k-j)} \\ &= \frac{1}{n}\cdot\frac{n}{k} + \frac{1}{n}\sum_{j=2}^{k} \frac{n^j}{k(k-1)\cdots(k+1-j)} = \frac{1}{n}\sum_{j=1}^{k} \frac{n^j}{k(k-1)\cdots(k+1-j)}. \end{aligned}$$

Rewriting (3) and (4) as $ks_k + nx_k = (k+1)s_{k+1} = 1 + ns_k$, we have

$$x_k = \frac{1}{n} + \frac{n-k}{n}s_k = \frac{1}{n^2}\left(n + (n-k)\sum_{j=1}^{k} \frac{n^j}{k(k-1)\cdots(k+1-j)}\right).$$

Setting $k = n-1$ yields $T_n = U_n$.

Also solved by P. Lalonde (Canada), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), and the proposer.

Forbidden Permutations

12300 [2022, 186]. *Proposed by H. A. ShahAli, Tehran, Iran.* Let n be an integer such that $n \geq 3$. Prove that there is no permutation π of $\{1, 2, \ldots, n\}$ such that $\pi(1), 2\pi(2), \ldots, n\pi(n)$ are distinct modulo n.

Solution by Allen Stenger, Boulder, CO. Assume that π is such a permutation. Note that $n\pi(n) \equiv 0 \pmod{n}$, so $\pi(k) \neq n$ for $k < n$. Thus π fixes n. Now restrict k so that $1 \leq k \leq n-1$. Define $r(k)$ by $k\pi(k) \equiv r(k) \pmod{n}$ with $1 \leq r(k) \leq n-1$. Both π and r permute $\{1, \ldots, n-1\}$.

Write $\gcd(a, b)$ for the greatest common divisor of a and b. Note that $ab \equiv c \pmod{n}$ implies $\gcd(a, n) \mid \gcd(c, n)$. Applying this observation when $\{a, b\} = \{k, \pi(k)\}$ and $c = r(k)$ yields

$$\gcd(k, n) \mid \gcd(r(k), n) \text{ and } \gcd(\pi(k), n) \mid \gcd(r(k), n).$$

The first divisibility gives $\gcd(k, n) \leq \gcd(r(k), n)$. Summing over k and observing that k and $r(k)$ run through the same values yields

$$\sum_{k=1}^{n-1} \gcd(k, n) \leq \sum_{k=1}^{n-1} \gcd(r(k), n) = \sum_{k=1}^{n-1} \gcd(k, n).$$

Thus we have $\gcd(k,n) = \gcd(r(k),n)$ for each k. Applying the same argument to $\pi(k)$ and $r(k)$ yields

$$\gcd(k,n) = \gcd(\pi(k),n) = \gcd(r(k),n). \tag{$*$}$$

We now prove that n is squarefree. Suppose that $p^2 \mid n$ for some prime p. When $k = p$, we have $\gcd(k,n) = p$, and then ($*$) implies also $\gcd(\pi(k),n) = p$ and $\gcd(r(k),n) = p$. Now $\pi(k)$ is a multiple of p, so $k\pi(k)$ is a multiple of p^2. Since $p^2 \mid n$ and $k\pi(k) \equiv r(k) \pmod{n}$, we have $p^2 \mid r(k)$. Thus $p^2 \mid \gcd(r(k),n) = p$, a contradiction.

Since $n \geq 3$ and n is squarefree, n cannot be a power of 2. Thus n is divisible by some odd prime p, and p is relatively prime to n/p. Let $S = \{n/p, 2n/p, \dots, (p-1)n/p\}$. The set S is the set of values of k such that $\gcd(k,n) = n/p$.

By ($*$), $\{k : k \in S\} = \{\pi(k) : k \in S\} = \{r(k) : k \in S\}$. Writing A for $\prod_{k\in S} k$, we then have

$$A = \prod_{k\in S} r(k) \equiv \prod_{k\in S} k\pi(k) = \left(\prod_{k\in S} k\right)\left(\prod_{k\in S} \pi(k)\right) = A^2 \pmod{p}.$$

However, $A = (n/p)^{p-1}(p-1)! \equiv 1 \cdot (-1) \pmod{p}$, where we have used Fermat's little theorem in the first factor and Wilson's theorem in the second factor. Now $A \equiv A^2$ becomes $-1 \equiv 1 \pmod{p}$, which is false when p is an odd prime.

Also solved by C. P. Anil Kumar (India), T. Beran & F. Fürnsinn & F. Lang & S. Schneider & M. Reibnegger & S. Yurkevich (Austria), J. Boswell & C. Curtis, N. Caro-Montoya (Brazil), W. Chang, A. De la Fuente, C. Farnsworth, N. Fellini (Canada), K. Gatesman, O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin, W. Janous (Austria), Y. Kim (Korea), O. P. Lossers (Netherlands), J. Manoharmayum (UK), M. Reid, T. Song, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Velásquez (Colombia), and the proposer.

A Skew-Symmetric Determinant of Sines

12302 [2022, 186]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let n be a positive integer, and let A be the $2n$-by-$2n$ skew-symmetric matrix with (j,k)-entry $\sin(j-k)/\sin(j+k)$. Prove

$$\det(A) = \prod_{1\leq j<k\leq 2n} \left(\frac{\sin(j-k)}{\sin(j+k)}\right)^2.$$

Solution by Pierre Lalonde, Plessisville, QC, Canada. We have

$$\frac{\sin(j-k)}{\sin(j+k)} = \frac{e^{i(j-k)} - e^{-i(j-k)}}{e^{i(j+k)} - e^{-i(j+k)}} = \frac{e^{2ij} - e^{2ik}}{e^{2ij}e^{2ik} - 1}.$$

For $j,k \geq 1$ define $a_{j,k} = (x_j - x_k)/(x_j x_k - 1)$, where each x_ℓ is an indeterminate. For a positive integer r, let $A_r = (a_{j,k})_{j,k=1}^r$, which we observe is skew-symmetric. We prove the more general result

$$\det(A_{2n}) = \prod_{1\leq j<k\leq 2n} a_{j,k}^2,$$

which yields the desired conclusion when x_ℓ is set to $e^{2i\ell}$ for all ℓ.

We prove the claim by induction on n. The case $n = 1$ is easily checked. Elementary algebra yields

$$a_{j,k} + a_{k,m} + a_{m,j} = -\frac{(x_j - x_k)(x_k - x_m)(x_m - x_j)}{(x_j x_k - 1)(x_k x_m - 1)(x_m x_j - 1)} = -a_{j,k}a_{k,m}a_{m,j}.$$

We now operate on A_r for $r = 2n$ with $n > 1$. Subtract the last row from every other row. Subtract the resulting last column from every other column. This does not change the value of the determinant. The (j,k)-entry for $j, k < r$ becomes

$$(a_{j,k} - a_{r,k}) - (a_{j,r} - a_{r,r}) = a_{j,k} + a_{k,r} + a_{r,j} = -a_{j,k}a_{k,r}a_{r,j} = a_{j,k}a_{k,r}a_{j,r}.$$

Hence for $j < r$, each element of row j contains $a_{j,r}$ as a factor and, for $k < r$, every element of column k contains $a_{k,r}$ (which equals $-a_{r,k}$) as a factor. Thus

$$\det(A_r) = \det\begin{bmatrix} & & & 1 \\ & A_{r-1} & & \vdots \\ & & & 1 \\ -1 & \dots & -1 & 0 \end{bmatrix} \prod_{1\le j<r} a_{j,r}^2.$$

Continuing the process, we subtract the penultimate row from rows 1 to $r-2$ and the penultimate column from columns 1 to $r-2$. Again, this does not change the determinant. We have

$$\det(A_r) = \det\begin{bmatrix} & & & 1 & 0 \\ & A_{r-2} & & \vdots & \vdots \\ & & & 1 & 0 \\ -1 & \dots & -1 & 0 & 1 \\ 0 & \dots & 0 & -1 & 0 \end{bmatrix} \prod_{1\le j<r-1} a_{j,r-1}^2 \prod_{1\le j<r} a_{j,r}^2.$$

Expansion of the determinant along the last row and the last column yields

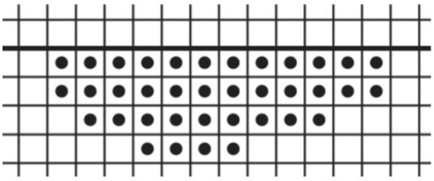
$$\det(A_r) = \det(A_{r-2}) \prod_{1\le j<r-1} a_{j,r-1}^2 \prod_{1\le j<r} a_{j,r}^2.$$

With $r = 2n$, the induction hypothesis completes the proof.

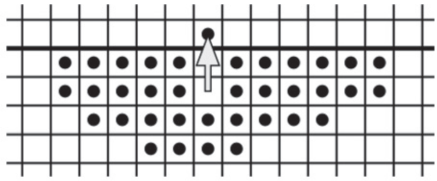
Also solved by T. Amdeberhan & S. B. Ekhad, N. Caro-Montoya (Brazil), O. P. Lossers (Netherlands), B. Ly, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), and the proposer.

CLASSICS

C19. *Due to John H. Conway, suggested by the editors.* A battlefield is modeled by an infinite grid of unit squares, whose centers are indexed by $\{(a,b) : a, b \in \mathbb{Z}\}$. The soldiers are modeled by pegs, which are placed initially at a finite number of squares (a, b) with $b \le 0$. The soldiers advance by jumping in the style of peg solitaire: A jump is permitted when there are three squares in the grid forming a 1-by-3 rectangle with one end square of this rectangle empty while the other two squares are occupied by pegs. Where this configuration exists, the peg on the end may jump over the peg in the middle and move to the empty end, while the peg in the middle is removed. How many pegs are needed in an initial configuration to allow a peg to advance to the square (0, 5)?



A possible initial configuration



A possible initial jump

Guessing Which of Two Numbers is Larger

C18. *Due to Thomas Cover; suggested by Richard Stanley.* Alice chooses two distinct numbers and writes each of them on a slip of paper. Bob selects one of the two slips at random and looks at the number on it. He must then choose to either keep that slip or switch to the other slip. Bob wins if he ends up with the slip with the larger number. Is there anything Bob can do to ensure that, no matter what numbers Alice chooses, his probability of winning is greater than 1/2?

Solution. Yes, there is a randomized strategy that achieves Bob's goal. Let $f : \mathbb{R} \to (0, 1)$ be a strictly increasing function. For example, f could be defined by

$$f(x) = \frac{1}{\pi} \arctan x + \frac{1}{2}.$$

If Bob sees the number x, he keeps his initial selection with probability $f(x)$ and switches to the other slip with probability $1 - f(x)$.

To see that this works, suppose Alice chooses numbers a and b, with $a < b$. Bob ends up with the larger number if he either chooses a initially and switches or chooses b initially and keeps it. Therefore his probability of success is $(1/2)(1 - f(a)) + (1/2) f(b)$, which equals $1/2 + (1/2)(f(b) - f(a))$. This is strictly larger than $1/2$.

Editorial Comment. One way to understand Bob's strategy is to imagine that f is the cumulative distribution function of some continuous random variable whose support is the entire real line. Bob selects a real number at random according to the cumulative distribution function. He then imagines that this number is on the other slip and acts accordingly, keeping the first slip if the number on the slip exceeds the random real and switching if the random real exceeds the number on the slip. If the random real is less than both of Alice's numbers, then no matter which slip Bob selects initially, he keeps it, so he wins with probability 1/2. Similarly, if the random real is greater than both of Alice's numbers, then Bob always switches, again winning with probability 1/2. But if the random real lands between Alice's numbers, an event of positive probability, then Bob wins no matter which slip he selects first.

How Alice chooses her numbers is irrelevant. Bob's strategy triumphs for all choices that Alice can make. The probability mentioned in the problem statement is not to be misread as a probability over Alice's possible choices.

If one does not permit Bob to randomize his responses—that is to say, if Bob's space of possible actions is limited to deterministic strategies governed by a choice function from $\mathbb{R}$ to {keep, switch}—then the answer to the question is negative. This illustrates the power of randomized strategies and also may explain why the affirmative answer seems to be paradoxical.

The fact that f is strictly increasing means that the larger the number that Bob sees, the more likely he is to keep it. This is an intuitively reasonable guideline for Bob to follow if he wants to end up with the larger number.

Notice that $\lim_{x\to\infty}(f(x+1) - f(x)) = 0$. Thus for every positive ϵ, if Alice chooses the numbers x and $x + 1$ for sufficiently large x, then Bob's probability of success will be less than $1/2 + \epsilon$.

An extensive review of this problem appears in A. Gnedin (2016), Guess the larger number, *Mathematica Applicanda*, 44(1): 183–207, where it is suggested that the phrasing of the problem as a guessing game is due to T. M. Cover (1987), Pick the largest number, *Open Problems in Communication and Computation*, New York, NY: Springer, p. 152.

Commuting Orthogonal Projections

12283 [2021, 856]. *Proposed by Yongge Tian, Shanghai Business School, Shanghai, China.* Let A and B be two n-by-n matrices that are orthogonal projections, that is, $A^2 = A = A^*$ and $B^2 = B = B^*$. Let $\sqrt{A+B}$ denote the positive semidefinite square root of $A + B$. Prove

$$\begin{aligned}\operatorname{trace}(A+B) - (2-\sqrt{2})\operatorname{rank}(AB) &\le \operatorname{trace}\sqrt{A+B}\\ &\le (\sqrt{2}-1)\operatorname{trace}(A+B) + (2-\sqrt{2})\operatorname{rank}(A+B),\end{aligned}$$

and show that equality holds simultaneously if and only if $AB = BA$.

Solution by Kyle Gatesman, student, Johns Hopkins University, Baltimore MD. For an n-by-n Hermitian matrix H and an integer $k \in \{1, \dots, n\}$, let $\lambda_k(H)$ be the kth smallest eigenvalue of H, with repetitions according to algebraic multiplicity. All eigenvalues of a Hermitian matrix are real, by the spectral theorem, so the ordering of these eigenvalues is well defined. Extend this notation to all integers k by letting $\lambda_k(H) = \infty$ for $k > n$ and $\lambda_k(H) = -\infty$ for $k < 1$. The spectral theorem also says that Hermitian matrices are diagonalizable, so algebraic and geometric multiplicity are the same for all eigenvalues. Thus the rank of a Hermitian matrix H equals $|\{k \in \{1, \dots, n\} \colon \lambda_k(H) \neq 0\}|$.

A projection matrix P satisfies $P^2 - P = 0$, so any eigenvalue λ of P satisfies $\lambda^2 - \lambda = 0$, which implies $\lambda \in \{0, 1\}$. The matrices A and B are Hermitian with solely nonnegative eigenvalues, so they are positive semidefinite. The sum of any two n-by-n positive semidefinite matrices is also positive semidefinite, so $A + B$ is positive semidefinite.

For an n-by-n orthogonal projection matrix P and an $x \in \mathbb{C}^n$,

$$(Px)^*(x - Px) = x^*P^*x - x^*P^*Px = x^*(P - P^2)x = 0,$$

so $(Px) \perp (x - Px)$, and thus $\|Px\|_2 = \sqrt{\|x\|_2^2 - \|x - Px\|_2^2} \le \|x\|_2$. All eigenvalues of $A + B$ are at most 2 because, for any nonzero vector $x \in \mathbb{C}^n$,

$$\|(A+B)x\|_2 \le \|Ax\|_2 + \|Bx\|_2 \le \|x\|_2 + \|x\|_2 = 2\|x\|_2.$$

The next few lemmas build up to a critical theorem about the eigenvalues of $(A+B)$ that are strictly between 0 and 2.

Lemma 1. *If (x, λ) is an eigenvector-eigenvalue pair for $A+B$ with $0<\lambda<2$, then $((A-B)x, 2-\lambda)$ is also an eigenvector-eigenvalue pair for $A+B$.*

Proof. Given $Ax+Bx=\lambda x$, multiplying on the left by A yields $A^2x+ABx=A(\lambda x)$, which implies $Ax+ABx=\lambda Ax$, or $ABx=(\lambda-1)Ax$. Similarly, multiplying on the left by B yields $BAx+B^2x=B(\lambda x)$, which implies $BAx+Bx=\lambda Bx$, or equivalently $BAx=(\lambda-1)Bx$. Letting $y=(A-B)x$, we obtain

$$\begin{aligned}(A+B)y &= (A+B)(A-B)x = (A-B+BA-AB)x \\ &= (A-B)x+(BAx-ABx) \;=\; y+((\lambda-1)Bx-(\lambda-1)Ax) \\ &= y-(\lambda-1)(A-B)x \;=\; y+(1-\lambda)y \;=\; (2-\lambda)y,\end{aligned}$$

so $((A-B)x, 2-\lambda)$ is an eigenvector-eigenvalue pair. Note that $(A-B)x$ cannot be the zero vector, because $(A-B)x=\vec{0} \iff Ax=Bx \iff \lambda x=(A+B)x=2Ax$, and the only possible eigenvalues of $2A$ are 0 and 2, which by assumption cannot equal λ. ■

Definition: For eigenvectors x and y of $A+B$ associated with eigenvalues strictly between 0 and 2, let y be a *dual* of x if y is a nonzero scalar multiple of $(A-B)x$. By Lemma 1, if y is a dual of x, then the associated eigenvalues of x and y sum to 2.

Lemma 2. *For any two eigenvectors x and y of $A+B$ associated with eigenvalues strictly between 0 and 2, if y is a dual of x, then x is a dual of y.*

Proof. Let λ be the eigenvalue associated with x. If y is a dual of x, then for some nonzero scalar γ we have $y=\gamma\cdot(A-B)x$. From the proof of Lemma 1, $(A+B)x=\lambda x$ implies $ABx=(\lambda-1)Ax$ and $BAx=(\lambda-1)Bx$. Now

$$\begin{aligned}(A-B)y=\gamma(A-B)^2x \;&=\; \gamma\left((A^2+B^2)x-(AB+BA)x\right) \\ &= \gamma\left((A+B)x-(\lambda-1)(A+B)x\right) \\ &= \gamma\left(\lambda x-(\lambda-1)\lambda x\right)=\gamma\lambda(2-\lambda)x.\end{aligned}$$

Since $\lambda\notin\{0,2\}$, the quantity $\gamma\lambda(2-\lambda)$ is nonzero. Thus x equals $1/\big(\gamma\lambda(2-\lambda)\big)$ $(A-B)y$ and is a dual of y. ■

Lemma 3. *Let $u_1,\dots,u_N$ and $v_1,\dots,v_N$ be eigenvectors of $A+B$ corresponding to eigenvalues strictly between 0 and 2, such that u_k and v_k are duals of each other for all $k\in\{1,\dots,N\}$. The vectors $u_1,\dots,u_N$ are linearly independent if and only if $v_1,\dots,v_N$ are linearly independent.*

Proof. By the symmetry of the duality relationship between u_k and v_k, it suffices to show that if $u_1,\dots,u_N$ are linearly dependent, then $v_1,\dots,v_N$ are also linearly dependent. Given $u_1,\dots,u_N$ and $v_1,\dots,v_N$, let $r_1,\dots,r_N$ be the (unique) nonzero scalars satisfying $v_k=r_k\cdot(A-B)u_k$ for all $k\in\{1,\dots,N\}$. Suppose some nonzero vector $(c_1,\dots,c_N)$ expresses dependence by $\sum_{k=1}^N c_ku_k=\vec{0}$. At least one entry in $(c_1/r_1,\dots,c_N/r_N)$ is nonzero, and

$$\sum_{k=1}^N \frac{c_k}{r_k}\cdot v_k=\sum_{k=1}^N\frac{c_k}{r_k}\cdot r_k\cdot(A-B)u_k=(A-B)\sum_{k=1}^N c_ku_k=\vec{0},$$

so $v_1,\dots,v_N$ are linearly dependent. ■

Let $m=\min\{k\in\mathbb{Z}: \lambda_k(A+B)>0\}$, and let $s=\max\{k\in\mathbb{Z}: \lambda_k(A+B)<2\}$. Since the rank of a positive semidefinite matrix equals the number of positive eigenvalues (counted with multiplicity), $\operatorname{rank}(A+B)=n-m+1$. This implies that the nullity of $A+B$ is $m-1$.

Theorem 1. *If $m \le k \le s$, then $\lambda_k(A+B) + \lambda_{m+s-k}(A+B) = 2$.*

Proof. Let W be the span of the eigenvectors of $A+B$ corresponding to eigenvalues in $(0, 2)$, and let I denote the interval of integers from m to s. Given a basis $\{u_m, \dots, u_s\}$ of W whose members are eigenvectors of $A+B$, for $k \in I$ let v_k be any dual of u_k. Since $u_m, \dots, u_s$ are linearly independent eigenvectors of $A+B$, by Lemma 3, $v_m, \dots, v_s$ are also linearly independent eigenvectors and thus also form a basis of W.

For an eigenvector x of $A+B$, let $\lambda^{(x)}$ denote the eigenvalue of $A+B$ associated with x. Index $\{u_m, \dots, u_s\}$ so that $\lambda^{(u_k)} = \lambda_k(A+B)$ for $k \in I$. For indices $j, k \in I$ with $j < k$, we have $\lambda^{(u_j)} \le \lambda^{(u_k)}$, so

$$\lambda^{(v_j)} = 2 - \lambda^{(u_j)} \ge 2 - \lambda^{(u_k)} = \lambda^{(v_k)}.$$

Thus the list $(\lambda^{(v_m)}, \dots, \lambda^{(v_s)})$ is precisely the reverse of the list $(\lambda^{(u_m)}, \dots, \lambda^{(u_s)})$. We conclude $\lambda^{(v_k)} = \lambda^{(u_{m+s-k})}$ for $k \in I$, so

$$2 = \lambda^{(u_k)} + \lambda^{(v_k)} = \lambda^{(u_k)} + \lambda^{(u_{m+s-k})} = \lambda_k(A+B) + \lambda_{m+s-k}(A+B). \qquad \blacksquare$$

Corollary 1. $\sum_{k=m}^{s} \lambda_k(A+B) = s - m + 1$.

Proof. By Theorem 1,

$$2\sum_{k=m}^{s}\lambda_k(A+B) = \sum_{k=m}^{s}(\lambda_k(A+B)+\lambda_{m+s-k}(A+B)) = \sum_{k=m}^{s} 2 = 2(s-m+1). \qquad \blacksquare$$

By Corollary 1,

$$\begin{aligned}
\operatorname{trace}(A+B) &= \sum_{k=s+1}^{n} 2 + \sum_{k=m}^{s}\lambda_k(A+B) + \sum_{k=1}^{m-1} 0 \\
&= 2(n-s) + (s-m+1) = (n-s) + (n-m+1) \\
&= (n-s) + \operatorname{rank}(A+B).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\operatorname{trace}\sqrt{A+B} &= \sum_{k=s+1}^{n} \sqrt{2} + \frac{1}{2}\sum_{k=m}^{s}\left(\sqrt{\lambda_k(A+B)} + \sqrt{\lambda_{m+s-k}(A+B)}\right) \\
&= \sqrt{2}(n-s) + \frac{1}{2}\sum_{k=m}^{s}\left(\sqrt{\lambda_k(A+B)} + \sqrt{2-\lambda_k(A+B)}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
&(\sqrt{2}-1)\operatorname{trace}(A+B) + (2-\sqrt{2})\operatorname{rank}(A+B) \\
&\quad = (\sqrt{2}-1)\left((n-s) + \operatorname{rank}(A+B) + \sqrt{2}\operatorname{rank}(A+B)\right) \\
&\quad = (\sqrt{2}-1)(n-s) + (\sqrt{2}-1)(\sqrt{2}+1)\operatorname{rank}(A+B) \\
&\quad = (\sqrt{2}-1)(n-s) + \operatorname{rank}(A+B) \\
&\quad = \sqrt{2}(n-s) - (n-s) + (n-m+1) = \sqrt{2}(n-s) + s - m + 1.
\end{aligned}$$

Hence the inequality on the right in the problem statement is equivalent to

$$\sum_{k=m}^{s} \frac{\sqrt{\lambda_k(A+B)} + \sqrt{2-\lambda_k(A+B)}}{2} \le s - m + 1.$$

Because the square root function is strictly concave over $(0, \infty)$, we in fact have the stronger inequality

$$\frac{\sqrt{\lambda_k(A+B)} + \sqrt{2 - \lambda_k(A+B)}}{2} \leq \sqrt{\frac{1}{2}\lambda_k(A+B) + \frac{1}{2}\big(2 - \lambda_k(A+B)\big)} = 1$$

for $m \leq k \leq s$. Thus the inequality on the right is true, and equality occurs if and only if $\lambda_k(A+B) = 1$ for $m \leq k \leq s$, which holds if and only if all eigenvalues of $A+B$ belong to the set $\{0, 1, 2\}$.

We next prove the inequality on the left. We use the well-known fact that if X and Y are n-by-n complex matrices, then XY and YX have the same characteristic polynomial and therefore the same spectrum. (See W. V. Parker (1953), The matrices AB and BA, this Monthly, 60(5): 316.)

Lemma 4. *All eigenvalues of AB are nonnegative real numbers.*

Proof. Since A is positive semidefinite, $\sqrt{A}B$ is well-defined. Since $AB = \sqrt{A}(\sqrt{A}B)$, we know that AB and $(\sqrt{A}B)\sqrt{A}$ have the same eigenvalues. Since $\sqrt{A}$ is also positive semidefinite, $\sqrt{A} = \sqrt{A}^*$. Hence for $x \in \mathbb{C}^n$,

$$x^*(\sqrt{A}B\sqrt{A})x = (\sqrt{A}x)^* B(\sqrt{A}x) \geq 0,$$

where the last step follows from the fact that B is positive semidefinite. Thus $\sqrt{A}B\sqrt{A}$ is positive semidefinite, so all its eigenvalues are nonnegative real numbers. Hence all eigenvalues of AB are also nonnegative real numbers. ■

Even if AB is not Hermitian, Lemma 4 implies that all eigenvalues of AB are real and thus have a well-defined ordering. Hence we can designate the kth smallest eigenvalue of the matrix AB as $\lambda_k(AB)$.

Lemma 5. *All eigenvalues of AB are at most* 1.

Proof. We showed earlier that $\|Px\|_2 \leq \|x\|_2$ for any n-by-n orthogonal projection matrix P and vector $x \in \mathbb{C}^n$. Thus for nonzero $x \in \mathbb{C}^n$ with $Bx \neq \vec{0}$,

$$\frac{\|ABx\|_2}{\|x\|_2} = \frac{\|A(Bx)\|_2}{\|(Bx)\|_2} \cdot \frac{\|Bx\|_2}{\|x\|_2} \leq 1 \cdot 1 = 1.$$

If $Bx = \vec{0}$, then $\|ABx\|_2/\|x\|_2 = 0 \leq 1$. For any eigenvector-eigenvalue pair (x, λ) of AB, we know $\|ABx\|_2/\|x\|_2 = \|\lambda x\|_2/\|x\|_2 = \lambda$, so $\lambda \leq 1$. ■

Lemma 6. $\operatorname{rank}(AB) \geq \operatorname{trace}(AB)$.

Proof. The nullity of AB is the geometric multiplicity of 0 as an eigenvalue, which is at most its algebraic multiplicity. Letting $m' = \min\{k \in \mathbb{Z} \colon \lambda_k(AB) > 0\}$, we have $\operatorname{rank}(AB) = n - \operatorname{nullity}(AB) \geq n - m' + 1$. By Lemma 5,

$$n - m' + 1 \geq \sum_{k=m'}^{n} \lambda_k(AB) = \sum_{k=1}^{n} \lambda_k(AB) = \operatorname{trace}(AB),$$

so $\operatorname{rank}(AB) \geq \operatorname{trace}(AB)$. Equality holds if and only if the geometric and algebraic multiplicities of the eigenvalue 0 of AB are equal and all eigenvalues of AB are 0 or 1. ■

By Lemma 6, the result $\operatorname{trace}(A+B) - (2-\sqrt{2})\operatorname{rank}(AB) \leq \operatorname{trace}\sqrt{A+B}$ follows from the stronger

$$(2-\sqrt{2})\operatorname{trace}(AB) \geq \operatorname{trace}(A+B) - \operatorname{trace}\sqrt{A+B}, \tag{1}$$

which we now prove. Our expressions for $\operatorname{trace}(A+B)$ and $\operatorname{trace}\sqrt{A+B}$ yield

$$\operatorname{trace}(A+B) - \operatorname{trace}\sqrt{A+B} = (2-\sqrt{2})(n-s) + \sum_{k=m}^{s}\left(\lambda_k(A+B) - \sqrt{\lambda_k(A+B)}\right)$$

$$= (2-\sqrt{2})(n-s) + \frac{1}{2}\sum_{k=m}^{s}\left(2 - \sqrt{\lambda_k(A+B)} - \sqrt{2-\lambda_k(A+B)}\right)$$

$$= (2-\sqrt{2})(n-s) + \frac{1}{2}\sum_{k=m}^{s}\left(2 - \sqrt{1+\alpha_k} - \sqrt{1-\alpha_k}\right),$$

where $\alpha_k = \lambda_k(A+B) - 1 \in (-1,1)$ for $m \le k \le s$. Since $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ and $(A+B)^2 = A^2 + B^2 + AB + BA = A + B + AB + BA$, we have

$$\operatorname{trace}(AB) = \frac{1}{2}\left(\operatorname{trace}((A+B)^2) - \operatorname{trace}(A+B)\right) = \frac{1}{2}\sum_{k=1}^{n}\left(\lambda_k(A+B)^2 - \lambda_k(A+B)\right)$$

$$= \frac{1}{2}\left((2^2-2)(n-s) + \sum_{k=m}^{s}\lambda_k(A+B)^2 - \sum_{k=m}^{s}\lambda_k(A+B) + (0^2-0)(m-1)\right)$$

$$= (n-s) + \frac{1}{4}\sum_{k=m}^{s}\left(\lambda_k(A+B)^2 + (2-\lambda_k(A+B))^2 - 2\right)$$

$$= (n-s) + \frac{1}{4}\sum_{k=m}^{s}\left((1+\alpha_k)^2 + (1-\alpha_k)^2 - 2\right) = (n-s) + \frac{1}{2}\sum_{k=m}^{s}\alpha_k^2,$$

where the third line follows from the second by Lemma 1. Thus (1) is equivalent to

$$(2-\sqrt{2})(n-s) + \frac{2-\sqrt{2}}{2}\sum_{k=m}^{s}\alpha_k^2 \ge (2-\sqrt{2})(n-s) + \frac{1}{2}\sum_{k=m}^{s}\left(2 - \sqrt{1+\alpha_k} - \sqrt{1-\alpha_k}\right),$$

which in turn is equivalent to

$$\sum_{k=m}^{s}\left((2-\sqrt{2})\alpha_k^2 + \sqrt{1+\alpha_k} + \sqrt{1-\alpha_k} - 2\right) \ge 0.$$

It suffices to prove the stronger inequality $(2-\sqrt{2})\alpha^2 + \sqrt{1+\alpha} + \sqrt{1-\alpha} - 2 \ge 0$ for $\alpha \in (-1,1)$. This stronger inequality is equivalent to

$$\sqrt{2} + (2-\sqrt{2})(1-\alpha^2) \le \sqrt{1+\alpha} + \sqrt{1-\alpha}.$$

Since both sides of this new inequality are nonnegative, we can square both sides to obtain the equivalent $2\sqrt{2}(2-\sqrt{2})\beta^2 + (2-\sqrt{2})^2\beta^4 \le 2\beta$, where $\beta = \sqrt{1-\alpha^2} \in (0,1]$. Since β is positive, we can divide by β to reduce to the equivalent inequality

$$2\sqrt{2}(2-\sqrt{2})\beta + (2-\sqrt{2})^2\beta^3 \le 2. \tag{2}$$

The function mapping β to $2\sqrt{2}(2-\sqrt{2})\beta + (2-\sqrt{2})^2\beta^3$ is strictly increasing on $[0,1]$, and at $\beta = 1$ the value is 2, so (2) holds. Thus the desired inequality

$$(2-\sqrt{2})\operatorname{trace}(AB) \ge \operatorname{trace}(A+B) - \operatorname{trace}\sqrt{A+B}$$

holds for all orthogonal projection matrices A and B, with equality if and only if $\alpha_k = 0$ for $m \le k \le s$, which happens if and only if all eigenvalues of $A+B$ lie in the set $\{0,1,2\}$.

This implies the original inequality on the left in the problem statement,

$$\operatorname{trace}(A+B) - (2-\sqrt{2})\operatorname{rank}(AB) \le \operatorname{trace}\sqrt{A+B},$$

with equality if and only if (a) all eigenvalues of $A + B$ lie in $\{0, 1, 2\}$, (b) all eigenvalues of AB lie in $\{0, 1\}$, and (c) 0 has equal algebraic and geometric multiplicity as an eigenvalue of AB. As we showed earlier, the original inequality on the right holds with equality if and only if (a) holds. We now show that (a) is equivalent to $AB = BA$. Finally, we show that $AB = BA$ implies (b) and (c) to complete the proof that $AB = BA$ is necessary and sufficient for equality in each inequality in the problem statement.

We analyze the three subspaces $\{x\colon (A + B)x = \lambda x\}$ for $\lambda \in \{0, 1, 2\}$. First, consider $\lambda = 0$. Since $A + B$ is positive semidefinite, $(A + B)x = \vec{0}$ implies $0 = x^*(A + B)x = x^*Ax + x^*Bx$, which implies $0 = x^*Ax = (Ax)^*(Ax)$ and $0 = x^*Bx = (Bx)^*(Bx)$, or $Ax = Bx = \vec{0}$. Thus $\{x : (A + B)x = \vec{0}\} = \ker A \cap \ker B$.

Next, consider $\lambda = 2$. If $(A + B)x = 2x$, then x satisfies equality in both halves of

$$\|(A + B)x\|_2 \le \|Ax\|_2 + \|Bx\|_2 \le \|x\|_2 + \|x\|_2,$$

which occurs if and only if $Ax = Bx = x$. Thus $\{x\colon (A + B)x = 2x\} = \mathrm{im} A \cap \mathrm{im} B$.

Finally, consider $\lambda = 1$. Note that

$$\begin{aligned}(A + B)x = x \iff\ & Bx = (I - A)x \\ & \implies A(Bx) = (A - A^2)x = \vec{0} \iff Bx \in \ker A.\end{aligned}$$

Similarly, $(A + B)x = x$ implies $Ax \in \ker B$. Thus a necessary condition for $(A + B)x = x$ is the existence of vectors $x_A \in \mathrm{im} A \cap \ker B$ and $x_B \in \mathrm{im} B \cap \ker A$ satisfying $x_A + x_B = x$. This condition is also sufficient, since if x admits such a decomposition, then

$$\begin{aligned}(A + B)x &= (A + B)(x_A + x_B) \\ &= Ax_A + Bx_A + Ax_B + Bx_B \ = \ x_A + \vec{0} + \vec{0} + x_B \ = \ x.\end{aligned}$$

Thus $\{x : (A + B)x = x\}$ is the direct sum of the orthogonal subspaces $\mathrm{im} A \cap \ker B$ and $\ker A \cap \mathrm{im} B$. The orthogonality of these subspaces crucially means that the subspace $\{x : (A + B)x = x\}$ has an orthonormal basis that can be partitioned into orthonormal bases of $\mathrm{im} A \cap \ker B$ and $\ker A \cap \mathrm{im} B$.

Since $A + B$ is unitarily diagonalizable (by the spectral theorem), $A + B$ admits an orthonormal basis of n eigenvectors, so the eigenvectors of $A + B$ are all associated with eigenvalues in the set $\{0, 1, 2\}$ if and only if there is an orthonormal basis of $\mathbb{C}^n$ that partitions into orthonormal bases of the pairwise orthogonal subspaces $\ker A \cap \ker B$, $\mathrm{im} A \cap \mathrm{im} B$, $\mathrm{im} A \cap \ker B$, and $\ker A \cap \mathrm{im} B$. This condition holds if and only if there is an n-by-n unitary matrix U whose columns all lie in $(\ker A \cup \mathrm{im} A) \cap (\ker B \cup \mathrm{im} B)$, which is equivalent to saying that the columns of U are eigenvectors of both A and B.

Thus the eigenvalues of $A + B$ all belong to $\{0, 1, 2\}$ if and only if A and B are simultaneously unitarily diagonalizable, meaning that there are diagonal matrices D_A and D_B and a single unitary matrix U satisfying $A = UD_AU^*$ and $B = UD_BU^*$. This holds if and only if $AB = BA$, by a well-known result in matrix analysis stating that two diagonalizable matrices commute if and only if they are simultaneously diagonalizable. (See R. A. Horn and C. R. Johnson (2013), Matrix Analysis, 2nd ed., Cambridge University Press.)

It remains to show that $AB = BA$ implies the conditions (b) and (c) stated earlier. As just mentioned, $AB = BA$ implies the existence of diagonal matrices D_A and D_B and a single unitary matrix U satisfying $A = UD_AU^*$ and $B = UD_BU^*$, so $AB = (UD_AU^*)(UD_BU^*) = U(D_AD_B)U^*$. Thus the matrix D_AD_B displays the eigenvalues of AB along its main diagonal, which is the elementwise product of the main diagonals of D_A and D_B. Since all entries in D_A and D_B are 0 or 1, all such elementwise products are also 0 or 1, so all eigenvalues of AB belong to $\{0, 1\}$, which implies (b). Finally, $AB = BA$

implies that AB is diagonalizable, which means that every eigenvalue of AB has equal algebraic and geometric multiplicity, implying (c).

Also solved by E. A. Herman, R. A. Horn, and the proposer.

Alternating Coefficients of Powers of Polynomials

12286 [2021, 946]. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* Let p be a prime number, and let m be a positive integer not divisible by p. Show that the coefficients of $(1+x+\cdots+x^{m-1})^{p-1}$ that are not divisible by p are alternately 1 and -1 modulo p. For example, $(1+x+x^2+x^3)^4 \equiv 1-x+x^4-x^6+x^8-x^{11}+x^{12} \pmod 5$.

Solution by Jayanta Manoharmayum, University of Sheffield, Sheffield, UK. Consider a formal power series with coefficients in $\mathbb{Z}_p$ given by $f(x)=\sum_{k=0}^{\infty} a_k x^k$ with $a_0=1$. Letting $(1-x)^{-1}f(x)=\sum_{n=0}^{\infty} b_n x^n$, we have $b_n=\sum_{k=0}^{n} a_k$. Hence $b_0=1$ and $b_n = b_{n-1}+a_n$ for $n \ge 1$. We conclude that the nonzero coefficients of f alternate between 1 and -1 if and only if each coefficient of $(1-x)^{-1}f(x)$ is 0 or 1.

It therefore suffices to show that each nonzero coefficient in the expansion of

$$(1-x)^{-1}(1+x+\cdots+x^{m-1})^{p-1}$$

is 1. Using $(a+b)^p=a^p+b^p$ (modulo p), we compute

$$(1+x+\cdots+x^{m-1})^{p-1}=\left(\frac{1-x^m}{1-x}\right)^p \cdot \frac{1-x}{1-x^m}=\frac{(1-x^{pm})(1-x)}{(1-x^p)(1-x^m)},$$

and thus

$$\begin{aligned}(1-x)^{-1}(1+x+\cdots+x^{m-1})^{p-1} &= \frac{1+x^m+\cdots+x^{m(p-1)}}{1-x^p} \\ &= \sum_{k=0}^{p-1} x^{km}\left(1+x^p+x^{2p}+\cdots\right).\end{aligned}$$

Since m and p are relatively prime, the exponents $0, m, 2m, \ldots, (p-1)m$ are distinct modulo p. It follows that each power of x appears in the series at most once, so each coefficient is either 0 or 1.

Editorial comment. The proof generalizes directly to show that, for m and n relatively prime, the nonzero coefficients of

$$\frac{(1-x^{mn})(1-x)}{(1-x^n)(1-x^m)}$$

alternate between 1 and -1.

Also solved by T. Amdeberhan & V. H. Moll, N. Caro-Montoya (Brazil), J.-P. Grivaux (France), N. Hodges (UK), Y. J. Ionin, J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), A. Pathak (India), M. Reid, A. Stadler (Switzerland), A. Stenger, R. Stong, B. Sury (India), R. Tauraso (Italy), M. Tetiva (Romania), M. Wildon (UK), L. Zhou, and the proposer.

An Application of the Jacobi Triple Product

12289 [2021, 946]. *Proposed by George E. Andrews, Pennsylvania State University, University Park, PA, and Mircea Merca, University of Craiova, Craiova, Romania.* Prove

$$\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{6n-1})(1-q^{6n-5}),$$

when $|q|<1$.

Solution by Rishabh Sarma, University of Florida, Gainesville, FL. Let L denote the left side of the desired identity. With $\omega = e^{2\pi i/3}$, we have

$$\frac{\omega^{n-1} + \omega^{-(n-1)}}{2} = \cos\left(\frac{2\pi(n-1)}{3}\right) = \cos\left(\pi - \frac{(2n+1)\pi}{3}\right) = -\cos\left(\frac{(2n+1)\pi}{3}\right).$$

When $m = -n-1$, we have $2m+1 = -(2n+1)$ and $m(m+1)/2 = n(n+1)/2$. This and the computation above yield

$$L = \sum_{n=-\infty}^{\infty} \cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2} = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\omega^{n-1} + \omega^{-(n-1)}\right) q^{n(n+1)/2}. \tag{1}$$

For $z \neq 0$ and $|q| < 1$, the Jacobi triple product identity states

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}). \tag{2}$$

Writing (2) using $q^{\frac{1}{2}}$ instead of q and then letting $z = \omega q^{\frac{1}{2}}$ and multiplying by ω^{-1}, we obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \omega^{n-1} q^{n(n+1)/2} &= \omega^{-1}\prod_{n=0}^{\infty}(1-q^{n+1})(1+\omega q^{n+1})(1+\omega^{-1}q^n)\\ &= \omega^{-1}(1+\omega^{-1})\prod_{n=0}^{\infty}(1-q^{n+1})(1+\omega q^{n+1})(1+\omega^{-1}q^{n+1}). \end{aligned}\tag{3}$$

Similarly, writing (2) using $q^{\frac{1}{2}}$ instead of q and then letting $z = \omega^{-1}q^{\frac{1}{2}}$ and multiplying by ω yields

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \omega^{-(n-1)} q^{n(n+1)/2} &= \omega\prod_{n=0}^{\infty}(1-q^{n+1})(1+\omega^{-1} q^{n+1})(1+\omega q^n)\\ &= \omega(1+\omega)\prod_{n=0}^{\infty}(1-q^{n+1})(1+\omega^{-1} q^{n+1})(1+\omega q^{n+1}). \end{aligned}\tag{4}$$

Note that $\omega^{-1}(1+\omega^{-1}) + \omega(1+\omega) = -2$. In addition,

$$(1+\omega x^n)(1+\omega^{-1}x^n) = 1 - x^n + x^{2n} = (1+x^{3n})/(1+x^n).$$

Thus substituting (3) and (4) into (1) yields

$$\begin{aligned} L &= -\frac{1}{2}\left(\omega^{-1}(1+\omega^{-1}) + \omega(1+\omega)\right)\prod_{n=0}^{\infty}(1-q^{n+1})(1+\omega^{-1}q^{n+1})(1+\omega q^{n+1})\\ &= \prod_{n=0}^{\infty}(1-q^{n+1})(1+\omega^{-1}q^{n+1})(1+\omega q^{n+1}) = \prod_{n=1}^{\infty}(1-q^n)\frac{1+q^{3n}}{1+q^n}. \end{aligned}$$

Finally, the claimed identity follows from

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{1+q^{3n}}{1+q^n} &= \prod_{n=1}^{\infty}\frac{(1-q^{6n})(1-q^n)}{(1-q^{3n})(1-q^{2n})}\\ &= \prod_{n=1}^{\infty}\frac{1-q^{2n-1}}{1-q^{6n-3}} = \prod_{n=1}^{\infty}(1-q^{6n-1})(1-q^{6n-5}), \end{aligned}$$

where in the second line we have in two stages canceled common factors in the numerator and denominator.

Also solved by T. Amdeberhan & V. H. Moll, K. Banerjee & M. G. Dastidar (Austria), A. Berkane (Algeria), H. Chen (US), R. Hemmecke (Austria), N. Hodges (UK), W. P. Johnson, P. Lalonde (Canada), K.-W. Lau (China), J. Manoharmayum (UK), R. Molinari, M. Omarjee (France), Z. Shen (China), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Wildon (UK), L. Zhou, and the proposer.

Analytic Solutions of a Functional Equation

12290 [2021, 946]. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* Find all analytic functions $f : \mathbb{C} \to \mathbb{C}$ that satisfy

$$|f(x+iy)|^2 = |f(x)|^2 + |f(iy)|^2$$

for all real numbers x and y.

Solution by Raymond Mortini, Université du Luxembourg, Esch-sur-Alzette, Luxembourg, and Rudolf Rupp, Technische Hochschule Nürnberg, Nürnberg, Germany. We show that the solutions are given by $f(z) = az$, $f(z) = a\sin(kz)$, and $f(z) = a\sinh(kz)$, where $a \in \mathbb{C}$ and $k \in \mathbb{R}$. It is easy to check that each of these satisfies the given equation, using the identities

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

and

$$\sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y.$$

Conversely, suppose that f is an analytic function satisfying the given equation. By setting $x = y = 0$ we see that $f(0) = 0$. Now let $h(x+iy) = |f(x+iy)|^2 = (f\overline{f})(x+iy)$. Note that

$$h_{xy}(x+iy) = \frac{\partial^2}{\partial y \partial x}\left(|f(x)|^2 + |f(iy)|^2\right) = 0.$$

Also, $f_x = f'$, $f_y = if_x = if'$, $f_{xy} = (f')_y = if''$, and $\overline{f}_y = \overline{(f_y)} = \overline{if_x} = -i\overline{f}_x$. Hence

$$0 = h_{xy} = f_{xy}\overline{f} + f\overline{f}_{xy} + f_x\overline{f}_y + f_y\overline{f}_x = 2\,\mathrm{Re}(f_{xy}\overline{f}) = -2\,\mathrm{Im}(f''\overline{f}).$$

Since $|f|^2 = f\overline{f}$ is a real-valued function, it follows that the meromorphic function $f''/f = f''\overline{f}/|f|^2$ is real-valued where defined. If we assume that f is not identically zero, then this can happen only when f''/f is a real constant λ.

The differential equation $f'' = \lambda f$ in $\mathbb{C}$ has solutions $az + d$ if $\lambda = 0$, $\alpha e^{\sqrt{\lambda}z} + \beta e^{-\sqrt{\lambda}z}$ if $\lambda > 0$, and $\alpha e^{i\sqrt{|\lambda|}z} + \beta e^{-i\sqrt{|\lambda|}z}$ if $\lambda < 0$. Since $f(0) = 0$, we have $d = 0$ or $\beta = -\alpha$. Setting $k = \sqrt{|\lambda|}$ yields the claimed expressions for $f(z)$.

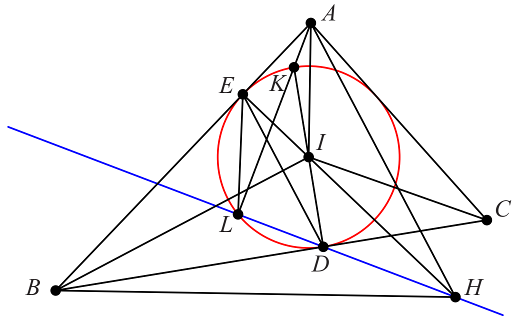
Also solved by O. Kouba (Syria), J. Manoharmayum (UK), R. Stong, and the proposer.

A Perpendicularity Involving the Incenter and Nagel Point

12291 [2021, 947]. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania, and Petru Braica, Satu Mare, Romania.* The *Nagel point* of a triangle is the point common to the three segments that join a vertex of the triangle to the point at which an excircle touches the opposite side. Let ABC be a triangle with incenter I and Nagel point J. Prove that AJ is perpendicular to the line through the orthocenters of triangles IAB and IAC.

Solution by Li Zhou, Polk State College, Winter Haven, FL. Suppose that the incircle ω of $\triangle ABC$ is tangent to BC at D and AB at E. Extend DI to intersect ω again at K, and let

AK intersect ω again at L. The tangent line to ω at K is parallel to BC, and therefore there is a homothety centered at A that sends this tangent line to BC. The image of ω under this homothety is the excircle tangent to BC, and the image of K is the point where this excircle is tangent to BC. Since the image of K is on the line AL, it follows that the Nagel point J lies on AL.



Let H be the intersection point of EI and LD. Since KD is a diameter of ω, $KL \perp LD$, and since AB is tangent to ω at E, $EI \perp AB$. It follows that A, E, L, and H lie on the circle with diameter AH. Therefore $\angle AHE = \angle ALE = \angle KDE$, and since $\triangle IDE$ is isosceles, $\angle KDE = \angle DEH$. Thus $AH \parallel ED$. Since BI is the perpendicular bisector of DE, we have $BI \perp AH$. Combining this with $IH \perp AB$, we conclude that H is the orthocenter of triangle IAB. Likewise, the orthocenter of $\triangle IAC$ is on LD as well, completing the proof.

Also solved by M. Bataille (France), C. Chiser (Romania), N. S. Dasireddy (India), I. Dimitrić, G. Fera (Italy), O. Geupel (Germany), J.-P. Grivaux (France), N. Hodges (UK), W. Janous (Austria), O. Kouba (Syria), J. H. Lindsey II, N. Osipov (Russia), C. G. Petalas (Greece), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, T. Zvonaru (Romania), Davis Problem Solving Group, UM6P Math Club (Morocco), and the proposer.

The Congruence Class of a Trigonometric Power

12292 [2021, 947]. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Let p be a prime number, and let $r = 1/(2\cos(4\pi/7))$. Evaluate $\lfloor r^{p+2} \rfloor$ modulo p.

Solution by UM6P Math Club, Mohammed VI Polytechnic University (UM6P), Ben Guerir, Morocco. For $p \notin \{2, 7\}$, when p is congruent to ± 1, ± 2, or ± 3 modulo 7, the value of $\lfloor r^{p+2} \rfloor$ is congruent modulo p to -12, -5, or 2, respectively. For the exceptions, $\lfloor r^4 \rfloor \equiv 1 \pmod 2$ and $\lfloor r^9 \rfloor \equiv 2 \pmod 7$.

Let $\theta = 2\pi/7$, so $r = 1/(2\cos 2\theta)$. Also let $s = 1/(2\cos\theta)$ and $t = 1/(2\cos 4\theta)$. For $x \in \{\theta, 2\theta, 4\theta\}$, we have $3x \equiv -4x \bmod 2\pi$, so $\sin 3x + \sin 4x = 0$. Using the multiple-angle formulas $\sin 3x = 3\sin x - 4\sin^3 x$ and

$$\sin 4x = 2\sin 2x \cos 2x = 4\sin x \cos x(\cos^2 x - \sin^2 x),$$

we compute

$$\begin{aligned} 0 = \frac{\sin 3x + \sin 4x}{\sin x} &= 3 - 4\sin^2 x + 4\cos^3 x - 4\cos x \sin^2 x \\ &= 8\cos^3 x + 4\cos^2 x - 4\cos x - 1. \end{aligned}$$

Thus r, s, and t satisfy $z^{-3} + z^{-2} - 2z^{-1} - 1 = 0$ and hence $z^3 + 2z^2 - z - 1 = 0$. Let $S_n = r^n + s^n + t^n$. From the equation in z satisfied by r, s, and t, we obtain the recurrence $S_{n+3} = -2S_{n+2} + S_{n+1} + S_n$. Since $(z - r)(z - s)(z - t) = z^3 + 2z^2 - z - 1$, the initial conditions are $S_0 = 3$, $S_1 = -2$, and

$$S_2 = S_1^2 - 2(rs + st + tr) = 4 - 2(-1) = 6.$$

The recurrence then implies that S_n is an integer for $n \geq 0$. Since $s \approx 0.80$ and $t \approx -0.55$, we have $s^n + t^n \in (0, 1)$ and $\lfloor r^n \rfloor = S_n - 1$ when $n \geq 1$.

Let $\omega = e^{2\pi i/7}$. Since $\cos x = (e^{ix} + e^{-ix})/2$, we have

$$1/r = \omega^2 + \omega^{-2}, \qquad 1/s = \omega + \omega^{-1}, \qquad \text{and} \qquad 1/t = \omega^4 + \omega^{-4}.$$

In the ring $\mathbb{F}_p[\omega]$ for prime p, we have $(x + y)^p = x^p + y^p$, so

$$1/r^p = \omega^{2p} + \omega^{-2p}, \qquad 1/s^p = \omega^p + \omega^{-p}, \qquad \text{and} \qquad 1/t^p = \omega^{4p} + \omega^{-4p}.$$

In order to compute S_{p+2}, we need the pth powers of r, s, and t. These are obtained by first "rationalizing the denominator" to express these quantities as polynomials in ω. For example,

$$s = \frac{1}{\omega + 1/\omega} = \frac{\omega}{\omega^2 + 1} = \frac{\omega(\omega^2 - 1)}{\omega^4 - 1} = \frac{\omega(\omega^2 - 1)(\omega^4 + 1)}{\omega^8 - 1} = \frac{\omega(\omega^2 - 1)(\omega^4 + 1)}{\omega - 1}.$$

Canceling $\omega - 1$ and expanding yields $s = \omega^6 + \omega^5 + \omega^2 + \omega$. Similarly, or by substituting ω^2 or ω^4 for ω in the expression for s, we have $r = \omega^5 + \omega^4 + \omega^3 + \omega^2$ and $t = \omega^6 + \omega^4 + \omega^3 + \omega$.

Raising these expressions to the pth power multiplies the exponents by p, and the exponents then reduce modulo 7. We need only consider three cases, since negating the exponents in these polynomials does not change the values.

(i) If $p \equiv \pm 1 \pmod 7$, then $r^p \equiv r$, $s^p \equiv s$, and $t^p \equiv t$ modulo p, reducing the computation to $S_{p+2} \equiv S_3 = -2S_2 + S_1 + S_0 = -12 - 2 + 3 = -11$ and $\lfloor r^{p+2} \rfloor \equiv -12$.

(ii) If $p \equiv \pm 2 \pmod 7$ with $p \neq 2$, then $r^p \equiv t$, $s^p \equiv r$, and $t^p \equiv s$ modulo p, reducing the computation to $S_{p+2} \equiv tr^2 + rs^2 + st^2 = -4$ (see comment below) and $\lfloor r^{p+2} \rfloor \equiv -5$.

(iii) If $p \equiv \pm 3 \pmod 7$, then $r^p \equiv s$, $s^p \equiv t$, and $t^p \equiv r$ modulo p, reducing the computation to $S_{p+2} \equiv sr^2 + ts^2 + rt^2 = 3$ (see comment below) and $\lfloor r^{p+2} \rfloor \equiv 2$.

Finally, if $p = 2$, then $\lfloor r^4 \rfloor = S_4 - 1 = 25 \equiv 1 \bmod p$, and if $p = 7$, then $\lfloor r^9 \rfloor = S_9 - 1 = -1461 \equiv 2 \bmod p$.

Editorial comment. The sums of powers of roots of a polynomial f (denoted S_k above) are called *Newton sums* owing to Newton's identities, which compute them in terms of the coefficients of f. In particular, $S_k = -(kc_k + \sum_{i=1}^{k-1} c_i S_{k-i})$, where $f(x) = \sum c_k x^{n-k}$ with $c_0 = 1$ and $c_k = 0$ for $k > n$. More than half a dozen proofs are known; those in this MONTHLY include **75** (1968), 396–397, **99** (1992), 749–751, and **110** (2003), 232–234. For further discussion, see artofproblemsolving.com/wiki/index.php/Newton's_Sums.

As in the solution above, most solvers needed to calculate α and β, where $\alpha = rs^2 + st^2 + tr^2$ and $\beta = rt^2 + sr^2 + ts^2$. Richard Stong noted that $\alpha + \beta = S_1 S_2 - S_3 = -1$ and $\alpha\beta = (rs + st + tr)^3 + rst \cdot S_1^3 = 12$, yielding $\{\alpha, \beta\} = \{-4, 3\}$. Allen Stenger noted that α and β must be integers and computed them to the nearest integer. Michael Reid expanded them in terms of ω. Roberto Tauraso noted that the sequence $\{S_n\}$ (OEIS A094648) is related to Catalan's constant.

Also solved by G. Fera (Italy), K. T. L. Koo (China), O. P. Lossers (Netherlands), M. Omarjee (France), M. Reid, A. Stenger, R. Stong, R. Tauraso (Italy), and the proposer.

A Real Identity

12293 [2022, 86]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.* Let n be a positive integer and r be a positive real number. Prove

$$\sum_{k=0}^{n} (-1)^k \left(\sum_{j=0}^{k} r^j \binom{n}{j} \right) \left(\sum_{j=0}^{k} (-r)^j \binom{n}{j} \right) = \left(\frac{(r+1)^n + (r-1)^n}{2} \right)^2.$$

Solution by Martin Widmer, University of London, London, UK. Let $a_0 = 0$, let

$$a_k = \binom{n}{0} + \binom{n}{2}r^2 + \cdots + \binom{n}{k-1}r^{k-1}$$

when $0 < k \le n+1$ and k is odd, and let

$$a_k = \binom{n}{1}r + \binom{n}{3}r^3 + \cdots + \binom{n}{k-1}r^{k-1}$$

when $0 < k \le n+1$ and k is even. We compute

$$(-1)^k \left(\sum_{j=0}^{k} r^j \binom{n}{j}\right)\left(\sum_{j=0}^{k}(-r)^j\binom{n}{j}\right) = (a_{k+1}+a_k)(a_{k+1}-a_k) = a_{k+1}^2 - a_k^2.$$

Finally,

$$\sum_{k=0}^{n}(a_{k+1}^2 - a_k^2) = a_{n+1}^2 = \left(\frac{(r+1)^n + (r-1)^n}{2}\right)^2.$$

Also solved by T. Amdeberhan & V. H. Moll, A. Berkane (Algeria), N. Caro-Montoya (Brazil), C. Curtis, K. Gatesman, O. Geupel (Germany), N. Hodges (UK), P. Lalonde (Canada), O. P. Lossers (Netherlands), J. Manoharmayum (UK), J. H. Nieto (Venezuela), M. Omarjee (France), E. Schmeichel, A. Stadler (Switzerland), R. Stong, M. Wildon (UK), L. Zhou, Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposers.

Exploring a Planet, Revisited, Revisited

12342 [2022, 785]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $v_1, \dots, v_n$ be unit vectors in $\mathbb{R}^d$. Prove that if u maximizes $\prod_{i=1}^{n}|v_i \cdot u|$ over all unit vectors $u \in \mathbb{R}^d$, then for all i, $|v_i \cdot u| \ge \sin(\pi/(2n))$.

Editorial comment. Several readers pointed out that this problem was presented and solved in Y. Zhao (2022), Exploring a planet, revisited, this MONTHLY 129(7): 678–680. As Zhao's note explains, the problem is connected to a conjecture of László Fejes Tóth (1973), Research problems: Exploring a planet, this MONTHLY 80(9): 494–498. A preprint of Zhao's note was posted to arxiv.org before we received this submitted problem, and the problem here is taken verbatim from Zhao's note. We regret the repetition without proper attribution.

CLASSICS

C18. *Due to Thomas Cover; suggested by Richard Stanley.* Alice chooses two distinct numbers and writes each of them on a slip of paper. Bob selects one of the two slips at random and looks at the number on it. He must then choose to either keep that slip or switch to the other slip. Bob wins if he ends up with the slip with the larger number. Is there anything Bob can do to ensure that, no matter what numbers Alice chooses, his probability of winning is greater than 1/2?

Choosing n Numbers With Sum 0 Modulo n

C17. *Due to Paul Erdős, Abraham Ginzburg, and Abraham Ziv; suggested by Gabriel Carroll and Yuri Ionin, independently.* Given $2n-1$ integers, show that it is possible to choose n of them that sum to a multiple of n.

Solution. Let the integers be $a_1, \dots, a_{2n-1}$. We first address the case of prime n. For $I \subset \{1, \dots, 2n-1\}$ with $|I| = n$, let $S_I = \left(\sum_{i\in I} a_i\right)^{n-1}$, and let $S = \sum_I S_I$. Thinking momentarily of $\{a_1, \ldots, a_{2n-1}\}$ as a set of indeterminates, the expressions S_I and S are homogenous polynomials of degree $n-1$. Each monomial in S will have k of the indeterminates represented for some k with $1 \le k \le n-1$, and each such monomial will arise with equal coefficient in S_I for precisely $\binom{2n-1-k}{n-k}$ choices of I. Since $\binom{2n-1-k}{n-k} \equiv 0$ (mod n), the coefficient of each monomial in S when like terms are gathered is a multiple of n. Hence $S \equiv 0 \pmod{n}$.

On the other hand, if the result of the problem is false, then for every I, we have $S_I \not\equiv 0$ (mod n) and so $S_I \equiv 1 \pmod{n}$ by Fermat's little theorem. Since there are $\binom{2n-1}{n}$ such sets I, and since

$$\binom{2n-1}{n} \equiv \frac{(2n-1)(2n-2)\cdots(n+1)}{(n-1)(n-2)\cdots 1} \equiv 1 \pmod{n},$$

we conclude that $S \equiv 1 \pmod{n}$, a contradiction. This proves the result when n is prime.

Finally, we extend the result to the case where n is any positive integer. We argue by induction on n. Suppose that $n = pm$ where p is prime, and suppose we are given a multiset $\{a_1, \ldots, a_{2n-1}\}$. Repeatedly applying the prime case of the result to p, we extract sets $I_1, \ldots, I_{2m-1}$ with $I_j \subset \{1, \ldots, 2n-1\} \setminus \left(\bigcup_{s=1}^{j-1} I_s\right)$ such that $|I_j| = p$ and $\sum_{i \in I_j} a_i \equiv 0 \pmod{p}$ for all $j \in \{1, \ldots, 2m-1\}$. The reason this is possible is that, as long as $j \le 2m-1$,

$$\left|\{1, \ldots 2n-1\} \setminus \left(\bigcup_{s=1}^{j-1} I_s\right)\right| = 2n-1-(j-1)p \ge 2n-1-(2m-2)p = 2p-1,$$

and so the result for p ensures a choice for I_j. Now, for $j \in \{1, \ldots, 2m-1\}$, let $b_j = (1/p)\sum_{i \in I_j} a_i$. Applying the induction hypothesis to m and $\{b_1, \ldots, b_{2m-1}\}$, we obtain a set $J \subset \{1, \ldots, 2m-1\}$ with $|J| = m$ and $\sum_{j \in J} b_j \equiv 0 \pmod{m}$. The set $\{a_i : i \in I_j, j \in J\}$ provides the subset of size n whose elements sum to a multiple of n.

Editorial comment. The result is from P. Erdős, A. Ginzburg, and A. Ziv (1961), Theorem in the additive number theory, *Bull. Res. Council Israel* 10F, 41–43. The number $2n-1$ in the problem statement is optimal, as the multiset with $n-1$ zeroes and $n-1$ ones has size $2n-2$ but no subset of size n summing to a multiple of n. Five separate proofs of the result for prime n appear in N. Alon and M. Dubiner (1993), Zero-sum sets of prescribed size, in *Combinatorics, Paul Erdős is Eighty* (Vol. 1), eds. D. Miklós, V. T. Sós, and T. Szőnyi, Keszthely, 33–50, where the argument given here is attributed to N. Zimmerman.

SOLUTIONS

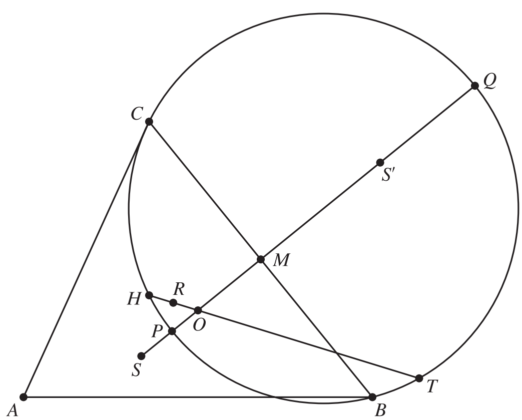
Four Concyclic Points

12280 [2021, 856]. *Proposed by Nguyen Duc Toan, Da Nang, Vietnam.* Let ABC be an acute scalene triangle with circumcenter O and orthocenter H. Let M and R be the midpoints of segments BC and OH, respectively, let S be the reflection across BC of the circumcenter of triangle BOC, and let T be the second point of intersection of the circumcircle of triangle BHC and line OH. Prove that M, R, S, and T are concyclic.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. The line OM passes through S, and the line OH passes through both R and T, so by the power law, to prove that the points M, S, R, and T are concyclic it suffices to show that $OR \cdot OT = OM \cdot OS$ and that O lies between R and T if and only if O lies between M and S.

Let α, β, and γ be the angles at vertices A, B, and C, respectively, of $\triangle ABC$, and let $a = BC$. By the inscribed angle theorem, $\angle BOC = 2\alpha$, and therefore $\angle BOM = \alpha$. Since $\angle BMO$ is a right angle, we have

$$OM = BM \cot(\angle BOM) = \frac{a}{2} \cot \alpha.$$



Since CH is perpendicular to AB, we have $\angle BCH = \pi/2 - \beta$. Likewise, $\angle CBH = \pi/2 - \gamma$, so

$$\angle BHC = \pi - (\pi/2 - \beta) - (\pi/2 - \gamma) = \beta + \gamma = \pi - \alpha.$$

Let P and Q be the intersection points of the circumcircle $\mathcal{C}$ of $\triangle BHC$ and the line OM, with P on the same side of BC as A and Q on the opposite side. Since $\angle BPC = \angle BHC = \pi - \alpha$, we have $\angle BPM = (\pi - \alpha)/2$ and

$$PM = \frac{a}{2}\cot\left(\frac{\pi-\alpha}{2}\right) = \frac{a}{2}\tan\left(\frac{\alpha}{2}\right).$$

Thus

$$OP = |OM - PM| = \frac{a}{2}\left|\cot\alpha - \tan\left(\frac{\alpha}{2}\right)\right|.$$

Similarly, $\angle BQC = \alpha$, $\angle BQM = \alpha/2$, $QM = (a/2)\cot(\alpha/2)$, and

$$OQ = OM + QM = \frac{a}{2}\left(\cot\alpha + \cot\left(\frac{\alpha}{2}\right)\right).$$

Thus by the power law for the circle $\mathcal{C}$,

$$OR \cdot OT = \frac{1}{2}OH \cdot OT = \frac{1}{2}OP \cdot OQ = \frac{a^2}{8}\left|\cot\alpha - \tan\left(\frac{\alpha}{2}\right)\right|\left(\cot\alpha + \cot\left(\frac{\alpha}{2}\right)\right). \quad (1)$$

Next, we claim that

$$OS = \frac{a}{2}\,|\cot\alpha + \cot(2\alpha)|, \quad (2)$$

and therefore

$$OM \cdot OS = \frac{a^2}{4}\cot\alpha\,|\cot\alpha + \cot(2\alpha)|. \quad (3)$$

Let S' be the circumcenter of $\triangle BOC$, so that S is the reflection of S' across BC. The calculation of OS depends on how α compares to $\pi/4$. Suppose first that $\alpha < \pi/4$. In that case, S' lies on the same side of BC as A, so S lies on the opposite side. By the inscribed angle theorem, $\angle BSC = \angle BS'C = 2\angle BOC = 4\alpha$ and $\angle BSM = 2\alpha$. Therefore $SM = BM\cot(\angle BSM) = (a/2)\cot(2\alpha)$, so

$$OS = OM + SM = \frac{a}{2}\big(\cot\alpha + \cot(2\alpha)\big).$$

Since $\cot\alpha$ and $\cot(2\alpha)$ are positive in this case, this agrees with (2).

If $\alpha = \pi/4$, then $S = S' = M$, so $OS = OM = (a/2)\cot\alpha$, which again agrees with (2) because $\cot(2\alpha) = 0$. Finally, suppose $\alpha > \pi/4$. In that case, S' lies on the opposite side of BC from A and S lies on the same side. Therefore $\angle BSC = \angle BS'C = 2\pi - 4\alpha$, $\angle BSM = \pi - 2\alpha$, $SM = (a/2)\cot(\pi - 2\alpha) = -(a/2)\cot(2\alpha)$, and

$$OS = |OM - SM| = \frac{a}{2}\,|\cot\alpha + \cot(2\alpha)|,$$

again confirming (2).

Combining (1) and (3), we see that establishing $OR \cdot OT = OM \cdot OS$ reduces to showing

$$\left|\cot\alpha - \tan\left(\frac{\alpha}{2}\right)\right|\left(\cot\alpha + \cot\left(\frac{\alpha}{2}\right)\right) = 2\cot\alpha\,|\cot\alpha + \cot(2\alpha)|.$$

After some rewriting (using $2\cot(2\alpha) = \cot\alpha - \tan\alpha$), we see that both sides are equal to $|3\cot^2\alpha - 1|$.

Finally, we must confirm the betweenness condition. For O to lie between R and T, it must be inside $\mathcal{C}$, which happens if and only if $\angle BOC > \angle BHC$. Using the equations $\angle BOC = 2\alpha$ and $\angle BHC = \pi - \alpha$, we see that this is equivalent to $\alpha > \pi/3$. For O to lie between M and S, we must have S on the same side of BC as O and $\angle BOC$ greater than $\angle BSC$. By previous reasoning, this holds if and only if $\alpha > \pi/4$ and $2\alpha > 2\pi - 4\alpha$, which again reduces to $\alpha > \pi/3$, thus completing the proof.

Editorial comment. The requirements that $\triangle ABC$ be acute and scalene are unnecessary, as long as some exceptional situations are accounted for. For example, if $AB = AC$, then the points M, R, S, and T are collinear, since all lie on the perpendicular bisector of BC. If $\angle BAC$ is a right angle, then $O = M$, $H = A$, the points M, R, and T are collinear, and S is undefined.

Also solved by M. Bataille (France), H. Chen (China), K. Gatesman, O. Geupel (Germany), J.-P. Grivaux (France), N. Hodges (UK), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, T. Wiandt, Davis Problem Solving Group, and the proposer.

Generating the Positive Rational Numbers

12282 [2021, 856]. *Proposed by George Stoica, Saint John, NB, Canada.* Prove that the multiplicative group generated by $\{\lfloor\sqrt{2}n\rfloor/n\colon\ n \in \mathbb{Z}^+\}$ is the group of positive rational numbers.

Solution by Stephen M. Gagola Jr., Kent State University, Kent, OH. Let M be the multiplicative group generated by the set of rationals in the problem statement. We first show that 3, 5, 7, 11, and 2 lie in M. To do this, we calculate here some small values of $\lfloor\sqrt{2}n\rfloor/n$.

n	3	4	5	7	8	11
$\lfloor\sqrt{2}\,n\rfloor/n$	$4/3$	$5/4$	$7/5$	$9/7$	$11/8$	$15/11$

The product of the first four entries in the second row is 3, so $3 \in M$. The equations $5 = 3(4/3)(5/4)$, $7 = 3(4/3)(5/4)(7/5)$, $11 = 3(3)(4/3)(5/4)(11/15)$, and $2 = 3(3)(4/3)(4/3)(11/8)(1/11)$ then show that 2, 5, 7, and 11 are also in M.

Let p be an odd prime number, and suppose that all primes less than p lie in M. Set $q = \lfloor\sqrt{2}\,p\rfloor$. Since $p < q < \sqrt{2}\,p$, if q is composite then all prime factors of q are less than p and belong to M. Therefore $q \in M$ and $p = q/(\lfloor\sqrt{2}\,p\rfloor/p) \in M$.

Hence we may assume that q is an odd prime. The definition of q yields $q < \sqrt{2}\,p < q+1$, which implies $\sqrt{2}\,q/2 < p < \sqrt{2}\,(q+1)/2$. Note that $(q+1)/2$ is an integer. Since the interval $[\sqrt{2}\,q/2, \sqrt{2}\,(q+1)/2]$ has length less than 1 and contains the integer p, we conclude $\lfloor\sqrt{2}\,(q+1)/2\rfloor = p$, and thus

$$\frac{p}{(q+1)/2} = \frac{\lfloor\sqrt{2}\,(q+1)/2\rfloor}{(q+1)/2} \in M.$$

Since $(q+1)/2 < p$, all prime factors of $(q+1)/2$ belong to M, and it follows that so does $(q+1)/2$ itself. Therefore $p \in M$.

Since M contains all prime numbers, and $1 = \lfloor\sqrt{2}\rfloor$, it follows that M is the group of all positive rationals.

Editorial comment. Gagola commented further that the set of "numerators," namely $\{\lfloor\sqrt{2}\,n\rfloor\colon\ n \in \mathbb{Z}^+\}$, also generates the group of positive rationals. First note inductively that if $n \in \mathbb{Z}^+$ and $(1+\sqrt{2})^n = a + b\sqrt{2}$, then $a^2 - 2b^2 = (-1)^n$. Let p be prime, and choose n odd and large enough so that $b > p$. We have $a - \sqrt{2}\,b = -1/(a + \sqrt{2}\,b)$, so $a + \varepsilon = \sqrt{2}\,b$, where $\varepsilon = 1/(a + \sqrt{2}\,b) < 1/b$. Therefore when $k < b$ we have $ka + k\varepsilon = \sqrt{2}\,kb$, where $k\varepsilon < 1$. Thus $\lfloor\sqrt{2}\,kb\rfloor = ka$, and all these integers belong to the group generated by the specified set. In particular, both a and pa belong to this group, and hence so does p.

Celia Schacht observed that the complete solution of this problem is the topic of I. Kátai and B. M. Phong, On the multiplicative group generated by $\{[\sqrt{2}n]/n : n \in \mathbb{N}\}$, *Acta Mathematica Hungarica* 145 (2015), no. 1, 80–87. She also cited two subsequent papers by the same authors with the same title (II and III) dealing with deeper questions: (2015), *Acta Scientiarum Mathematicarum (Szeged)* 81 no. 3–4, 431–436, and (2015), *Acta Mathematica Hungarica* 147 no. 1, 247–254.

Also solved by J. Boswell & C. Curtis, A. Dixit & S. Pathal (India), K. Gatesman, N. Hodges (UK), O. P. Lossers (Netherlands), M. Reid, C. Schacht, A. Stadler (Switzerland), D. Terr, and the proposer.

A Triangle with Perimeter 2021

12284 [2021, 857]. *Proposed by Zachary Franco, Houston, TX.* Let ABC be a triangle with circumcenter O, incenter I, orthocenter H, sides of integer length, and perimeter 2021. Suppose that the perpendicular bisector of OH contains A and I. Find the length of BC.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela. We show that $BC = 679$.

Let a, b, and c denote the lengths of sides BC, CA, and AB, respectively. We first show that $a^2 = b^2 + c^2 \pm bc$.

Since either $\angle ABC$ or $\angle ACB$ is acute, by symmetry we may assume that $\angle ABC$ is acute. Let P be the foot of the altitude from A, and let M be the midpoint of AC. Let R be the circumradius of $\triangle ABC$, and let $h = AP$ and $u = BP$. Note that since A lies on the perpendicular bisector of OH, $AH = AO = R$.

We first consider the case in which $\triangle ABC$ is acute. Triangles ABP, CHP, and AOM are similar, since they all have a right angle and an angle equal to $\angle ABC$. Therefore

$$\frac{AP}{BP} = \frac{CP}{HP} \qquad \text{and} \qquad \frac{AP}{AB} = \frac{AM}{AO}. \tag{1}$$

Since $\triangle ABC$ is acute, H is between A and P and P is between B and C, so $CP = a - u$ and $HP = h - R$. Hence the equations in (1) become

$$\frac{h}{u} = \frac{a-u}{h-R} \qquad \text{and} \qquad \frac{h}{c} = \frac{b/2}{R}. \tag{2}$$

From the first of these equations we get $hR + au = h^2 + u^2 = c^2$, and the second yields $hR = bc/2$; combining these, we have $au = c^2 - bc/2$.

Applying the Pythagorean theorem to $\triangle ACP$ yields $h^2 = b^2 - (a-u)^2$, which implies $c^2 - u^2 = b^2 - (a-u)^2$ and therefore $a^2 = b^2 - c^2 + 2au$. Substituting $au = c^2 - bc/2$, we conclude $a^2 = b^2 + c^2 - bc$.

If $\angle ACB$ is obtuse, then similar reasoning can be used, except that now P is between A and H and C is between B and P. Therefore $CP = u - a$ and $HP = R - h$. The equations in (2) still hold, so we again obtain $a^2 = b^2 + c^2 - bc$.

If $\angle ACB$ is a right angle, then $H = C$ and O is the midpoint of AB, so $c = 2R = 2b$. Thus by the Pythagorean theorem, $a^2 = c^2 - b^2 = b^2 + c^2 - 2b^2 = b^2 + c^2 - bc$.

It is not possible for $\angle BAC$ to be a right angle, because that would imply $H = A$, contradicting $AH = R$. Finally, we consider the case in which $\angle BAC$ is obtuse. In this case, A is between P and H and P is between B and C, so $CP = a - u$, $HP = h + R$, and the first equation in (1) becomes $h/u = (a-u)/(h+R)$. Imitating the earlier algebraic reasoning then leads to the equation $a^2 = b^2 + c^2 + bc$. This completes the proof that $a^2 = b^2 + c^2 \pm bc$.

Since $a + b + c = 2021$, we have $(2021 - b - c)^2 = b^2 + c^2 \pm bc$, which simplifies to

$$tbc - 2 \cdot 2021b - 2 \cdot 2021c + 2021^2 = 0, \tag{3}$$

where t is either 1 or 3. We see that $2021 \mid tbc$. Since $2021 = 43 \cdot 47$, either $47 \mid b$ and $43 \mid c$ or $43 \mid b$ and $47 \mid c$ (neither b nor c can be divisible by both 43 and 47, because b and c are both less than 2021). By symmetry, we may assume $b = 47x$ and $c = 43y$, where $0 < x < 43$ and $0 < y < 47$. Substituting into (3) gives

$$txy - 94x - 86y + 2021 = 0. \tag{4}$$

If $t = 1$, then equation (4) can be rearranged to read

$$(x - 86)(y - 94) = 3 \cdot 43 \cdot 47.$$

Since $43 \nmid x$ and $47 \nmid y$, we must have either $x - 86 = \pm 3 \cdot 47$ or $y - 94 = \pm 3 \cdot 43$. But these are all inconsistent with $0 < x < 43$ and $0 < y < 47$, so this case is impossible.

Thus $t = 3$, and equation (4) is equivalent to

$$(3x - 86)(3y - 94) = 43 \cdot 47.$$

In this case the only possibilities are $3x - 86 = \pm 47$ and $3y - 94 = \pm 43$. Since $3x - 86 = 47$ does not give an integer value for x, we must have $3x - 86 = -47$ and $3y - 94 = -43$, so $x = 13$, $y = 17$, $b = 611$, $c = 731$, and finally $a = 2021 - b - c = 679$.

Editorial comment. An alternative proof of the equation $a^2 = b^2 + c^2 \pm bc$ uses the well-known equation $AH = 2R|\cos\alpha|$, where α denotes the measure of $\angle BAC$. This implies that the perpendicular bisector of OH passes through A if and only if $\cos\alpha = \pm 1/2$, which means α is either $\pi/3$ or $2\pi/3$. The law of cosines then gives $a^2 = b^2 + c^2 \pm bc$.

Note that the condition that the perpendicular bisector of OH contains I was not used in the solution above. In fact, this condition is implied by $\alpha = \pi/3$ but contradicted by $\alpha = 2\pi/3$. To see this, let V be the midpoint of the arc BC of the circumcircle of $\triangle ABC$ that does not contain A. If $\alpha = \pi/3$, then $\overrightarrow{OV} = \overrightarrow{AH}$, and therefore $AHVO$ is a rhombus. It follows that the perpendicular bisector of OH is AV, which bisects $\angle BAC$ and therefore passes through I. On the other hand, if $\alpha = 2\pi/3$, then $\overrightarrow{OV} = -\overrightarrow{AH}$. It follows that AV is parallel to OH, and therefore the perpendicular bisector of OH, which passes through A, does not pass through I.

Also solved by F. R. Ataev (Uzbekistan), M. Bataille (France), H. Chen (China), C. Chiser (Romania) & N. Ivaschescu (Canada), G. Fera (Italy), K. Gatesman, O. Geupel (Germany), N. Hodges (UK), W. Janous (Austria), K.-W. Lau (China), O. P. Lossers (Netherlands), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. S. Tiberio, M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), Davis Problem Solving Group, Eagle Problem Solvers, Fejéntaláltuka Szeged Problem Solving Group (Hungary), and the proposer.

A Sum and Integral That Cannot Be Interchanged

12285 [2021, 857]. *Proposed by Atul Dixit, Indian Institute of Technology, Gandhinagar, India.* Prove

$$\sum_{m=1}^{\infty} \int_0^{\infty} \frac{t\cos t}{t^2 + m^2u^2}\,dt = \int_0^{\infty} \left(-\frac{\pi}{2u}\cos t + \sum_{m=1}^{\infty} \frac{t\cos t}{t^2 + m^2u^2} \right) dt$$

for $u > 0$.

Solution by Albert Stadler, Herrliberg, Switzerland. Integrating by parts twice, always taking the antiderivative that vanishes at 0, we get

$$\int_0^{\infty} \frac{t\cos t}{t^2 + m^2u^2} dt = \int_0^{\infty} \frac{(t^2 - m^2u^2)\sin t}{(t^2 + m^2u^2)^2} dt = \int_0^{\infty} \frac{2t(t^2 - 3m^2u^2)}{(t^2 + m^2u^2)^3}(1 - \cos t) dt.$$

Since this last integrand satisfies the bound

$$\left| \frac{2t(t^2 - 3m^2u^2)}{(t^2 + m^2u^2)^3}(1 - \cos t) \right| \le \frac{12t}{(t^2 + m^2u^2)^2}$$

and

$$\sum_{m=1}^{\infty}\int_0^{\infty}\frac{t}{(t^2+m^2u^2)^2}\,dt=\sum_{m=1}^{\infty}\frac{1}{2m^2u^2}<\infty,$$

the dominated convergence theorem applies. Hence we can interchange summation and integration to get

$$\sum_{m=1}^{\infty}\int_0^{\infty}\frac{t\cos t}{t^2+m^2u^2}\,dt=\int_0^{\infty}\sum_{m=1}^{\infty}\frac{2t(t^2-3m^2u^2)}{(t^2+m^2u^2)^3}(1-\cos t)\,dt.$$

Next we integrate by parts twice "in the other direction," this time choosing in each case the antiderivative that vanishes as $t\to\infty$. To choose the right antiderivative in the second integration by parts, we use the partial fraction decomposition of coth (see I. S. Gradshteyn and I. M. Ryzhik (2007), Table of Integrals, Series, and Products, 7th ed., Burlington, MA: Academic Press, p. 44, equation 1.421.4), to compute

$$\lim_{t\to\infty}\sum_{m=1}^{\infty}\frac{t}{t^2+m^2u^2}=\lim_{t\to\infty}\left[\frac{\pi}{2u}\coth\left(\frac{\pi t}{u}\right)-\frac{1}{2t}\right]=\frac{\pi}{2u}.$$

This leads to the calculation

$$\begin{aligned}\sum_{m=1}^{\infty}\int_0^{\infty}\frac{t\cos t}{t^2+m^2u^2}\,dt&=\int_0^{\infty}(1-\cos t)\,d\left(-\sum_{m=1}^{\infty}\frac{t^2-m^2u^2}{(t^2+m^2u^2)^2}\right)\\&=\int_0^{\infty}\sin t\sum_{m=1}^{\infty}\frac{t^2-m^2u^2}{(t^2+m^2u^2)^2}\,dt\\&=\int_0^{\infty}\sin t\,d\left(\frac{\pi}{2u}-\sum_{m=1}^{\infty}\frac{t}{t^2+m^2u^2}\right)\\&=\int_0^{\infty}\left(-\frac{\pi}{2u}\cos t+\sum_{m=1}^{\infty}\frac{t\cos t}{t^2+m^2u^2}\right)dt.\end{aligned}$$

Editorial comment. As the solution above shows, one integration by parts is sufficient to get a situation where it is valid to pull the sum inside the integral, but to justify the interchange one must give more careful bounds (as was done by O. P. Lossers). The proposer and N. Hodges showed that the given integral I can be evaluated explicitly in terms of the digamma function ψ as

$$I=\frac{1}{2}\log\left(\frac{u}{2\pi}\right)-\frac{1}{4}\left(\psi\left(\frac{iu}{2\pi}\right)+\psi\left(-\frac{iu}{2\pi}\right)\right).$$

This follows from identifying the final sum as in the solution above and using Gradshteyn & Ryzhik (3.951.6).

Also solved by N. Hodges (UK), O. P. Lossers (Netherlands), J. Van Casteren & L. Kempeneers (Belgium), and the proposer.

Another Consequence of Euler's Identity

12287 [2021, 946]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Prove

$$\sum_{n=1}^{\infty}\left(n\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^2-\frac{1}{n}\right)=\frac{3}{2}-\frac{1}{2}\zeta(2)+\frac{3}{2}\zeta(3),$$

where ζ is the Riemann zeta function, defined by $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let $S_n = \sum_{k=n}^{\infty} 1/k^2$. We rewrite the summand of the series in the form

$$nS_n^2 - \frac{1}{n} = T_n - T_{n-1} + U_n,$$

where $T_0 = 0$ and we can evaluate both $\sum U_n$ and $\lim_{n\to\infty} T_n$. By the telescoping of the partial sums, the desired series converges to $\lim T_n + \sum U_n$.

In order to obtain a suitable T_n, we define T_n in terms of the sequence S. Let $T_n = a_n S_{n+1}^2 + b_n S_{n+1}$, where a_n and b_n will be chosen later. Since $S_{n+1} = S_n - 1/n^2$,

$$\begin{aligned} T_n - T_{n-1} &= a_n \left(S_n - \frac{1}{n^2}\right)^2 + b_n \left(S_n - \frac{1}{n^2}\right) - a_{n-1} S_n^2 - b_{n-1} S_n \\ &= (a_n - a_{n-1}) S_n^2 + \frac{-2a_n}{n^2} S_n + \frac{a_n}{n^4} + (b_n - b_{n-1}) S_n - \frac{b_n}{n^2}. \end{aligned}$$

To make the coefficient on S_n^2 be n, set $a_n = n(n+1)/2$. Now

$$\begin{aligned} T_n - T_{n-1} &= nS_n^2 + \frac{-(n+1)}{n} S_n + \frac{n+1}{2n^3} + (b_n - b_{n-1}) S_n - \frac{b_n}{n^2} \\ &= nS_n^2 - \frac{1}{n} + (b_n - b_{n-1} - (1 + 1/n)) S_n + E_n, \end{aligned}$$

where $E_n = (n+1)/(2n^3) - b_n/n^2 + 1/n$. To eliminate the coefficient on S_n, set $b_0 = 0$ and $b_n = b_{n-1} + 1 + 1/n$ for $n \ge 1$. Thus $b_n = n + H_n$, where H_n is the nth harmonic number $\sum_{i=1}^{n} 1/i$. Now

$$T_n - T_{n-1} = nS_n^2 - \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{2n^3} - \frac{H_n}{n^2}.$$

Since $T_0 = 0$, summing this identity yields

$$\sum_{n=1}^{m} \left(nS_n^2 - \frac{1}{n}\right) = T_m + \sum_{n=1}^{m} \frac{H_n}{n^2} - \sum_{n=1}^{m} \frac{1}{2n^2} - \sum_{n=1}^{m} \frac{1}{2n^3}.$$

Letting $m \to \infty$ and using the Euler identity $\sum_{n=1}^{\infty} H_n/n^2 = 2\zeta(3)$, we obtain

$$\sum_{n=1}^{\infty} \left(nS_n^2 - \frac{1}{n}\right) = \lim_{m\to\infty} T_m + 2\zeta(3) - \frac{1}{2}\zeta(2) - \frac{1}{2}\zeta(3).$$

Returning to the definition of T_m, we now have

$$T_m = \frac{m(m+1)}{2} S_{m+1}^2 + (m + H_m) S_{m+1}. \tag{$*$}$$

To compute the limit, we use $1/k - 1/(k+1) < 1/k^2 < 1/(k-1) - 1/k$ to obtain $1/m < S_m < 1/(m-1)$, and thus $\lim_{m\to\infty} mS_m = 1$. Hence the first term in $(*)$ tends to $1^2/2$ and the second term in $(*)$ tends to 1 (since $H_m/m \to 0$). This gives $\lim_{m\to\infty} T_m = 1/2 + 1 = 3/2$, and the desired result follows.

Editorial comment. A simple proof of Euler's formula for $\zeta(3)$ appears in an editorial comment following problem 12091 [2019, 180; 2020, 853] in this MONTHLY. The solution to problem 2136 from Mathematics Magazine by Kelly D. McLenithan (2023) *Math. Mag.*

96, 90–91 uses similar methods, in particular the same identity of Euler, which it observes is the $q = 2$ case of Euler's 1775 result

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^q} = (q+2)\zeta(q+1) - \sum_{m=1}^{q-2} \zeta(m+1)\zeta(q-m).$$

Also solved by T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), O. Bordellés (France), P. Bracken, B. Bradie, B. S. Burdick, H. Chen, A. De la Fuente, G. Fera (Italy), O. Geupel (Germany), E. A. Herman, N. Hodges (UK), M. Hoffman, K.-W. Lau (China), O. P. Lossers (Netherlands), J. Manoharmayum (UK), C. Morin (France), M. Omarjee (France), C. Sanford, A. Stadler (Switzerland), A. Stenger, S. M. Stewart (Saudi Arabia), R. Tauraso (Italy), T. Wiandt, UM6P Math Club (Morocco), and the proposer.

The Dirichlet Integral in Disguise

12288 [2021, 946]. *Proposed by Seán Stewart, Bomaderry, Australia.* Prove

$$\int_0^{\infty} \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx = \frac{\pi}{5}.$$

Solution by Ming-Can Fan, Huizhou University, Guangdong, China. The substitution $x = 1/t$ yields

$$\begin{aligned}\int_0^{\infty} \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx &= \int_0^{\infty} \left(\frac{t^2 - \sin^2 t}{t^3}\right)^2 dt \\ &= \int_0^{\infty} \frac{2t^2 - 2\sin^2 t}{t^4}\, dt + \int_0^{\infty} \frac{\sin^4 t - t^4}{t^6}\, dt.\end{aligned}$$

Using $2\sin^2 t = 1 - \cos(2t)$ and integration by parts three times yields

$$\begin{aligned}\int_0^{\infty} \frac{2t^2 - 2\sin^2 t}{t^4}\, dt &= \int_0^{\infty} \frac{2t^2 - 1 + \cos(2t)}{t^4}\, dt \\ &= -\frac{2t^2 - 1 + \cos(2t)}{3t^3}\Bigg|_0^{\infty} + \int_0^{\infty} \frac{4t - 2\sin(2t)}{3t^3}\, dt \\ &= -\frac{2t - \sin(2t)}{3t^2}\Bigg|_0^{\infty} + \int_0^{\infty} \frac{2 - 2\cos(2t)}{3t^2}\, dt \\ &= -\frac{2 - 2\cos(2t)}{3t}\Bigg|_0^{\infty} + \int_0^{\infty} \frac{4\sin(2t)}{3t}\, dt = \frac{4}{3}\cdot\frac{\pi}{2} = \frac{2\pi}{3},\end{aligned}$$

where in the last step we have used the well-known fact that $\int_0^{\infty} \sin(at)/t\, dt = \pi/2$ for all $a > 0$. (The case $a = 1$ is known as the *Dirichlet integral*, and the general formula follows via the substitution $u = at$.) Similarly, using $\sin^4 t = (3 + \cos(4t) - 4\cos(2t))/8$ and integration by parts five times, we get

$$\begin{aligned}\int_0^{\infty} \frac{\sin^4 t - t^4}{t^6}\, dt &= \int_0^{\infty} \frac{(3 + \cos(4t) - 4\cos(2t))/8 - t^4}{t^6}\, dt \\ &= \int_0^{\infty} \frac{2\sin(2t) - 16\sin(4t)}{15t}\, dt = \frac{2}{15}\cdot\frac{\pi}{2} - \frac{16}{15}\cdot\frac{\pi}{2} = -\frac{7\pi}{15}.\end{aligned}$$

Hence

$$\int_0^{\infty} \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^2 dx = \frac{2\pi}{3} - \frac{7\pi}{15} = \frac{\pi}{5}.$$

Also solved by U. Abel & V. Kushnirevych (Germany), T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), P. Bracken, H. Chen, Ó. Ciaurri (Spain), G. A. Edgar, G. Fera (Italy), M. L. Glasser, G. C. Greubel, J.-P. Grivaux (France), N. Grivaux (France), J. A. Grzesik, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), J. Londoño & J. Quintero (Colombia), O. P. Lossers (Netherlands), J. Manoharmayum (UK), K. D. McLenithan, R. Mortini (Luxembourg) & R. Rupp (Germany), A. Natian, M. Omarjee (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Van Casteren & L. Kempeneers (Belgium), E. I. Verriest, M. Wildon (UK), UM6P Math Club (Morocco), and the proposer.

CLASSICS

C17. *Due to Paul Erdős, Abraham Ginzburg, and Abraham Ziv; suggested by Gabriel Carroll and Yuri Ionin, independently.* Given $2n - 1$ integers, show that it is possible to choose n of them that sum to a multiple of n.

Parallel Mountain Climbers

C16. *Suggested by the editors.* Two hikers start together at the bottom of a mountain and climb to the summit but along different trails, which may go up and down along the way. Show that it is possible for them to complete their respective hikes in such a way that they are at the same elevation at every moment.

Solution. Suppose that the length of the first trail is a and the length of the second trail is b. For $s \in [0, a]$, let $f(s)$ represent the elevation of the first trail at distance s from the start, and for $t \in [0, b]$, let $g(t)$ represent the elevation of the second trail at distance t from the start. (We take "bottom" and "summit" in the problem statement to mean that neither trail dips below the initial elevation or rises above the final elevation.) The ordered pair (s, t) describes simultaneous positions of the two hikers. We require a plan that maintains $f(s) = g(t)$. The initial position is $(0, 0)$, and we want to show that (a, b) can be reached.

When $f(s) = g(t)$ but s is not a local extremum of f and t is not a local extremum of g, the two hikers can both move higher or both move lower. Hence the crucial points are those pairs (s, t) such that at least one coordinate is a local extremum of the corresponding function. We call the local extrema other than $(0, 0)$ or (a, b) *internal*.

We define a graph with these crucial points as its vertices. Two crucial points are adjacent if the hikers can move from one to the other without encountering any other crucial point. In particular, if one hiker is at an internal local extremum, then that hiker can move forward or backward while the other matches the change in elevation, until one of them reaches another local extremum. Thus such vertices have degree 2 in the graph. If both hikers are at an internal local extremum, then the degree is 0 or 4.

The vertices $(0, 0)$ and (a, b) have degree 1 in the graph; they are the only vertices with odd degree. Since every graph has an even number of vertices of odd degree, $(0, 0)$ and (a, b) must lie in the same component of the graph, and hence there is a path for the two hikers to reach the summit while maintaining the same elevation.

Editorial Comment. It is necessary that the functions f and g achieve their common global minimum at 0 and their common global maximum at a and b, respectively. If the trail for one hiker but not the other dips below the starting elevation (or passes above the summit), the hikers cannot achieve the required goal. In addition, we assume that f and g have only finitely many critical points.

The problem has appeared in many places and in various forms. The earliest formulation in terms of mountain climbers may be in J. V. Whittaker (1966), A mountain-climbing problem, *Canadian J. Math.*, 18: 873–882, but the essential idea appears in T. Homma (1952), A theorem on continuous functions, *Kodai Math. Sem. Reports*, 4(1): 13–16.

SOLUTIONS

Four Inequalities, One Proof

12275 [2021, 755]. *Proposed by Yun Zhang, Xi'an, China.* Let x, y, and z be positive real numbers with $x + y + z = 3$. Prove each of the following inequalities.

(a) $x^5y^5z^5(x^4 + y^4 + z^4) \le 3$.

(b) $x^8y^8z^8(x^5 + y^5 + z^5) \le 3$.

(c) $x^{11}y^{11}z^{11}(x^6 + y^6 + z^6) \le 3$.

(d) $x^{16}y^{16}z^{16}(x^7 + y^7 + z^7) \le 3$.

Solution by Kyle Gatesman, Johns Hopkins University, Baltimore, MD. More generally, we are interested in finding the maximum value of the function f defined by $f(x, y, z) = x^py^pz^p(x^q + y^q + z^q)$ subject to the constraints $x, y, z > 0$ and $x + y + z = 3$, where we allow p and q to be arbitrary real numbers with $p > 0$ and $q > 2$. We prove that the condition

$$(p + q)^q(q - 1)^{q-1} \le (2p + q)p^{q-1}(q + 1)^{q-1} \tag{1}$$

is sufficient to guarantee that the unique optimal solution is $(x, y, z) = (1, 1, 1)$, which implies that the maximum value of $f(x, y, z)$ is 3. The inequalities **(a)**, **(b)**, **(c)**, and **(d)** in the problem statement follow from this result.

Let S denote the closed simplex $\{(x, y, z) \in \mathbb{R}^3 : x, y, z \ge 0 \text{ and } x + y + z = 3\}$, so that our optimization domain is the relative interior of S. By the extreme value theorem, the continuous function f attains its supremum on S at one or more points in S. Since the value of f is positive in the interior of S and zero on the boundary, the supremum of f over S is attained in the interior. Therefore, any global maximizer (x, y, z) of f over S satisfies $\nabla f(x, y, z) = \lambda\nabla(x + y + z - 3) = (\lambda, \lambda, \lambda)$ for some λ. Letting $\alpha = x^q + y^q + z^q$, we have

$$\nabla f(x, y, z) = x^py^pz^p\left(\frac{qx^q + p\alpha}{x}, \frac{qy^q + p\alpha}{y}, \frac{qz^q + p\alpha}{z}\right),$$

so a necessary condition for optimality is

$$\frac{qx^q + p\alpha}{x} = \frac{qy^q + p\alpha}{y} = \frac{qz^q + p\alpha}{z}. \tag{2}$$

Temporarily fix $\alpha > 0$, and let $g(t) = (qt^q + p\alpha)/t$. Since

$$g''(t) = q(q-1)(q-2)t^{q-3} + 2p\alpha/t^3$$

and since $p > 0$ and $q > 2$, we have $g''(t) > 0$ for all $t > 0$, so g is strictly convex over $(0, \infty)$. Thus, for any constant c, the equation $g(t) = c$ admits at most two distinct solutions in t. The numbers x, y, and z must be solutions to such an equation, so x, y, and z cannot all be distinct. By symmetry, we may assume that $z = x$.

Let $u = y/x$, so that $\alpha = x^q(u^q + 2)$. Condition (2) is equivalent to

$$\frac{qx^q + px^q(u^q + 2)}{x} = \frac{qu^q x^q + px^q(u^q + 2)}{ux},$$

which simplifies to

$$pu^{q+1} - (p+q)u^q + (2p+q)u - 2p = 0.$$

This condition is satisfied when $u = 1$, which corresponds to $y = x$. To show that $u = 1$ is the only solution when (1) holds, it suffices to show that the function h defined by $h(u) = pu^{q+1} - (p+q)u^q + (2p+q)u - 2p$ is strictly increasing (and therefore injective) over $(0, \infty)$.

Observe that

$$h'(x) = p(q+1)u^q - (p+q)qu^{q-1} + 2p + q \quad \text{and}$$

$$h''(x) = pq(q+1)u^{q-1} - (p+q)q(q-1)u^{q-2} = pq(q+1)u^{q-2}\left(u - \frac{(p+q)(q-1)}{p(q+1)}\right).$$

Let $u_0 = (p+q)(q-1)/(p(q+1))$. Clearly $h''(u)$ is negative for $u \in (0, u_0)$ and positive for $u \in (u_0, \infty)$, so $h'(u)$ attains its minimum value at $u = u_0$. Therefore, h is strictly increasing if and only if $h'(u_0) \geq 0$. This is equivalent to

$$\left(\frac{(p+q)(q-1)}{p(q+1)}\right)^{q-1} \big((p+q)(q-1) - (p+q)q\big) + 2p + q \geq 0,$$

which is equivalent to (1). Hence, when (1) holds, $u = 1$ is the only value of u for which (x, ux, x) can be a maximizer of f over S. It follows that the only possible maximizer of f over all of S is $(1, 1, 1)$.

Editorial comment. It is not hard to show that, for fixed $q > 2$, inequality (1) holds for all sufficiently large p. In fact, in each of **(a)**–**(d)**, the value of p is the smallest positive integer for which (1) holds.

There are several other ways to prove these inequalities. As indicated by multiple solvers, one could use the pqr-method, which involves rewriting the inequalities in terms of $x + y + z$, $xy + yz + zx$, and xyz (often denoted p, q, and r; see Chapter 14 in Z. Cvetkovski, (2012), *Inequalities: Theorems, Techniques, and Selected Problems*, Berlin: Springer). Alternatively, one can rewrite all four inequalities in the form $f \geq 0$, where $f(x, y, z) = (x + y + z)^{k+1} - 3^k(xyz)^p(x^q + y^q + z^q)$. Assuming without loss of generality that $x \geq y \geq z$, we can write $x = u + v + w$, $y = u + v$, and $z = u$ for $u, v, w \geq 0$. For inequalities **(a)**–**(d)**, taking $k = 18$, 28, 38, and 54, respectively, Albert Stadler used *Mathematica* to verify that $f(u + v + w, u + v, u) \geq 0$. These are the smallest values of k for which Stadler's method works.

Also solved by P. Bracken, D. Henderson, N. Hodges (UK), W. Janous (Austria), K.-W. Lau (China), P. W. Lindstrom, A. Stadler (Switzerland), R. Stong, J. Vukmirović (Serbia), J. Yan (China), L. Zhou, and Fejéntaláltuka Szeged Problem Solving Group (Hungary).

A Complicated Way to Write 1

12276 [2021, 755]. *Proposed by Joe Santmyer, Las Cruces, NM.* Prove

$$\sum_{n=2}^{\infty} \frac{1}{n+1} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{2^{i-1}(i-1)!(n-2i)!} = 1.$$

Solution by Allen Stenger, Boulder, CO. Letting a_n denote the inner sum, we see that a_n is the coefficient of x^{n-2} in the product

$$\left(\sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} \right) \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} \right).$$

Since the product equals $e^{x^2/2} e^x$, we have

$$\sum_{n=2}^{\infty} a_n x^n = x^2 e^{x^2/2+x}.$$

Integrating both sides from 0 to 1 yields

$$\sum_{n=2}^{\infty} \frac{a_n}{n+1} = \int_0^1 x^2 e^{x^2/2+x}\, dx = (x-1)e^{x^2/2+x}\Big|_0^1 = 1,$$

justified by computing $f'(x) = x^2 e^{x^2/2+x}$ when $f(x) = (x-1)e^{x^2/2+x}$.

Also solved by T. Amdeberhan & V. H. Moll, M. Bataille (France), A. Berkane (Algeria), C. Burnette, Ó. Ciaurri (Spain), A. De la Fuente, G. Fera (Italy), K. Gatesman, M. L. Glasser, J. W. Hagood, E. A. Herman, N. Hodges (UK), W. Janous (Austria), O. Kouba (Syria), O. P. Lossers (Netherlands), D. Pinchon (France), E. Schmeichel, A. Stadler (Switzerland), S. M. Stewart (Saudi Arabia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), L. Zhou, and the proposer.

A Matrix Rank Restriction

12277 [2021, 756]. *Proposed by Cristian Chiser, Elena Cuza College, Craiova, Romania.* Let A, B, and C be three pairwise commuting 2-by-2 real matrices. Show that if at least one of the matrices $A - B$, $B - C$, and $C - A$ is invertible, then the matrix

$$A^2 + B^2 + C^2 - AB - AC - BC$$

cannot have rank 1.

Solution by Jacob Boswell & Chip Curtis, Missouri Southern State University, Joplin, MO. Set $M = A^2 + B^2 + C^2 - AB - AC - BC$, $D = A - B$, and $E = A - C$. By symmetry, we may assume that D is invertible. All of the named matrices pairwise commute. Thus

$$M = D^2 - DE + E^2.$$

Multiplying on the left by $(D^{-1})^2$ yields

$$N = I - X + X^2, \tag{$*$}$$

where $N = (D^{-1})^2 M$ and $X = D^{-1}E$. Since D is invertible, M has rank 1 if and only if N has rank 1.

We conclude by showing that N cannot have rank 1. To the contrary, assume that N has 0 as an eigenvalue with multiplicity 1. Let $\mathbf{v}$ be an eigenvector of N with eigenvalue 0. Since N and X commute, $NX\mathbf{v} = XN\mathbf{v} = 0$, so $X\mathbf{v}$ must be a multiple of $\mathbf{v}$, since the eigenspace of 0 for N is one-dimensional. Thus, $\mathbf{v}$ is an eigenvector of X for some eigenvalue λ. Multiplying both sides of $(*)$ on the right by $\mathbf{v}$ gives $\mathbf{0} = \left(1 - \lambda + \lambda^2\right)\mathbf{v}$. It follows that $\lambda^2 - \lambda + 1 = 0$. Since X is a real matrix, its complex eigenvalues occur in conjugate pairs, so both roots of the polynomial p given by $p(x) = x^2 - x + 1$ are eigenvalues of X, and p is the characteristic polynomial of X. This yields $I - X + X^2 = 0$, or $N = 0$, a contradiction.

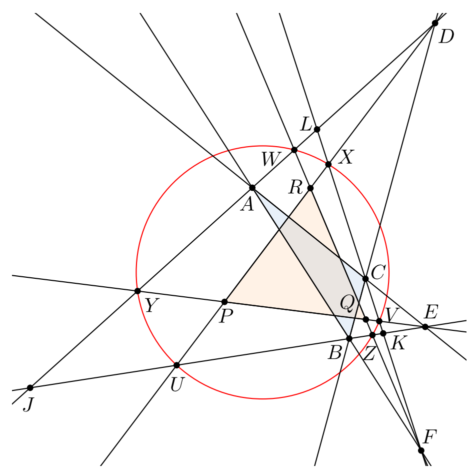
Also solved by G. Bourgeois (France), S. M. Gagola Jr., K. Gatesman, J.-P. Grivaux (France), J. W. Hagood, E. A. Herman, E. J. Ionaşcu, K. T. L. Koo (China), J. H. Lindsey II, O. P. Lossers (Netherlands), K. D. McLenithan, M. Omarjee (France), A. Pathak, A. Stadler (Switzerland), R. Stong, J. Stuart & R. Horn, R. Tauraso (Italy), L. Zhou, UM6P Math Club (Morocco), and the proposer.

An Equilateral Triangle and a Circle

12278 [2021, 756]. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam.* Let ABC be a scalene triangle, and let its external angle bisectors at A, B, and C meet BC, CA, and AB at D, E, and F, respectively. Let l, m, and n be lines through D, E, and F that (internally) trisect angles $\angle ADB$, $\angle BEC$, and $\angle CFA$, respectively, with the angle between l and AD equal to 1/3 of $\angle ADB$, the angle between m and BE equal to 1/3 of $\angle BEC$, and the angle between n and CF equal to 1/3 of $\angle CFA$.
(a) Show that l, m, and n form an equilateral triangle.
(b) The lines l, m, and n each intersect AD, BE, and CF. Of these nine points of intersection, three are the points D, E, and F. Show that the other six lie on a circle.

Solution by Li Zhou, Polk State College, Winter Haven, FL.

(a) We use A, B, and C to denote both the vertices of $\triangle ABC$ and the interior angles at those vertices. We may assume $A < B < C$. We also use D and E to denote $\angle CDA$ and $\angle BEC$, respectively. Let J be the intersection of AD and BE, K the intersection of BE and CF, and L the intersection of CF and AD. Let P be the intersection of l and m, Q the intersection of m and n, and R the intersection of n and l.



By construction, the three triangles $\triangle JAB$, $\triangle CKB$, and $\triangle CAL$ are all similar to $\triangle JKL$, with interior angles $(\pi - C)/2$, $(\pi - A)/2$, and $(\pi - B)/2$ at J, K, and L, respectively. Also,

$$D = \pi - \angle ACD - \angle DAC = \pi - (\pi - C) - \frac{\pi - A}{2} = C - \frac{B + C}{2} = \frac{C - B}{2},$$

and similarly $E = (C - A)/2$. Therefore,

$$\angle RPQ = \angle LJK + \frac{D}{3} + \frac{E}{3} = \frac{\pi - C}{2} + \frac{C - B}{6} + \frac{C - A}{6} = \frac{3\pi - (A+B+C)}{6} = \frac{\pi}{3}.$$

Similarly, $\angle PQR = \angle QRP = \pi/3$, so $\triangle PQR$ is equilateral.

(b) Suppose that l intersects BE at U and CF at X, m intersects CF at V and AD at Y, and n intersects AD at W and BE at Z. Applying the law of sines to $\triangle JBA$, we get

$$\frac{JA}{\sin((\pi - B)/2)} = \frac{JB}{\sin((\pi - A)/2)}, \tag{1}$$

and applying it to $\triangle JBD$ and $\triangle JEA$ yields

$$\frac{JD}{\sin((\pi - B)/2)} = \frac{JB}{\sin D}, \quad \frac{JE}{\sin((\pi - A)/2)} = \frac{JA}{\sin E}. \tag{2}$$

Also, $\angle JUD = \pi - \angle PUE = \angle EPU + \angle UEP = (2\pi + E)/3$, and similarly $\angle EYJ = (2\pi + D)/3$. Therefore applying the law of sines to $\triangle JUD$ and $\triangle EYJ$ gives us

$$\frac{JU}{\sin(D/3)} = \frac{JD}{\sin((2\pi + E)/3)} \quad \text{and} \quad \frac{JY}{\sin(E/3)} = \frac{JE}{\sin((2\pi + D)/3)}. \tag{3}$$

Combining (1), (2), and (3) yields

$$\frac{JU}{JY} = \frac{\sin E \, \sin(D/3) \, \sin((2\pi + D)/3)}{\sin D \, \sin(E/3) \, \sin((2\pi + E)/3)}.$$

By the triple-angle formula,

$$\begin{aligned} \sin D &= \sin(D/3)(3\cos^2(D/3) - \sin^2(D/3)) \\ &= 4\sin(D/3)\sin((2\pi + D)/3)\sin((\pi + D)/3), \end{aligned}$$

and the same is true if angle D is replaced with angle E. Thus,

$$\frac{JU}{JY} = \frac{\sin((\pi + E)/3)}{\sin((\pi + D)/3)}.$$

Finally,

$$\angle ZWJ = \pi - \angle DWR = \angle WRD + \angle RDW = (\pi + D)/3,$$

and similarly $\angle JZW = (\pi + E)/3$. Therefore, the law of sines applied to $\triangle JZW$ yields

$$\frac{JW}{\sin((\pi + E)/3)} = \frac{JZ}{\sin((\pi + D)/3)},$$

and hence

$$\frac{JW}{JZ} = \frac{\sin((\pi + E)/3)}{\sin((\pi + D)/3)} = \frac{JU}{JY}.$$

We conclude that U, Y, W, and Z lie on a circle. Likewise, V, Z, U, and X lie on a circle, and W, X, V, and Y lie on a circle. If the three circles are distinct, then the three radical axes UZ, VX, and WY are concurrent. But these axes are JK, KL, and LJ, which are not concurrent. Therefore, two of the three circles are the same, so the six points are all on the same circle.

Editorial comment. Zhou points out that applying Pascal's theorem to the hexagon $UXVYWZ$ shows that D, E, and F are collinear. Thus, by the theorem of Desargues, JP, KQ, and LR are concurrent.

Also solved by C. R. Pranesachar (India), R. Stong, and the proposer.

A Stirling Identity

12279 [2021, 856]. *Proposed by Brad Isaacson, Brooklyn, NY.* Let $S(m, k)$ denote the number of partitions of a set with m elements into k nonempty blocks. (These are the *Stirling numbers of the second kind*.) Let j and n be positive integers of opposite parity with $j < n$. Prove

$$\sum_{r=j}^{n} \frac{(-1)^r (r-1)! \binom{r}{j} S(n, r)}{2^r} = 0.$$

Solution I by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Note that $r!S(n, r)$ is the number of surjective mappings from a set with n elements onto a set with r elements. Therefore, by inclusion-exclusion,

$$r!S(n, r) = \sum_{k=1}^{r} (-1)^{r-k} \binom{r}{k} k^n = \left[\frac{d^n}{dt^n} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} e^{kt} \right]_{t=0} = \left[\frac{d^n}{dt^n} (e^t - 1)^r \right]_{t=0}.$$

Let $a(n, j)$ denote the sum in question. Since $S(n, r) = 0$ for $r > n$,

$$a(n, j) = \left[\frac{d^n}{dt^n} \sum_{r=j}^{\infty} \frac{(-1)^r}{r2^r} \binom{r}{j} \left(e^t - 1\right)^r \right]_{t=0} = \left[\frac{d^n}{dt^n} \frac{1}{j} \sum_{r=j}^{\infty} \binom{r-1}{j-1} \left(\frac{1-e^t}{2} \right)^r \right]_{t=0}.$$

(The interchange of the derivative and summation can be justified by showing that the series of derivatives converges uniformly on an interval around 0.) From the negative binomial expansion, $\sum_{r=j}^{\infty} \binom{r-1}{j-1} x^r = x^j/(1-x)^j$. Hence,

$$a(n, j) = \frac{1}{j} \cdot \left[\frac{d^n}{dt^n} \left(\frac{1-e^t}{1+e^t} \right)^j \right]_{t=0}.$$

Since $(1 - e^t)/(1 + e^t)$ is odd, so is

$$\frac{d^n}{dt^n} \left(\frac{1-e^t}{1+e^t} \right)^j$$

when j and n have opposite parity. Therefore, $a(n, j) = 0$ in this case.

Solution II by Tewodros Amdeberhan and Victor H. Moll, Tulane University, New Orleans, LA. Let $a(n, j)$ denote the sum in question. We proceed by induction on n, beginning with $a(2, 1) = -1/2 + 2/4 = 0$. Grouping the partitions by whether n is a part by itself, $S(n, r) = rS(n-1, r) + S(n-1, r-1)$. With the standard conventions that $\binom{r}{j} = 0$ for $j < 0$ or $j > r$ and that $S(n, r) = 0$ for $r > n$, we use the recurrence and reindexing to obtain

$$\begin{aligned} a(n, j) &= \sum_{r=0}^{n} \frac{(-1)^r (r-1)! \binom{r}{j} S(n, r)}{2^r} \\ &= \sum_{r=0}^{n} \frac{(-1)^r (r-1)! \binom{r}{j} \big(rS(n-1, r) + S(n-1, r-1) \big)}{2^r} \\ &= \sum_{r=0}^{n} \frac{(-1)^r r! \big(\binom{r}{j} - \binom{r+1}{j}/2 \big) S(n-1, r)}{2^r}. \end{aligned}$$

Via three applications of the binomial recurrence,

$$\binom{r}{j} - \frac{1}{2}\binom{r+1}{j} = \binom{r}{j} - \frac{1}{2}\binom{r}{j} - \frac{1}{2}\binom{r}{j-1}$$
$$= \frac{1}{2}\binom{r-1}{j} + \frac{1}{2}\binom{r-1}{j-1} - \frac{1}{2}\binom{r-1}{j-1} - \frac{1}{2}\binom{r-1}{j-2} = \frac{1}{2}\binom{r-1}{j} - \frac{1}{2}\binom{r-1}{j-2}.$$

Substituting this identity into the previous expression for the sum yields

$$a(n,j) = \sum_{r=0}^{n} \frac{(-1)^r r!\left(\binom{r-1}{j}/2 - \binom{r-1}{j-2}/2\right)S(n-1,r)}{2^r}$$
$$= \sum_{r=0}^{n} \frac{(-1)^r (r-1)!\left(\frac{j+1}{2}\binom{r}{j+1} - \frac{j-1}{2}\binom{r}{j-1}\right)S(n-1,r)}{2^r}$$
$$= \frac{j+1}{2}\,a(n-1,j+1) - \frac{j-1}{2}\,a(n-1,j-1).$$

By convention $a(n-1,n) = 0$, and the rightmost term is 0 when $j = 1$. In all other cases, when j and n have opposite parity, the induction hypothesis implies $a(n-1, j \pm 1) = 0$. We conclude $a(n,j) = 0$.

Also solved by U. Abel & V. Kushnirevych (Germany), A. Berkane (Algeria), A. De la Fuente, O. P. Lossers (Netherlands), J. H. Nieto (Venezuela), A. Stadler (Switzerland), R. Tauraso (Italy), M. Wildon (UK), UM6P Math Club (Morocco), and the proposer.

A Hyperbolic Logarithmic Integral

12281 [2021, 856]. *Proposed by Paolo Perfetti, University of Rome Tor Vergata, Rome, Italy.* Evaluate

$$\int_0^\infty \left(\frac{\cosh x}{\sinh^2 x} - \frac{1}{x^2}\right)(\ln x)^2\,dx.$$

Solution by Michel Bataille, Rouen, France. Let I be the integral to be evaluated. We show that $I = (\ln 2)(2\gamma - \ln 2 - 2\ln\pi)$, where γ is Euler's constant.

Suppose $0 < a < b$. Integrating by parts gives

$$\int_a^b \left(\frac{\cosh x}{\sinh^2 x} - \frac{1}{x^2}\right)(\ln x)^2\,dx = F(b) - F(a) - 2\int_a^b \frac{\ln x}{x}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)dx,$$

where $F(x) = (\ln x)^2(1/x - 1/\sinh x)$. Since

$$\lim_{b\to\infty} F(b) = \lim_{b\to\infty}\left(\frac{(\ln b)^2}{b} - \frac{2(\ln b)^2}{e^b - e^{-b}}\right) = 0 - 0 = 0$$

and

$$\lim_{a\to 0^+} F(a) = \lim_{a\to 0^+} a(\ln a)^2 \cdot \frac{\sinh a - a}{a^2 \sinh a} = 0 \cdot \frac{1}{6} = 0,$$

we conclude

$$I = -2\int_0^\infty \frac{\ln x}{x}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)dx.$$

It is known that for $x \neq 0$,

$$\frac{1}{\sinh x} = \frac{1}{x} + \sum_{n=1}^{\infty}(-1)^n \frac{2x}{x^2 + n^2\pi^2}$$

(see I. S. Gradshteyn, I. M. Ryzhik (2007), *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press, p. 27, equation 1.217.2). Hence,

$$I = \int_0^\infty \left(\sum_{n=1}^\infty \frac{4(-1)^n \ln x}{x^2 + n^2\pi^2} \right) dx.$$

Next we show that we can reverse the order of the integration and summation in this formula. For $0 < x \le 1$ and N a positive integer, we have

$$\left| \sum_{n=1}^N \frac{4(-1)^n \ln x}{x^2+n^2\pi^2} \right| \le \sum_{n=1}^N \left| \frac{4(-1)^n \ln x}{x^2+n^2\pi^2} \right| \le \sum_{n=1}^N \frac{-4\ln x}{n^2\pi^2} \le -\frac{4\ln x}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} = -\frac{2\ln x}{3},$$

and $\int_0^1 -(2/3)\ln x\, dx < \infty$. It follows, by the dominated convergence theorem, that

$$\int_0^1 \sum_{n=1}^\infty \frac{4(-1)^n \ln x}{x^2+n^2\pi^2}\, dx = \sum_{n=1}^\infty \int_0^1 \frac{4(-1)^n \ln x}{x^2+n^2\pi^2}\, dx. \tag{4}$$

Similarly, for $x \ge 1$ and N a positive integer,

$$\left| \sum_{n=1}^N \frac{4(-1)^n \ln x}{x^2+n^2\pi^2} \right| \le \frac{4\ln x}{x^2+\pi^2} \le \frac{4\ln x}{x^2}$$

and $\int_1^\infty 4(\ln x)/x^2\, dx < \infty$, so

$$\int_1^\infty \sum_{n=1}^\infty \frac{4(-1)^n \ln x}{x^2+n^2\pi^2}\, dx = \sum_{n=1}^\infty \int_1^\infty \frac{4(-1)^n \ln x}{x^2+n^2\pi^2}\, dx. \tag{5}$$

Combining (4) and (5), we have

$$I = \int_0^\infty \sum_{n=1}^\infty \frac{4(-1)^n \ln x}{x^2+n^2\pi^2}\, dx = 4\sum_{n=1}^\infty (-1)^n \int_0^\infty \frac{\ln x}{x^2+n^2\pi^2}\, dx. \tag{6}$$

To evaluate the integral on the right side of (6), we first use the substitution $u = x/(n\pi)$, as follows:

$$\begin{aligned} \int_0^\infty \frac{\ln x}{x^2+n^2\pi^2}\, dx &= \frac{1}{n^2\pi^2} \int_0^\infty \frac{\ln x}{(x/(n\pi))^2+1}\, dx \\ &= \frac{1}{n\pi}\left(\int_0^\infty \frac{\ln(n\pi)}{u^2+1}\, du + \int_0^\infty \frac{\ln u}{u^2+1}\, du \right) \\ &= \frac{\ln(n\pi)}{2n} + \frac{1}{n\pi}\int_0^\infty \frac{\ln u}{u^2+1}\, du. \end{aligned}$$

The last integral above vanishes, as can be seen by making the substitution $t = 1/u$:

$$\int_0^\infty \frac{\ln u}{u^2+1}\, du = \int_0^\infty \frac{-\ln t}{1/t^2+1} \cdot \frac{1}{t^2}\, dt = -\int_0^\infty \frac{\ln u}{u^2+1}\, du.$$

Substituting into (6), we obtain

$$I = 4\sum_{n=1}^\infty (-1)^n \frac{\ln(n\pi)}{2n} = 2\left(\sum_{n=1}^\infty \frac{(-1)^n \ln n}{n} + \ln\pi \sum_{n=1}^\infty \frac{(-1)^n}{n} \right).$$

Finally, we use the formulas $\sum_{n=1}^\infty (-1)^{n-1}/n = \ln 2$ and

$$\sum_{n=1}^\infty (-1)^{n-1}(\ln n)/n = (\ln 2)^2/2 - \gamma \ln 2$$

(see the solution to problem 873 in *Coll. Math. J.* 40(2), March 2009, pp. 136–137) to conclude

$$I = 2\left(\gamma \ln 2 - \frac{(\ln 2)^2}{2} - \ln \pi \ln 2\right) = (\ln 2)(2\gamma - \ln 2 - 2\ln \pi).$$

Also solved by U. Abel & V. Kushnirevych (Germany), T. Amdeberhan & V. H. Moll, A. Berkane (Algeria), N. Bhandari (Nepal), K. N. Boyadzhiev, P. Bracken, H. Chen, G. Fera (Italy), M. L. Glasser, N. Hodges (UK), J. E. Kampmeyer, L. Kempeneers & J. Van Casteren (Belgium), O. Kouba (Syria), M. Omarjee (France), A. Stadler (Switzerland), A. Stenger, S. M. Stewart (Saudi Arabia), M. Štofka (Slovakia), R. Stong, R. Tauraso (Italy), Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

CLASSICS

C16. *Suggested by the editors.* Two hikers start together at the bottom of a mountain and climb to the summit but along different trails, which may go up and down along the way. Show that it is possible for them to complete their respective hikes in such a way that they are at the same elevation at every moment.

Costly Positive Integers

C15. *Suggested by Joel Spencer, New York University, New York, NY.* A *construction chain* for n is a sequence $a_1, \ldots, a_k$ where $a_1 = 1$, $a_k = n$, and each entry in the sequence is either the sum or the product of two previous, possibly identical, elements from the sequence. The *cost* of a construction chain is the number of entries that are the sum (but not the product) of preceding entries. For example, 1, 2, 3, 6, 12, 144, 1728, 1729 is a construction chain for 1729; its cost is 3, because the elements 2, 3, and 1729 require addition. Let $c(n)$ be the minimal cost of a construction chain for n. Prove that c is unbounded.

Solution. We show that, given n, the total number of construction chains for numbers less than or equal to n and with cost K or less is at most $K(1 + \log_2 n)^{2K^2}$. Since this is less than n for large n, some integer does not have a construction chain with cost K or less.

Suppose that $a_1, \ldots, a_k$ is a construction chain for m with $m \le n$ having cost s, with $0 \le s \le K$. Let $b_1, \ldots, b_{s+1}$ be the subsequence of $a_1, \ldots, a_k$ with $b_1 = a_1 = 1$ consisting of all entries that were produced using addition. For $2 \le i \le s + 1$,

$$b_i = \prod_{j=1}^{i-1} b_j^{e_j} + \prod_{j=1}^{i-1} b_j^{f_j},$$

where e_j and f_j are nonnegative integers. Note that e_j and f_j are in $\{0, 1, \ldots, \lfloor \log_2 n \rfloor\}$. Hence, the number of choices for b_i with $2 \le i \le s + 1$ is bounded above by $(1+\log_2 n)^{2(i-1)}$. This is at most $(1+\log_2 n)^{2s}$. Hence, the number of possible sequences $b_1, \ldots, b_{s+1}$ is at most $(1 + \log_2 n)^{2s^2}$, which in turn is bounded by $(1 + \log_2 n)^{2K^2}$. Summing over all costs s from 1 to K yields at most $K(1 + \log_2 n)^{2K^2}$, as claimed.

Editorial Comment. We do not know the origin of this problem.

If the number of primes were finite, we could calculate them all with finitely many additions of 1, and then any composite could be computed with zero additional cost. Therefore a corollary of the problem is that the number of primes is infinite. It is challenging to compute $c(n)$. Work of Joseph DeVincentis, Stan Wagon, and Alan Zimmermann has led to results on the cost function for n beyond one million. For $k \ge 0$, let M_k be the least n such that $c(n) = k$. The sequence $M_0, M_1, \ldots$ begins 1, 2, 3, 7, 23, 719, 1169951. See oeis.org/A355015 and also the related oeis.org/A354914.

SOLUTIONS

A Recurrence Yielding Factorials

12265 [2021, 658]. *Proposed by Ross Dempsey, student, Princeton University, Princeton, NJ.* For a fixed positive integer k, let $a_0 = a_1 = 1$ and $a_n = a_{n-1} + (k-n)^2 a_{n-2}$ for $n \geq 2$. Show that $a_k = (k-1)!$.

Solution by Jovan Vukmirović, Belgrade, Serbia, and UM6P Math Club, Mohammed VI Polytechnic University, Ben Guerir, Morocco, independently. Let $b_n = a_n + (k-n-1)a_{n-1}$. Note $b_{k-1} = a_{k-1}$. In general, $a_n = a_{n-1} + (k-n)^2 a_{n-2}$ implies

$$b_n = a_n - a_{n-1} + (k-n)a_{n-1} = (k-n)\big((k-n)a_{n-2} + a_{n-1}\big) = (k-n)b_{n-1}.$$

Therefore,

$$\begin{aligned} b_n &= (k-n)b_{n-1} \\ &= (k-n)(k-n+1)b_{n-2} = \cdots \\ &= (k-n)(k-n+1)\cdots(k-2)b_1. \end{aligned}$$

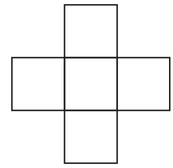
In particular, $b_k = 0$. Since $b_1 = k-1$,

$$a_k = b_k + a_{k-1} = a_{k-1} = b_{k-1} = (k-1)!.$$

Also solved by M. R. Bacon & C. K. Cook, B. Bradie, A. C. Castrillón (Colombia), H. Chen (China), A. De la Fuente, H. Y. Far, K. Gatesman, J. F. Gonzalez & F. A. Velandia (Colombia), J.-P. Grivaux (France), E. A. Herman, N. Hodges (UK), E. J. Ionaşcu, O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Martin (Germany), A. Natian, M. Omarjee (France), C. R. Pranesachar (India), M. Reid, J. L. Guerra & A. J. Rosenthal, K. Sarma (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), J. Vinuesa (Spain), M. Wallner (Austria), H. Widmer (Switzerland), M. Wildon (UK), L. Zhou, Davis Problem Solving Group, and the proposer.

Arbitrarily Disconnectable Polyominos

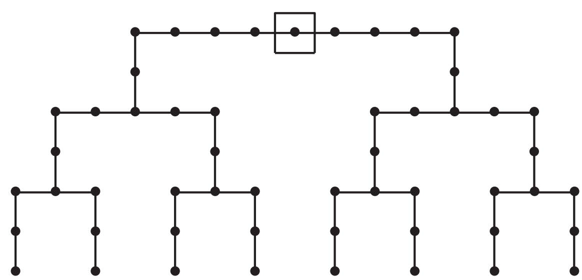
12266 [2021, 658]. *Proposed by Haoran Chen, Xi'an Jiaotong–Liverpool University, Suzhou, China.* A union of a finite number of squares from a grid is called a *polyomino* if its interior is simply connected. Given a polyomino P and a subpolyomino Q, we write $c(P, Q)$ for the number of components that remain when Q is removed from P. Let $f(k) = \max_P \min_Q c(P, Q)$, where the maximum is taken over all polyominoes and the minimum is taken over all subpolyominoes Q of P of size k. For example, $f(2) \geq 3$, because any domino removed from the pentomino at right breaks the pentomino into 3 pieces. Is f bounded?



Solution by Richard Stong, Center for Communications Research, San Diego, CA. We show that f is unbounded. With any polyomino P we can associate a graph G by taking a vertex for each square of P and making vertices adjacent when their squares share a side. We use only polyominos P where the resulting graph G is a tree. The removed subpolyomino Q will correspond to a subtree H, so that the graph associated with $Q - P$ will be $G - V(H)$, and they will have the same number of components.

We use a polyomino whose associated graph is a subdivision of a complete binary tree. Let $G_{h,N}$ be the subdivision of the complete binary tree with height h in which each edge is replaced by a path of length N. For fixed h, we prove that $G_{h,N}$ is the graph associated with some polyomino when N is sufficiently large. It then suffices to show that when m is fixed, for sufficiently large h and N there is a choice of k such that deleting the vertices of any k-vertex subtree of $G_{h,N}$ results in at least m components.

Let T_h be the complete binary tree of height h, with 2^h leaves. We initially represent a subdivision of T_h and can then lengthen paths appropriately to obtain $G_{h,N}$. The vertices of T_h at distance j from the leaves will initially be on the line $y = 2j$, and the root will be at $(0, 2h)$. For $h = 0$, place the root at the origin. For $h \geq 1$, having embedded a subdivision of T_{h-1} with leaves on the horizontal axis (with consecutive leaves separated by 2), take two copies and shift one rightward to have leaves at odd points $(1, 0)$ through $(2^h - 1, 0)$, and shift the other leftward to have leaves at odd points $(-1, 0)$ through $(-2^h + 1, 0)$. The roots of the two copies will now be at $(2^{h-1}, 2h - 2)$ and $(-2^{h-1}, 2h - 2)$. Place the root of T_h at $(0, 2h)$. The edge from $(0, 2h)$ to its right child is represented by a path from $(0, 2h)$ to $(2^{h-1}, 2h)$ and then down two steps to $(2^{h-1}, 2h - 2)$; the path to $(-2^{h-1}, 2h - 2)$ is the reflection of this. Here is T_3:



This construction requires $N \geq 2^{h-1} + 2$. The vertical steps involved in a given level can be lengthened by the same amount to produce an embedding of $G_{h,N}$ associated with a polyomino $P_{h,N}$. The vertical steps have length at least 2 to avoid unwanted edges in the associated graph.

Let a 2-*power sum* be an integer of the form $\sum_i \varepsilon_i 2^{a_i}$, where $\varepsilon_i \in \{1, -1\}$ and a_i is a nonnegative integer for all i. When we consider deleting the vertices of a subtree, the following claim is helpful.

Claim: *Given a positive integer m, there is a positive integer t such that if $|t-u| \le 2m$, then any expression of u as a 2-power sum has more than m terms.*

To prove the claim, we show that when R is sufficiently large, there are congruence classes modulo 2^R that can serve as t. Powers of 2 and their negations take on only $2R+1$ distinct values modulo 2^R. Hence sums of m such terms take on at most $(2R+1)^m$ values modulo 2^R. Within $2m$ units of such values there are at most $(4m+1)(2R+1)^m$ congruence classes. Since a polynomial in R grows more slowly than 2^R, when R is sufficiently large we can pick t from any of the remaining congruence classes.

Fix m, and let t be an integer as guaranteed by the claim. Choose h so that $2^{h+1}-2>t$. Let $G=G_{h,N}$ for some large N, and let $k=tN$. We claim that for any subtree H of G with k vertices, $G-V(H)$ has at least m components. Since m is arbitrary, this makes f unbounded.

Let v be a vertex of H closest to the root of G. Let the distance from v to the leaves below it be $rN+s$, where $0 \le s < N$. The subtree of G rooted at v has $1+s+(2^{r+1}-2)N$ vertices. Let S be the set of vertices w in G such that w is not in H but the parent of w is in H. Let r_iN+s_i be the distance from the ith vertex of S to the leaves below it, where $0 \le s_i < N$. The vertices of H are precisely the descendants of v that are not descendants of vertices in S. Thus

$$Nt = k = |V(H)| = 1+s+(2^{r+1}-2)N - \sum_i \big(1+s_i+(2^{r_i+1}-2)N\big).$$

Let $u = 2^{r+1} - \sum_i 2^{r_i+1}$. The difference between t and u is

$$(1+s-2N)/N - \sum_i (1+s_i-2N)/N.$$

Since $0 \le s_i < N$ and $0 \le s < N$, each term lies between -2 and 2. Hence $|t-u| \le 2m$ if $|S| < m$. Since u is a 2-power sum with $|S|+1$ terms, the choice of t yields $|S| \ge m$. That is, $G-V(H)$ has at least m components.

Also solved by the proposer.

Balanced Colorings of Graphs

12268 [2021, 658]. *Proposed by Samina Boxwala Kale, Nowrosjee Wadia College, Pune, India, Vašek Chvátal, Concordia University, Montreal, Canada, Donald E. Knuth, Stanford University, Stanford, CA, and Douglas B. West, University of Illinois, Urbana, IL.*
(a) Show that there is an easy way to decide whether the edges of a graph can each be colored red or green so that at each vertex the number of incident edges with one color differs from the number having the other color by at most 1.
(b) Show that it is NP-hard to decide whether the vertices of a graph can each be colored red or green so that at each vertex the number of neighboring vertices with one color differs from the number having the other color by at most 1.

Solution by Edward Schmeichel, San Jose State University, San Jose, CA. In both **(a)** and **(b)**, we call a coloring of the specified type a *balanced coloring*. The existence of balanced colorings in one component does not affect their existence in others, so we can apply the criterion for connected graphs to each component.

(a) A connected graph G fails to have a balanced edge-coloring if and only if all vertices have even degree and the number of edges is odd.

If all vertices have even degree and G has a balanced edge-coloring, then the subgraphs in the two colors have the same degree at each vertex and hence the same number of edges, which is impossible when the number of edges is odd.

If the vertices have even degree and the number of edges is even, then assigning colors alternately along an Eulerian circuit gives half of the edges at each vertex to each color.

If some vertex has odd degree, then the number of vertices with odd degree is even, and adding one vertex v and making it adjacent to all the vertices of odd degree produces a connected graph G' with all vertex degrees even. In G' there is an Eulerian circuit starting and ending at v. Assigning colors alternately along the circuit gives each vertex other than v the same number of edges of each color, and then deleting the edges at v produces a balanced edge-coloring of G.

(b) We show that if there is a polynomial-time algorithm to test whether a balanced vertex coloring exists, then there is a polynomial-time algorithm for the following well-known NP-hard problem.

NOT-ALL-EQUAL 3SAT: Given variables $x_1, \dots, x_n$ and clauses $c_1, \dots, c_m$, where each clause is a set of three "literals" (variables or their complements), is there a truth assignment to the variables so that each clause contains both a true literal and a false literal?

Given an instance I of NOT-ALL-EQUAL 3SAT, we construct a graph G such that I is satisfiable if and only if G has a balanced vertex coloring. For each clause c_i, create a set S_i of three independent vertices labeled by the literals in c_i, together with a vertex σ_i adjacent to all three vertices in S_i. Let $S = \bigcup_i S_i$. Note that S is an independent set of size $3m$; labels may appear on more than one vertex.

Next we add vertices and edges to G to ensure that in a balanced vertex coloring, vertices in S having the same label will have the same color, while vertices with complementary labels will have opposite colors. Think of green as representing TRUE and red as representing FALSE.

For each instance of two vertices v and w in S with identical labels, add a star with four edges, with each of v and w adjacent to two leaves of the star, giving those leaves degree 2. The leaves of the star need neighbors of opposite colors, so v and w must have the same color in a balanced vertex coloring.

For each instance of two vertices v and w in S with complementary labels, add two new vertices, with v and w adjacent to both. The new vertices have degree 2, and hence v and w must have opposite colors in a balanced vertex coloring.

If G has a balanced vertex coloring, then the balance condition at each σ_i guarantees that each clause has a vertex of each color. Thus a balanced vertex coloring of G converts to a satisfying truth assignment for I.

Conversely, given a satisfying truth assignment for I, using green on vertices labeled with true literals and red on vertices labeled with false literals fulfills the balance condition at each σ_i. Each vertex of S is adjacent to an even number of added vertices, and we can color the added vertices so that each vertex of S has the same number of neighbors of each color among the added vertices. Since each vertex of S is adjacent to only one vertex of the form σ_i, we can then color the vertices of that form arbitrarily to complete a balanced vertex coloring of G.

Editorial comment. In G. P. Cornuéjols (1988), General Factors of Graphs, *J. Comb. Th. B* 45, 185–198, it is shown that for any nonnegative integer k, there is a polynomial-time algorithm to decide whether the edges of a graph can be colored red or green so that at each vertex the numbers of incident edges of the two colors differ by at most k. For part **(b)**, Mark Wildon reduced a variant of the Subset Sum problem to the given coloring problem.

Also solved by R. Stong, M. Wildon (UK), and the proposers.

Integrating an Absolute Value

12271 [2021, 659]. *Proposed by Steven Deckelman, University of Wisconsin–Stout, Menomonie, WI.* Let n be a positive integer. Evaluate

$$\int_0^{2\pi} \left| \sin\left((n-1)\theta - \frac{\pi}{2n}\right) \cos(n\theta) \right| d\theta.$$

Solution by Jovan Vukmirović, Belgrade, Serbia. Let I_n denote the requested integral. We show that

$$I_n = \begin{cases} \dfrac{4n}{2n-1} \cot\left(\dfrac{\pi}{2n}\right) - \dfrac{4(n-1)}{2n-1} \cot\left(\dfrac{\pi}{2(n-1)}\right), & \text{if } n \text{ is even;} \\ \dfrac{4n}{2n-1} \csc\left(\dfrac{\pi}{2n}\right) - \dfrac{4(n-1)}{2n-1} \csc\left(\dfrac{\pi}{2(n-1)}\right), & \text{if } n \text{ is odd.} \end{cases}$$

Since the integrand is periodic with period π, the substitution $\theta = x - \pi/(2n)$ gives

$$I_n = 2\int_{-\pi/2}^{\pi/2} \left| \cos((n-1)x) \sin(nx) \right| dx.$$

Let $f_n(x) = \cos\big((n-1)x\big) \sin(nx)$. Since $|f_n(x)|$ is an even function, we have

$$I_n = 4\int_0^{\pi/2} |f_n(x)|\, dx.$$

Note that the function F_n defined by

$$F_n(x) = -\frac{1}{2}\left(\cos x + \frac{1}{2n-1} \cos\big((2n-1)x\big)\right)$$

is an antiderivative of f_n. When $x \in [0, \pi/2]$ we have $f_1(x) \geq 0$, so

$$I_1 = 4\int_0^{\pi/2} f_1(x)\, dx = 4\big(F_1(\pi/2) - F_1(0)\big) = 4.$$

Now suppose $n \geq 2$. The positive values of x where $\cos\big((n-1)x\big)$ changes sign are given by $c_k = (2k-1)\pi/\big(2(n-1)\big)$, and the values where $\sin(nx)$ changes sign are given by $d_k = k\pi/n$, for $k = 1, 2, \ldots$. Setting $m = \lfloor n/2 \rfloor$, we have

$$0 < c_1 < d_1 < c_2 < \cdots < c_m \leq d_m \leq \frac{\pi}{2} < d_{m+1} < c_{m+1},$$

so $f_n(x)$ is negative for $c_k < x < d_k$, $k = 1, \ldots, m$, and nonnegative at all other points in $[0, \pi/2]$. Hence

$$\int_0^{\pi/2} |f_n(x)|\, dx = F_n(\pi/2) - F_n(0) - 2\sum_{k=1}^{m} \big(F_n(d_k) - F_n(c_k)\big),$$

and the desired integral is given by

$$I_n = \frac{4n}{2n-1} + \sum_{k=1}^{m} \left(\frac{8n}{2n-1} \cos\left(\frac{k\pi}{n}\right) - \frac{8(n-1)}{2n-1} \cos\left(\frac{(2k-1)\pi}{2(n-1)}\right) \right).$$

To simplify the sum, we apply the identity

$$\cos(a+b) + \cos(a+2b) + \cdots + \cos(a+mb) = \frac{\sin\big(a + (2m+1)b/2\big) - \sin(a + b/2)}{2\sin(b/2)}$$

(easily verified by induction on m) to get

$$I_n = \frac{4n}{2n-1} \cdot \frac{\sin\left((2m+1)\pi/(2n)\right)}{\sin\left(\pi/(2n)\right)} - \frac{4(n-1)}{2n-1} \cdot \frac{\sin\left(m\pi/(n-1)\right)}{\sin\left(\pi/(2(n-1))\right)}.$$

Since m is equal to $n/2$ if n is even and $(n-1)/2$ if n is odd, we obtain the desired formula for I_n.

Note that $I_n \to 8/\pi$ as $n \to \infty$.

Also solved by G. Fera (Italy), D. Henderson, N. Hodges (UK), O. Kouba (Syria), O. P. Lossers (Netherlands), A. Natian, A. Stadler (Switzerland), M. Štofka (Slovakia), R. Stong, E. I. Verriest, and the proposer.

Lists Whose Consecutive Terms Sum to Powers of 2

12272 [2021, 755]. *Proposed by H. A. ShahAli, Tehran, Iran, and Stan Wagon, Macalester College, St. Paul, MN.*

(a) For which integers n with $n \geq 3$ do there exist distinct positive integers $a_1, \ldots, a_n$ such that $a_i + a_{i+1}$ is a power of 2 for all $i \in \{1, \ldots, n\}$? (Here subscripts are taken modulo n, so that $a_{n+1} = a_1$.)

(b) What is the answer if the word “positive” is removed from part **(a)**?

Solution by Rory Molinari, Michigan. For (**a**) there is no such n, but for (**b**) there exist such lists for all n except $n = 4$.

(**a**) Suppose that $a_1, \ldots, a_n$ is such a list. By symmetry, we may assume $a_1 < a_2$. Let $a_i + a_{i+1} = 2^{c_i}$ for all i. Since $a_{i-1} \neq a_{i+1}$, we have $c_{i-1} \neq c_i$. If $a_{i-1} < a_i$ and $a_i > a_{i+1}$, then

$$a_i > \max\{2^{c_{i-1}}/2, 2^{c_i}/2\} \geq \min\{2^{c_{i-1}}, 2^{c_i}\},$$

from which $\min\{a_{i-1}, a_{i+1}\}$ is negative. Hence $a_1 < \cdots < a_n < a_1$, a contradiction.

(**b**) Suppose that distinct integers $a_1, \ldots, a_4$ exist such that $a_i + a_{i+1} = 2^{c_i}$. By symmetry, we may assume that a_1 is the smallest. Now

$$0 < a_3 - a_1 = 2^{c_2} - 2^{c_1} = 2^{c_3} - 2^{c_4}.$$

Consequently, c_1 and c_4 are both the exponent of the greatest power of 2 dividing $a_3 - a_1$. Hence $c_1 = c_4$, which yields $a_2 = a_4$, a contradiction.

For $n = 3$, one such list is $(3, -1, 5)$.

Let $\alpha_i = 1 - 2^i$ and $\beta_i = 3 \cdot 2^i - 1$. For even n at least 6, with $k = n/2$, consider the list

$$(1, 3, \alpha_1, \beta_1, \ldots, \alpha_{k-2}, \beta_{k-2}, \alpha_{k-1}, 2^k - 1).$$

Since $1+3=4$, $3+\alpha_1=2$, $\alpha_i+\beta_i=2^{i+1}$, $\beta_i+\alpha_{i+1}=2^i$, $\alpha_{k-1}+2^k-1=2^{k-1}$, and $2^k - 1 + 1 = 2^k$, every sum of two cyclically consecutive elements is a power of 2. Since $0 > \alpha_1 > \cdots > \alpha_{k-1}$ and $3 < \beta_1 < \cdots < \beta_{k-2} < 2^k - 1$, the terms are distinct.

When $n = 2k - 1 \geq 5$, it suffices to use the list for $2k$ with the term α_1 deleted, since $3 + \beta_1 = 8$.

Editorial comment. Yuri Ionin strengthened the conclusion in part (**a**), using induction to prove that positive integers $a_1, \ldots, a_n$ chosen so that cyclically $a_i + a_{i+1}$ is always a power of 2 has at most $\lceil (n+1)/2 \rceil$ distinct elements and that this bound is sharp.

Also solved by C. Curtis & J. Boswell, S. M. Gagola Jr., K. Gatesman, O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin, M. D. Meyerson, M. Reid, A. Stadler (Switzerland), R. Tauraso (Italy), F. A. Velandia & J. F. Gonzalez (Colombia), J. Yan (China), Fejéntaláltuka Szeged Problem Solving Group (Hungary), and the proposer. Part (**a**) also solved by H. Chen (China), O. P. Lossers (Netherlands), R. Martin (Germany), L. Zhou, and the UM6P Math Club (Morocco).

Zeta Function Inequalities from Convexity

12273 [2021, 755]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let ζ be the Riemann zeta function, defined by $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$. For $s > 1$, prove the following inequalities:

$$\sum_{\text{prime } p} \frac{1}{p^s - 0.5} < \log \zeta(s), \qquad \sum_{\text{prime } p} \frac{1}{p^s} < \log \frac{\zeta(s)}{\sqrt{\zeta(2s)}}, \qquad \sum_{\text{prime } p} \frac{1}{p^s + 0.5} < \log \frac{\zeta(s)}{\zeta(2s)}.$$

Composite solution by Allen Stenger, Boulder, CO, and Li Zhou, Polk State College, Winter Haven, FL. We prove the more general inequality

$$\sum_p \frac{1}{p^s + \alpha} < \log \frac{\zeta(s)}{(\zeta(2s))^{\alpha+1/2}}, \tag{$*$}$$

where $-1/2 \le \alpha \le 1/2$ and the sum is over all primes. The three requested inequalities are for $\alpha \in \{-1/2, 0, 1/2\}$.

The Euler product formula for $\zeta(s)$ with $s > 1$ is $\zeta(s) = \prod_p 1/(1 - p^{-s})$, where the product is taken over all primes. Hence the right side of ($*$) is the logarithm of

$$\prod_p \frac{1}{1 - p^{-s}} \Bigg/ \left(\prod_p \frac{1}{1 - p^{-2s}} \right)^{\alpha+1/2},$$

which simplifies to $\prod_p (1 - p^{-2s})^{\alpha+1/2}/(1 - p^{-s})$, where the products are over all primes. Letting $R = \log\big((1 - p^{-2s})^{\alpha+1/2}/(1 - p^{-s})\big)$, we obtain the desired inequality term-by-term by proving $R > 1/(p^s + \alpha)$. We compute

$$\begin{aligned}
R &= (\alpha + 1/2) \log \big((1 - p^{-s})(1 + p^{-s})\big) - \log(1 - p^{-s}) \\
&= (\alpha - 1/2) \log \left(\frac{p^s - 1}{p^s} \right) + (\alpha + 1/2) \log \left(\frac{p^s + 1}{p^s} \right) \\
&= \frac{1 - 2\alpha}{2} \big(\log p^s - \log(p^s - 1) \big) + \frac{1 + 2\alpha}{2} \big(\log(p^s + 1) - \log p^s \big) \\
&= \int_{p^s - 1}^{p^s} \frac{1 - 2\alpha}{2x} dx + \int_{p^s}^{p^s + 1} \frac{1 + 2\alpha}{2x} dx.
\end{aligned}$$

We obtain lower bounds on these integrals using the left side of the Hermite–Hadamard inequality

$$f\left(\frac{a + b}{2} \right) \le \frac{1}{b - a} \int_a^b f(x) dx \le \frac{f(a) + f(b)}{2}$$

for convex f, with the inequalities being strict when f is strictly convex. Applying the Hermite–Hadamard inequality to both integrals in the final expression for R yields

$$R > \frac{1 - 2\alpha}{2p^s - 1} + \frac{1 + 2\alpha}{2p^s + 1}.$$

Letting $u = p^s$, it now suffices to prove

$$\frac{1/2 - \alpha}{u - 1/2} + \frac{1/2 + \alpha}{u + 1/2} \ge \frac{1}{u + \alpha}$$

for $-1/2 \le \alpha \le 1/2$ and $u \ge 2$. Letting $g(\alpha)$ denote the left side minus the right side in this inequality, we compute $g''(\alpha) = -2/(u + \alpha)^3 < 0$. Thus g is a concave function, and its minimum on the interval $[-1/2, 1/2]$ occurs at an endpoint. Since $g(-1/2) = g(1/2) = 0$, we have $g(\alpha) \geq 0$ throughout the interval, and the result follows.

Editorial comment. The proof above uses only the left side of the Hermite–Hadamard inequality. Applying the right side to the convex function e^x yields

$$\frac{e^b - e^a}{b - a} < \frac{e^b + e^a}{2}.$$

For $b = 2/u$ and $a = 0$, this reduces to $e^{2/u} - 1 < (e^{2/u} + 1)/u$. For $u > 1$, we can rearrange and take logarithms to obtain $2/u < \log\big((u+1)/(u-1)\big)$. The proposer used this last inequality to show that one can start from any of the specified sums in the problem and build up to the desired expression in terms of the zeta function without a decrease at any step of the process. For example,

$$\sum_p \frac{2}{2p^s - 1} < \sum_p \log\left(\frac{2p^s - 1 + 1}{2p^s - 1 - 1}\right) = \sum_p \log\left(\frac{p^s}{p^s - 1}\right) = \log \zeta(s).$$

This solution proceeds in the opposite direction from the solution presented above.

Also solved by H. Chen, D. Fleischman, K. Gatesman, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), K. Nelson, M. Omarjee (France), D. Pinchon (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), T. Wiandt, J. Yan (China), Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

A Trigonometric Logarithmic Integral

12274 [2021, 755]. *Proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.* Evaluate

$$\int_0^1 \frac{\arctan x}{1 + x^2}\left(\ln\left(\frac{2x}{1 - x^2}\right)\right)^2 dx.$$

Solution by Michel Bataille, Rouen, France. Let I be the integral to be evaluated. We show that $I = \pi^4/128$.

The change of variables $x = \tan(u/2)$ readily leads to

$$I = \frac{1}{4}\int_0^{\pi/2} u(\ln\tan u)^2\, du.$$

Using the substitution $u = \pi/2 - v$ we obtain

$$\int_{\pi/4}^{\pi/2} u(\ln\tan u)^2\, du = \int_0^{\pi/4}\left(\frac{\pi}{2} - v\right)(\ln(\cot v))^2\, dv = \int_0^{\pi/4}\left(\frac{\pi}{2} - v\right)(\ln\tan v)^2\, dv,$$

from which we deduce

$$I = \frac{1}{4}\left(\int_0^{\pi/4} u(\ln\tan u)^2\, du + \int_0^{\pi/4}\left(\frac{\pi}{2} - u\right)(\ln\tan u)^2\, du\right) = \frac{\pi}{8}\int_0^{\pi/4}(\ln\tan u)^2\, du.$$

Finally, the substitution $u = \arctan t$ gives

$$I = \frac{\pi}{8}\int_0^1 \frac{(\ln t)^2}{1 + t^2}\, dt = \frac{\pi}{8}\sum_{n=0}^{\infty}(-1)^n \int_0^1 t^{2n}(\ln t)^2\, dt$$

$$= \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi}{4}\beta(3) = \frac{\pi}{4}\cdot\frac{\pi^3}{32} = \frac{\pi^4}{128},$$

where β is the Dirichlet beta function.

Editorial comment. Seán M. Stewart derived the more general formula

$$\int_0^1 \frac{\arctan x}{1+x^2}\left(\ln\left(\frac{2x}{1-x^2}\right)\right)^{2n} dx = \frac{\pi}{8}(2n)!\beta(2n+1).$$

The integral $\int_0^1 (\ln t)^2/(1+t^2)\,dt$ also made an appearance in the solution of Problem 12158 [2020, 86; 2021, 757] from this MONTHLY.

Also solved by T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), A. Berkane (Algeria), N. Bhandari (Nepal), P. Bracken, J. V. Casteren & L. Kempeneers (Belgium), H. Chen (China), H. Chen, A. Dixit (India), G. Fera (Italy), K. Gatesman, M. L. Glasser, H. Grandmontagne (France), N. Grivaux (France), J. A. Grzesik (Canada), E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), J. E. Kampmeyer III, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Magliano, K. D. McLenithan & S. C. Mortenson, A. Natian, M. Omarjee (France), D. Pinchon (France), A. Stadler (Switzerland), A. Stenger, S. M. Stewart (Saudi Arabia), M. Štofka (Slovakia), R. Stong, M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), J. Yan (China), L. Zhou, Fejéntaláltuka Szeged Problem Solving Group (Hungary), Missouri Problem Solving Group, UM6P Math Club (Morocco), and the proposer.

CLASSICS

C15. *Suggested by Joel Spencer, New York University, New York, NY.* A *construction chain* for n is a sequence $a_1, \dots, a_k$ where $a_1 = 1$, $a_k = n$, and each entry in the sequence is either the sum or the product of two previous, possibly identical, elements from the sequence. The *cost* of a construction chain is the number of entries that are the sum (but not the product) of preceding entries. For example, 1, 2, 3, 6, 12, 144, 1728, 1729 is a construction chain for 1729; its cost is 3, because the elements 2, 3, and 1729 require addition. Let $c(n)$ be the minimal cost of a construction chain for n. Prove that c is unbounded.

Coprimality in Pascal's Triangle

C14. *Due to Paul Erdős and George Szekeres; suggested by the editors.* Show that no two entries chosen from the interior of any row of Pascal's triangle are relatively prime.

Solution. Suppose $0 < a < b < n$. The identity

$$\binom{n}{a}\binom{n-a}{b-a} = \binom{n}{b}\binom{b}{a} \qquad (*)$$

is easily verified (both sides count committees of size b with a subcommittee of size a chosen from a set of n people). It follows that if $\binom{n}{a}$ and $\binom{n}{b}$ are relatively prime, then $\binom{n}{a}$ divides $\binom{b}{a}$. This contradicts $\binom{b}{a} < \binom{n}{a}$.

Editorial Comment. The result is from Paul Erdős and George Szekeres (1978), Some number theoretic problems on binomial coefficients, *Aust. Math. Soc. Gazette* 597–99 (available on-line at combinatorica.hu/∼p_erdos/1978-46.pdf). There the following stronger result is proved: If $0 < a < b \le n/2$ and $d = \gcd\left(\binom{n}{a}, \binom{n}{b}\right)$, then $d \ge 2^a$. To see this, note that $(*)$ implies $\binom{n}{a}/d$ divides $\binom{b}{a}$, which in turn implies $d \ge \binom{n}{a}\big/\binom{b}{a}$. Since this last expression is equal to

$$\left(\frac{n}{b}\right)\left(\frac{n-1}{b-1}\right)\cdots\left(\frac{n-a+1}{b-a+1}\right),$$

and since each of these factors is at least 2, we have $d \ge 2^a$. This inequality is strict when $a > 1$.

SOLUTIONS

The Laplace Transform Simplifies an Integral

12260 [2021, 563]. *Proposed by Seán M. Stewart, Bomaderry, Australia.* Prove

$$\int_0^\infty \frac{\sin^2 x - x\sin x}{x^3}\,dx = \frac{1}{2} - \log 2.$$

Solution by Tewodoros Amdeberham, Tulane University, New Orleans, LA, and Akalu Tefera, Grand Valley State University, Allendale, MI. The Laplace transform $\mathcal{L}$ defined by $\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$ has the property

$$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty \mathcal{L}[f](s)\cdot \mathcal{L}^{-1}[g](s)\,ds.$$

Applying this with $f(x) = \sin^2 x - x\sin x = 1/2 - (1/2)\cos(2x) - x\sin x$ and $g(x) = 1/x^3$ leads to

$$\begin{aligned}
\int_0^\infty \frac{\sin^2 x - x\sin x}{x^3}\,dx &= \int_0^\infty \mathcal{L}\left[\frac{1}{2} - \frac{1}{2}\cos(2x) - x\sin x\right](s)\cdot \mathcal{L}^{-1}\left[\frac{1}{x^3}\right](s)\,ds \\
&= \int_0^\infty \left(\frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} - \frac{2s}{(s^2+1)^2}\right)\cdot \frac{s^2}{2}\,ds \\
&= \int_0^\infty \frac{s}{s^2+4} - \frac{s}{s^2+1} + \frac{s}{(s^2+1)^2}\,ds \\
&= \left[\frac{\log(s^2+4) - \log(s^2+1)}{2} - \frac{1}{2(s^2+1)}\right]_0^\infty = \frac{1}{2} - \log 2.
\end{aligned}$$

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), G. E. Bilodeau, K. N. Boyadzhiev, P. Bracken, B. Bradie, A. C. Castrillón, H. Chen, C. Degenkolb, A. De la Fuente, H. Y. Far, G. Fera (Italy), A. Garcia (France), M. L. Glasser, R. Gordon, H. Grandmontagne (France), G. C. Greubel, N. Grivaux (France), P. Haggstrom (Australia), L. Han (US) &

X. Tan (China), D. Henderson, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), W. P. Johnson, A. M. Karparvar (Iran), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), J. Magliano, K. McLenithan, I. Mező (China), M. Omarjee (France), D. Pinchon (France), S. Sharma (India), P. Shi (China), A. Stadler (Switzerland), J. L. Stitt, R. Stong, R. Tauraso (Italy), Y. Tsyban (Saudi Arabia), J. Van Casteren & L. Kempeneers (Belgium), E. I. Verriest, M. Vowe (Switzerland), S. Wagon, T. Wiandt, H. Widmer (Switzerland), M. Wildon (UK), L. Zhou, Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

Counting Equilateral Triangles in Hypercubes

12261 [2021, 563]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let a_n be the number of equilateral triangles whose vertices are chosen from the vertices of the n-dimensional cube. Compute $\lim_{n\to\infty} na_n/8^n$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The limit is $1/(3\sqrt{3}\,\pi)$.

Let the n-dimensional hypercube have vertex set $\{0,1\}^n$. For vertices A, B, C chosen from this set, let I be the set of coordinates where A differs from both B and C, let J be the set of coordinates where B differs from both A and C, and let K be the set of coordinates where C differs from both A and B. Since $\|A-B\|^2 = |I|+|J|$, $\|B-C\|^2 = |J|+|K|$, and $\|C-A\|^2 = |K|+|I|$, the vertices in $\{A,B,C\}$ form an equilateral triangle if and only if $|I|=|J|=|K|$. Conversely, choose a vertex A and three disjoint sets of indices I, J, K, each of positive size k. Define B to differ from A in coordinates $I\cup J$ and C to differ from A in coordinates $I\cup K$. The resulting triangle ABC is equilateral, and each equilateral triangle arises in 3! ways. Thus,

$$a_n = \frac{2^n}{6}\sum_{k=1}^{\lfloor n/3\rfloor}\binom{n}{3k}\frac{(3k)!}{(k!)^3}. \qquad (*)$$

Stirling's formula gives

$$\frac{(3k)!}{(k!)^3} = \frac{\sqrt{3}}{2\pi k}\cdot 3^{3k}\left(1+O\left(\frac{1}{k}\right)\right),$$

which we can write equivalently as

$$\frac{(3k)!}{(k!)^3} = \frac{3\sqrt{3}}{2\pi(3k+1)}\cdot 3^{3k}\left(1+O\left(\frac{1}{k}\right)\right).$$

Since $\binom{n}{3k}\le 2^n$ and $(3k)!/(k!)^3\le 3^{3k}$, any term in the sum $(*)$ with $k<n/6$ contributes less than $2^n\cdot 2^n\cdot 3^{n/2}$ to a_n. This value, which simplifies to $(4\sqrt{3})^n$, is $o(8^n)$. Therefore, in computing $\lim_{n\to\infty} na_n/8^n$, the sum of the estimates has relative error $O(1/n)$. Also, starting the sum at $k=0$ has no impact on the limit. Thus

$$\begin{aligned}\frac{na_n}{8^n} &= \frac{(n+1)a_n}{8^n}\left(1+O\left(\frac{1}{n}\right)\right) = \frac{\sqrt{3}}{4^{n+1}\pi}\left(\sum_{k=0}^{\lfloor n/3\rfloor}\frac{n+1}{3k+1}\binom{n}{3k}3^{3k}\right)\left(1+O\left(\frac{1}{n}\right)\right)\\ &= \frac{1}{4^{n+1}\sqrt{3}\,\pi}\left(\sum_{k=0}^{\lfloor n/3\rfloor}\binom{n+1}{3k+1}3^{3k+1}\right)\left(1+O\left(\frac{1}{n}\right)\right).\end{aligned}$$

Letting $\omega = e^{2\pi i/3}$ and using $|3\omega+1| = |3\omega^{-1}+1| = \sqrt{7} < 4$, it follows that

$$\begin{aligned}\frac{na_n}{8^n} &= \frac{1}{4^{n+1}\sqrt{3}\,\pi}\cdot\frac{(3+1)^{n+1}+\omega^{-1}(3\omega+1)^{n+1}+\omega(3\omega^{-1}+1)^{n+1}}{3}\left(1+O\left(\frac{1}{n}\right)\right)\\ &= \frac{1}{3\sqrt{3}\,\pi}\left(1+O\left(\frac{1}{n}\right)\right).\end{aligned}$$

Therefore, the requested limit is $1/(3\sqrt{3}\pi)$.

Also solved by U. Abel & V. Kushnirevych (Germany), H. Chen (China), H. Chen (US), R. Dempsey, G. Fera & G. Tescaro (Italy), N. Hodges (UK), M. Omarjee (France), D. Pinchon (France), R. Tauraso (Italy), L. Zhou, and the proposer.

A Trigonometric Generating Function

12262 [2021, 563]. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.* For a nonnegative integer m, let

$$A_m = \sum_{k=0}^{\infty} \left(\frac{1}{(6k+1)^{2m+1}} - \frac{1}{(6k+5)^{2m+1}} \right).$$

Prove $A_0 = \pi\sqrt{3}/6$ and, for $m \ge 1$,

$$2A_m + \sum_{n=1}^{m} \frac{(-1)^n \pi^{2n}}{(2n)!} A_{m-n} = \frac{(-1)^m (4^m+1)\sqrt{3}}{2(2m)!} \left(\frac{\pi}{3}\right)^{2m+1}.$$

Solution by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria. The sequence $(A_m)_{m\ge 0}$ is bounded, so for $x \in (-1, 1)$ we may define

$$\begin{aligned} F(x) &= \sum_{m=0}^{\infty} A_m x^{2m} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{x^{2m}}{(6k+1)^{2m+1}} - \frac{x^{2m}}{(6k+5)^{2m+1}} \right) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{x^{2m}}{(6k+1)^{2m+1}} - \frac{x^{2m}}{(6k+5)^{2m+1}} \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{6k+1}{(6k+1)^2 - x^2} - \frac{6k+5}{(6k+5)^2 - x^2} \right). \end{aligned}$$

Setting $\alpha = (1+x)/6$ and $\beta = (1-x)/6$, we have

$$\begin{aligned} &\frac{6k+1}{(6k+1)^2 - x^2} - \frac{6k+5}{(6k+5)^2 - x^2} \\ &\qquad = \frac{1}{2} \left(\frac{1}{6k+1+x} + \frac{1}{6k+1-x} - \frac{1}{6k+5+x} - \frac{1}{6k+5-x} \right) \\ &\qquad = \frac{1}{12} \left(\frac{1}{\alpha+k} + \frac{1}{\beta+k} + \frac{1}{\beta-k-1} + \frac{1}{\alpha-k-1} \right). \end{aligned}$$

Next we use the partial fraction expansion of the cotangent, which is

$$\pi \cot(\pi z) = \sum_{k=0}^{\infty} \left(\frac{1}{z+k} + \frac{1}{z-k-1} \right),$$

when z is not an integer. Applying this with $z = \alpha$ and $z = \beta$ gives

$$\begin{aligned} F(x) &= \frac{\pi}{12}\big(\cot(\pi\alpha) + \cot(\pi\beta)\big) = \frac{\pi}{12} \cdot \frac{\sin\big(\pi(\alpha+\beta)\big)}{\sin(\pi\alpha)\sin(\pi\beta)} \\ &= \frac{\pi}{6} \cdot \frac{\sin(\pi(\alpha+\beta))}{\cos\big(\pi(\alpha-\beta)\big) - \cos\big(\pi(\alpha+\beta)\big)} = \frac{\pi}{6} \cdot \frac{\sin(\pi/3)}{\cos(\pi x/3) - \cos(\pi/3)} \\ &= \frac{\pi\sqrt{3}}{6} \cdot \frac{1}{2\cos(\pi x/3) - 1}. \end{aligned}$$

From $(\cos(2\theta)+\cos\theta)(2\cos\theta-1)=\cos(3\theta)+1$, with $\theta=\pi x/3$, we conclude

$$\big(1+\cos(\pi x)\big)F(x)=\frac{\pi\sqrt{3}}{6}\left(\cos\left(\frac{2\pi x}{3}\right)+\cos\left(\frac{\pi x}{3}\right)\right),$$

and hence

$$\left(2+\sum_{n=1}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}x^{2n}\right)\sum_{n=0}^{\infty}A_n x^{2n}=\frac{\pi\sqrt{3}}{6}\sum_{m=0}^{\infty}\frac{(-1)^m(4^m+1)\pi^{2m}}{3^{2m}(2m)!}x^{2m}.$$

Comparing the coefficients of x^{2m} on both sides, we get $A_0=\pi\sqrt{3}/6$ and, for $m\geq 1$,

$$2A_m+\sum_{n=1}^{m}\frac{(-1)^n\pi^{2n}}{(2n)!}A_{m-n}=\frac{(-1)^m(4^m+1)\sqrt{3}}{2(2m)!}\left(\frac{\pi}{3}\right)^{2m+1},$$

as desired.

Editorial comment. Omran Kouba also noted that by using

$$\left(2\cos\left(\frac{\pi x}{3}\right)-1\right)F(x)=\frac{\pi\sqrt{3}}{6},$$

we obtain the alternative recurrence

$$A_m=\sum_{n=1}^{m}\frac{2(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}A_{m-n}.$$

Also solved by K. F. Andersen (Canada), P. Bracken, H. Chen, G. Fera (Italy), M. L. Glasser, G. C. Greubel, E. A. Herman, N. Hodges (UK), O. P. Lossers (Netherlands), K. Nelson, A. Stadler (Switzerland), M. Štofka (Slovakia), R. Tauraso (Italy), and the proposer.

A Concurrency from A Conic Inscribed in A Triangle

12263 [2021, 564]. *Proposed by Dong Luu, Hanoi National University of Education, Hanoi, Vietnam.* In triangle ABC, let D, E, and F be the points at which the incircle of ABC touches the sides BC, CA, and AB, respectively. Let D', E', and F' be three other points on the incircle with E' and F' on the minor arc EF and D' on the major arc EF and such that AD', BE', and CF' are concurrent. Let X, Y, and Z be the intersections of lines EF and $E'F'$, lines FD and $F'D'$, and lines DE and $D'E'$, respectively. Prove that AX, BY, and CZ are either concurrent or parallel.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. It is well known that AD, BE, and CF intersect at a point G, the Gergonne point of $\triangle ABC$. We choose homogeneous coordinates such that $A=(1:0:0)$, $B=(0:1:0)$, $C=(0:0:1)$, and $G=(1:1:1)$. It follows that $D=(0:1:1)$, $E=(1:0:1)$, and $F=(1:1:0)$, and the equation of the incircle is $x^2+y^2+z^2-2xy-2xz-2yz=0$.

Since the point of intersection of the lines AD', BE', and CF' lies in the interior of $\triangle ABC$, we can take its coordinates to be $(a^2:b^2:c^2)$, with $a,b,c>0$. This gives $D'=(x:b^2:c^2)$ for some x satisfying the quadratic equation

$$x^2+b^4+c^4-2xb^2-2xc^2-2b^2c^2=0.$$

Of its two solutions $x=(b-c)^2$ and $x=(b+c)^2$, we must choose $x=(b-c)^2$ for D' to be on the major arc EF. Note that since $D\neq D'$, we have $b\neq c$. In the same way we find $E' = (a^2 : (c-a)^2 : c^2)$ and $F' = (a^2 : b^2 : (a-b)^2)$, and a, b, and c are distinct. A somewhat tedious but elementary computation gives

$$X = (a(c-b) : b(c-a) : c(a-b)),$$

$$Y = (a(b-c) : b(a-c) : c(a-b)),$$

$$Z = (a(b-c) : b(c-a) : c(b-a)),$$

so the lines AX, BY, and CZ intersect at the point $(a(b-c) : b(c-a) : c(a-b))$.

Editorial comment. Lossers observed that the solution above works if the incircle is replaced with any ellipse tangent to the sides of the triangle. Li Zhou generalized the problem further by showing that the result holds for any conic tangent to the lines containing the sides of the triangle, with suitable adjustments to the restrictions on the positions of D', E', and F'.

Also solved by L. Zhou and the proposer.

Irreducible Polynomials in Two Variables

12264 [2021, 564]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Let P_d be the set of all polynomials of the form $\sum_{0\le i,j\le d} a_{i,j}x^iy^j$ with $a_{i,j} \in \{1,-1\}$ for all i and j. Prove that there is a positive integer d such that more than 99 percent of the elements of P_d are irreducible in the ring of polynomials with integer coefficients.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The number 2 is a primitive root modulo the prime p when the smallest value of m such that p divides $2^m - 1$ is $p-1$. Hence the field $\mathbb{F}_{2^{p-1}}$ is the extension of $\mathbb{F}_2$ of lowest degree that contains a primitive pth root of unity modulo 2. It follows that the minimal polynomial of any primitive pth root of unity modulo 2 has degree at least $p-1$. Since the primitive pth roots of unity are the roots of the polynomial $(x^p-1)/(x-1)$ (which equals $x^{p-1} + \cdots + x + 1$ and has degree $p-1$) it follows that this polynomial is irreducible modulo 2. Thus all polynomials of the form $a_0 + a_1x + \cdots + a_{p-1}x^{p-1}$ with all $a_i \in \{-1, 1\}$ (or indeed with all a_i odd) are irreducible over $\mathbb{Z}$.

If $\sum_{0\le i,j\le p-1} a_{i,j}x^iy^j \in P_{p-1}$ is reducible, say as $F(x,y)G(x,y)$, then

$$F(x,0)G(x,0) = a_{0,0} + a_{1,0}x + \cdots + a_{p-1,0}x^{p-1}.$$

Since this polynomial in x is irreducible, $F(x,0)$ or $G(x,0)$ (we may assume $F(x,0)$) has degree $p-1$ as a polynomial in x. Looking at the term with highest degree in x in $F(x,y)G(x,y)$, we conclude that $G(x,y)$ is a constant polynomial in x, and hence we can write $G(x,y)$ as $G(y)$. Swapping the roles of x and y, we find symmetrically that (since $G(y)$ cannot be constant), $G(y)$ has degree $p-1$ and $F(x,y)$ is constant in y, so we write it as $F(x)$. Thus all reducible polynomials in P_{p-1} have the form $F(x)G(y)$. Since $F(0)G(0) = \pm 1$, we conclude $F(0), G(0) \in \{-1, 1\}$, Looking at the terms with degree 0 in x and y yields that all coefficients of $F(x)$ are in $\{1, -1\}$.

Finally, there are 2^p choices for each of F and G, but this double counts the product FG as the product $(-F)(-G)$. Thus there are exactly 2^{2p-1} reducible polynomials in P_{p-1}.

In particular, taking $p = 5$ and noting that 2 is a primitive root modulo 5, we see that only 2^9 of the 2^{25} elements of P_4 are reducible, which is less than 1% of the total number of polynomials in P_4. The fraction only decreases as p increases.

Also solved by S. M. Gagola Jr., O. P. Lossers (Netherlands), D. Pinchon (France), and the proposer.

Combining the Cauchy–Schwarz and AM–GM Inequalities

12267 [2021, 658]. *Proposed by Michel Bataille, Rouen, France.* Let x, y, and z be non-negative real numbers such that $x + y + z = 1$. Prove

$$(1-x)\sqrt{x(1-y)(1-z)} + (1-y)\sqrt{y(1-z)(1-x)} + (1-z)\sqrt{z(1-x)(1-y)} \geq 4\sqrt{xyz}.$$

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. It is clear that the required inequality holds if any of x, y, or z is zero; it is an equality if two of them are zero. Now suppose that x, y, and z are all positive. Dividing by $\sqrt{xyz}$ and using the fact that $x + y + z = 1$, we see that the inequality is equivalent to

$$\frac{(y+z)\sqrt{(x+z)(x+y)}}{\sqrt{yz}} + \frac{(x+z)\sqrt{(x+y)(y+z)}}{\sqrt{xz}} + \frac{(x+y)\sqrt{(y+z)(x+z)}}{\sqrt{xy}} \geq 4.$$

The Cauchy–Schwarz inequality gives $\sqrt{(x+z)(x+y)} \geq x + \sqrt{yz}$, and by the AM–GM inequality, $y + z \geq 2\sqrt{yz}$. Applying these, we obtain

$$\frac{(y+z)\sqrt{(x+z)(x+y)}}{\sqrt{yz}} \geq \frac{(y+z)(x+\sqrt{yz})}{\sqrt{yz}} = \frac{(y+z)x}{\sqrt{yz}} + y + z \geq 2x + y + z = x + 1.$$

Combining this with similar inequalities for the other two terms, we get

$$\frac{(y+z)\sqrt{(x+z)(x+y)}}{\sqrt{yz}} + \frac{(x+z)\sqrt{(x+y)(y+z)}}{\sqrt{xz}} + \frac{(x+y)\sqrt{(y+z)(x+z)}}{\sqrt{xy}}$$
$$\geq (x+1) + (y+1) + (z+1) = 4,$$

as required. When x, y, and z are positive, equality holds only if $x = y = z = 1/3$.

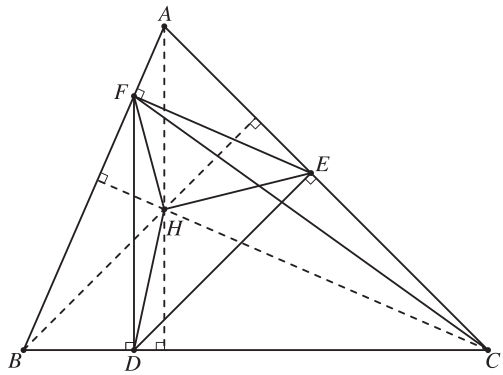
Also solved by A. Alt, F. R. Ataev (Uzbekistan), A. Berkane (Algeria), P. Bracken, H. Chen (China), H. Chen, C. Chiser (Romania), N. S. Dasireddy (India), M. Dinča (Romania), H. Y. Far, G. Fera (Italy), A. Garcia (France), O. Geupel (Germany), P. Haggstrom (Australia), D. Henderson, N. Hodges (UK), F. Holland (Ireland), E. J. Ionaşcu, W. Janous (Austria), A. M. Karparvar (Iran), P. Khalili, K. T. L. Koo (Hong Kong), O. Kouba (Syria), K.-W. Lau (Hong Kong), S. Lee (Korea), O. P. Lossers (Netherlands), J. F. Loverde, A. Mhanna (Lebanon), M. Reid, V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), J. F. Gonzalez & F. A. Velandia (Colombia), M. Vowe (Switzerland), J. Vukmirović (Serbia), H. Widmer (Switzerland), L. Wimmer (Germany), L. Zhou, UM6P MathClub (Morocco), and the proposer.

A Triangle Inscribed in a Similar Triangle

12269 [2021, 659]. *Proposed by Mehmet Şahin and Ali Can Güllü, Ankara, Turkey.* Let ABC be an acute triangle. Suppose that D, E, and F are points on sides BC, CA, and AB, respectively, such that FD is perpendicular to BC, DE is perpendicular to CA, and EF is perpendicular to AB. Prove

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = 1.$$

Solution I by Michael Reid, University of Central Florida, Orlando, FL. For a polygon $PQ \cdots Z$, let $(PQ \cdots Z)$ denote its area. Let H be the orthocenter of $\triangle ABC$. Since the triangle is acute, H lies in its interior. Both CH and EF are perpendicular to AB, so they are parallel, and therefore $(CEF) = (HEF)$. Thus



$$\frac{AF}{AB} = \frac{(AFC)}{(ABC)} = \frac{(AFE)+(CEF)}{(ABC)} = \frac{(AFE)+(HEF)}{(ABC)} = \frac{(HEAF)}{(ABC)}.$$

Similarly, $BD/BC = (HFBD)/(ABC)$ and $CE/CA = (HDCE)/(ABC)$, so

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = \frac{(HEAF)+(HFBD)+(HDCE)}{(ABC)} = \frac{(ABC)}{(ABC)} = 1.$$

Solution II by Li Zhou, Polk State College, Winter Haven, FL. By Miquel's theorem, the circumcircles of triangles AFE, BDF, and CED concur at a point, the Miquel point M. Note that since $\angle AFE$ is a right angle, AE is a diameter of the circumcircle of $\triangle AFE$, and therefore $\angle AME$ is also a right angle. Similarly, $\angle BMF$ and $\angle CMD$ are right angles.

Since $\angle MFE$ and $\angle MAE$ are subtended by the same arc of the circumcircle of $\triangle AFE$, they are equal. Similarly, $\angle MED = \angle MCD$ and $\angle MDF = \angle MBF$. Also, $\angle MAE = \angle MED$, since both are complementary to $\angle MEA$, and similarly $\angle MCD = \angle MDF$. We conclude that all six of the angles $\angle MFE$, $\angle MAE$, $\angle MED$, $\angle MCD$, $\angle MDF$, and $\angle MBF$ are equal. This means that M is a Brocard point of both $\triangle ABC$ and $\triangle DEF$. Let ω denote the measure of all six angles, which is the Brocard angle. It is well known that $\cot\omega = \cot A + \cot B + \cot C$.

Triangles MEF and MAB are similar, since corresponding sides are perpendicular. Hence $EF/AB = EM/AM$, so

$$\frac{AF}{AB} = \frac{AF}{EF}\cdot\frac{EF}{AB} = \cot A \cdot \frac{EM}{AM} = \cot A \tan\omega.$$

Similarly, $BD/BC = \cot B \tan\omega$ and $CE/CA = \cot C \tan\omega$, so

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = (\cot A + \cot B + \cot C)\tan\omega = \cot\omega\tan\omega = 1.$$

Editorial comment. Several readers noted that the result can be extended to obtuse triangles by allowing one of the points D, E, and F to lie on an extension of a side of $\triangle ABC$ and using signed distances.

It was not required to construct $\triangle DEF$, or even to show that such a triangle exists. However, Solution II shows how to construct the unique such triangle. Let M be the Brocard point of $\triangle ABC$ such that $\angle MAC$, $\angle MBA$, and $\angle MCB$ all have the same measure ω. Triangle DEF is the image of triangle CAB under a rotation of $\pi/2$ radians about M followed by a dilation centered at M with ratio $\tan\omega$.

Also solved by M. Bataille (France), R. B. Campos (Spain), H. Chen (China), C. Chiser (Romania), M. Dincă, G. Fera (Italy), D. Fleischman, K. Gatesman, O. Geupel (Germany), E. A. Herman, N. Hodges (UK),

E. J. Ionaşcu, Y. J. Ionin, W. Janous (Austria), W. Ji (China), M. Goldenberg & M. Kaplan, A. M. Karparvar (Iran), P. Khalili, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), J. McHugh, M. D. Meyerson, J. Minkus, M. R. Modak (India), C. G. Petalas (Greece), C. R. Pranesachar (India), I. Retamoso, V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), J. Vukmirović (Serbia), T. Wiandt, H. Widmer (Switzerland), L. Wimmer (Germany), T. Zvonaru (Romania), Davis Problem Solving Group, Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

A Refinement of a Putnam Problem

12270 [2021, 659]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $a_0 = 1$, and let $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 0$. Show that the sequence whose nth term is $e^{a_n} - n - (1/2)\ln n$ converges.

Solution by Kuldeep Sarma, Tezpur University, Tezpur, India. Define $u_n = e^{a_n}$, and note that $u_{n+1} = u_n e^{1/u_n}$. Since the sequence $\{u_n\}$ is positive and strictly increasing, it must either converge to a positive limit or diverge to $+\infty$. If the sequence converges to L, then the recurrence relation gives $L = Le^{1/L}$, which is impossible; therefore $\lim_{n\to\infty} u_n = +\infty$.

Note that $\lim_{n\to\infty}(u_{n+1} - u_n) = \lim_{n\to\infty} u_n(e^{1/u_n} - 1) = 1$. Therefore, by the Stolz–Cesàro theorem, $\lim_{n\to\infty} u_n/n = 1$. It follows that

$$\lim_{n\to\infty} \frac{u_{n+1} - u_n - 1}{1/n} = \lim_{n\to\infty} \frac{u_n^2(e^{1/u_n} - 1 - 1/u_n)}{u_n/n} = \frac{1/2}{1} = \frac{1}{2}.$$

By the Stolz–Cesàro theorem again,

$$\begin{aligned}\lim_{n\to\infty} \frac{u_n - n}{\ln n} &= \lim_{n\to\infty} \frac{(u_{n+1} - (n+1)) - (u_n - n)}{\ln(n+1) - \ln n} \\ &= \lim_{n\to\infty} \frac{u_{n+1} - u_n - 1}{1/n} \cdot \frac{1/n}{\ln(1+1/n)} = \frac{1}{2} \cdot 1 = \frac{1}{2}.\end{aligned}$$

Combining the recurrence relation for u_n with the Maclaurin series for the exponential function, for $n \geq 1$ we have

$$u_{n+1} = u_n + 1 + \frac{1}{2u_n} + O\left(\frac{1}{u_n^2}\right) = u_n + 1 + \frac{1}{2n} - \frac{u_n - n}{2nu_n} + O\left(\frac{1}{u_n^2}\right).$$

From previous observations, we know that

$$\frac{u_n - n}{2nu_n} \sim \frac{\ln n}{4n^2} \quad \text{and} \quad \frac{1}{u_n^2} \sim \frac{1}{n^2},$$

so

$$u_{n+1} = u_n + 1 + \frac{1}{2n} + O\left(\frac{\ln n}{n^2}\right).$$

Since $\sum_{n=1}^{\infty} \ln n/n^2$ converges, we conclude that $\sum_{n=1}^{N-1}(u_{n+1} - u_n - 1 - 1/(2n))$ converges as $N \to \infty$. For $N \geq 2$,

$$\sum_{n=1}^{N-1}\left(u_{n+1} - u_n - 1 - \frac{1}{2n}\right) = u_N - u_1 - (N-1) - \frac{H_{N-1}}{2},$$

where we write H_k for the kth harmonic number $\sum_{i=1}^{k} 1/i$. Therefore

$$e^{a_N} - N - \frac{1}{2}\ln N = \sum_{n=1}^{N-1}\left(u_{n+1} - u_n - 1 - \frac{1}{2n}\right) + u_1 - 1 - \frac{1}{2N} + \frac{1}{2}(H_N - \ln N).$$

The desired result follows, since $H_N - \ln N \to \gamma$ as $N \to \infty$.

Editorial comment. Several solvers noted similarities between this problem and MONTHLY Problem 11837 [2015, 391; 2017, 91], which asks for a proof that the sequence $\{a_n - \ln n\}$ decreases monotonically to 0. The earlier MONTHLY problem is a refinement of Problem B4 of the 73rd William Lowell Putnam Mathematical Competition, which simply asks whether $\{a_n - \ln n\}$ has a finite limit. Indeed, since $a_n - \ln n = \ln(u_n/n)$, it follows from the above solution that $\lim_{n\to\infty}(a_n - \ln n) = 0$. This solves the Putnam problem and part of the earlier MONTHLY problem.

Also solved by M. Bataille (France), A. Berkane (Algeria), P. Bracken, H. Chen, N. Grivaux (France), X. Tang (China) & L. Han (US), E. A. Herman, N. Hodges (UK), E. J. Ionaşcu, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), S. Omar (Morocco), E. Omey (Belgium), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), J. Vukmirović (Serbia), J. Yan (China), UM6P Math Club (Morocco), and the proposer.

CLASSICS

C14. *Due to Paul Erdős and George Szekeres; suggested by the editors.* Show that no two entries chosen from the interior of any row of Pascal's triangle are relatively prime.

Visiting Every Region on a Sphere Exactly Once

C13. *Due to Leo Moser; suggested by the editors.* Let n be a multiple of 4, and consider an arrangement of n great circles on the sphere, no three concurrent, dividing the sphere into regions. Show that there is no path on the sphere that visits each region once and only once and never passes through an intersection point of two of the great circles.

Solution. The great circles define a graph G: the vertices are the intersection points of the circles, and the edges are the arcs of the circles joining vertices. Let H be the graph of the corresponding map: the vertices are the regions of G, and edges connect adjacent regions across an edge of G. Because any two great circles intersect twice, G has $n(n-1)$ vertices. Because every vertex of G has four neighbors, G has $2n(n-1)$ edges. By Euler's formula $V - E + F = 2$ relating the numbers of vertices, edges, and faces of a connected graph on the sphere, G has $n(n-1)+2$ faces. This is the number of vertices of H and is even.

Since every edge in H crosses a great circle, and every cycle in H must cross each great circle an even number of times to return to the original region, every cycle in H has even length. Hence H is bipartite, meaning that we can color each vertex of H red or blue in such a way that all edges connect a red vertex and a blue vertex.

The regions of G containing diametrically opposite points on the sphere lie on opposite sides of every great circle. Hence every path joining the vertices for these points crosses every great circle an odd number of times. Since n is even, this implies that such a path has even length, so the vertices representing antipodal regions are colored the same. It follows that H has an even number of vertices of each color.

If H has a path that visits each vertex, then H must have the same number of vertices of each color. Since the two color classes have the same even size, the number of vertices in H is a multiple of 4. However, that number is $n(n-1)+2$, which is not divisible by 4.

Editorial comment. This problem appeared in this MONTHLY as problem E788 [1947, 471; 1948, 366] and is due to Leo Moser. There is an essentially unique arrangement of n great circle arcs on a sphere when $n \le 5$, and for $n \in \{2, 3, 5\}$ each of these arrangements does permit a Hamiltonian path, in fact a Hamiltonian circuit. When $n = 6$, some arrangements permit Hamiltonian paths and some do not.

SOLUTIONS

Two Zeta Sums that Sum to Zeta of Two

12246 [2021, 376]. *Proposed by Seán Stewart, Bomaderry, Australia.* Let ζ be the Riemann zeta function, defined for $n \geq 2$ by $\zeta(n) = \sum_{k=1}^{\infty} 1/k^n$. Let H_n be the nth harmonic number, defined by $H_n = \sum_{k=1}^{n} 1/k$. Prove

$$\sum_{n=2}^{\infty} \frac{\zeta(n)}{n^2} + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n) H_n}{n} = \frac{\pi^2}{6}.$$

Composite solution by Khristo N. Boyadzhiev, Ohio Northern University, Ada, OH, and Stephen Kaczkowski, South Carolina Governor's School for Science and Mathematics, Hartsville, SC. The factor $1/n^2$ in the first sum suggests relevance of the dilogarithm function Li_2, defined by $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$. Henceforth let $L(x) = \mathrm{Li}_2(\mathrm{x})$. It is well known that $L(1) = \pi^2/6$.

For $|x| \leq 1/2$, let

$$M(x) = L(x) + L\Big(\frac{x}{x-1}\Big) + \frac{1}{2}\big(\ln(1-x)\big)^2.$$

From the power series expansions of $\ln(1-x)$, we find that $M'(x) = 0 = M(0)$ whenever $|x| < 1/2$. Thus we have the functional equation $M(x) = 0$, known as Landen's identity. For $x = -1/k$ with $k \geq 2$, this becomes

$$L\Big(-\frac{1}{k}\Big) + L\Big(\frac{1}{k+1}\Big) + \frac{1}{2}\ln^2\Big(1 + \frac{1}{k}\Big) = 0. \qquad (*)$$

For the first sum in the desired statement, we obtain

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{1}{k^n} = \sum_{k=1}^{\infty}\sum_{n=2}^{\infty} \frac{1}{n^2}\Big(\frac{1}{k}\Big)^n = \sum_{k=1}^{\infty}\left(L\Big(\frac{1}{k}\Big) - \frac{1}{k}\right).$$

Here the interchange of summations is valid since every summand is positive. Note that the subtraction of $1/k$ is essential for the convergence.

If the second sum in the statement is the similarly convergent sum

$$\sum_{k=1}^{\infty}\left(\frac{1}{k} - L\Big(\frac{1}{k+1}\Big)\right),$$

then the result follows, since the combined sum over k telescopes to $L(1)$.

From the power series of $-\ln(1-x)$ and $1/(1-x)$, we have

$$\frac{-\ln(1-x)}{1-x} = \sum_{n=1}^{\infty} H_n x^n.$$

Integration then yields

$$\sum_{n=1}^{\infty}\left(H_{n+1}-\frac{1}{n+1}\right)\frac{x^{n+1}}{n+1}=\frac{1}{2}\ln^2(1-x),$$

which we rewrite as

$$\sum_{n=2}^{\infty}\frac{H_n x^n}{n}=\frac{1}{2}\ln^2(1-x)+L(x)-x.$$

Pending justification of the interchange of summations, we compute

$$\begin{aligned}\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)H_n}{n} &= \sum_{n=2}^{\infty}\frac{H_n}{n}\sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)^n = \sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\frac{H_n}{n}\left(-\frac{1}{k}\right)^n \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{2}\ln^2\left(1+\frac{1}{k}\right)+L\left(-\frac{1}{k}\right)+\frac{1}{k}\right) = \sum_{k=1}^{\infty}\left(\frac{1}{k}-L\left(\frac{1}{k+1}\right)\right).\end{aligned}$$

Here the last step uses the functional equation $(*)$ for L.

It remains to justify the interchange of summations. The double summation with the inner sum over k may be written as

$$\sum_{n=2}^{\infty}\frac{H_n}{n}(-1)^n+\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{H_n}{n}\left(-\frac{1}{k}\right)^n.$$

Since H_n/n is decreasing, the first sum converges. Next,

$$\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{H_n}{n}\left(\frac{1}{k}\right)^n<\sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\left(\frac{1}{k}\right)^n=\sum_{k=2}^{\infty}\left(\frac{1}{k}\right)^2\left(\frac{k}{k-1}\right)<\infty.$$

Thus the double summation is absolutely convergent. It follows that the interchange is valid, which completes the proof.

Editorial comment. Many solvers (including the proposer) relied on some version of the known identity

$$\ln\Gamma(1-x)=\gamma x+\sum_{n=2}^{\infty}\frac{\zeta(n)}{n}x^n,$$

where γ is Euler's constant. The proposer also showed that the two sums are, respectively, $-\gamma+J$ and $\pi^2/6+\gamma-J$, where

$$J=\int_0^1\frac{\ln\Gamma(1-x)}{x}\,dx.$$

T. Apostol famously proved $\zeta(2)=\pi^2/6$ by making a change of variable in a double integral for $\zeta(2)$. Solvers Hervé Grandmontagne and Richard Stong, independently, showed that each of the two sums here summing to $\pi^2/6$ has a usable representation as a double integral. Grandmontagne used well-known integrals for $\zeta(n)$, H_n, and $1/n^2$ to write the two sums as

$$\sum_{n=2}^{\infty}\int_0^1\int_0^1\frac{y^{n-1}\ln(1-y)(\ln x)^{n-1}}{(1-x)(n-1)!}\,dx\,dy$$

and

$$\sum_{n=2}^{\infty}\int_0^1\int_0^1\frac{(-y)^{n-1}\ln(1/y)(\ln x)^{n-1}}{(1-x)(n-1)!}\,dx\,dy.$$

After interchanging summation and integration, some simplification leads to

$$\int_0^1 \int_0^1 \frac{(x^{1-y}-1)\ln y}{1-x}\,dx\,dy + \int_0^1 \int_0^1 \frac{(x^{-y}-1)\ln(1/y)}{1-x}\,dx\,dy = \int_0^1 \int_0^1 x^{-y}\ln(1/y)\,dx\,dy.$$

The two double integrals on the left are the two sums in the posed problem, and the double integral on the right equals $\zeta(2)$. However, while the interchange of summation and integration needed to complete this proof can be justified, it does require a fair amount of work, especially for the first double integral.

Also solved by F. R. Ataev (Uzbekistan), A. Berkane (Algeria), P. Bracken, B. Bradie, B. S. Burdick, H. Chen (US), G. Fera (Italy), M. L. Glasser, R. Gordon, H. Grandmontagne (France), G. C. Greubel, A. M. Karparvar (Iran), O. Kouba (Syria), Z. Lin (China), C. Sanford, K. Sarma (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Wildon (UK), and the proposer.

An Angle Bisector That Bisects a Segment

12253 [2021, 467]. *Proposed by Alexandru Gîrban, Constanţa, Romania, and Bogdan D. Suceavă, Fullerton, CA.* Let ABC be a triangle, and let D and E be the contact points of the incircle of ABC with the segments BC and CA, respectively. Let M be the intersection of the line DE and the line through A parallel to BC. Prove that the bisector of $\angle ABC$ passes through the midpoint of DM.

Solution by Haoran Chen, Suzhou, China. Let F be the tangency point of the incircle with AB, and let N be the intersection of the bisector of $\angle ABC$ with AM. By three applications of the tangent segment theorem, $AE = AF$, $BF = BD$, and $CD = CE$. Since AM is parallel to BC, $\triangle CDE$ and $\triangle AME$ are similar, and therefore $AM = AE$. Also, $\angle ANB = \angle CBN = \angle ABN$, so $\triangle ABN$ is isosceles and $AN = AB > AF = AE = AM$. Thus M is between A and N, and $MN = AN - AM = AB - AF = BF = BD$. It follows that BN and DM intersect at a point P such that $\triangle PBD$ and $\triangle PNM$ are congruent, and hence $PD = PM$.

Also solved by M. Bataille (France), J. Cade, C. Chiser (Romania), P. De (India), C. de la Losa (France), I. Dimitrić, M. Dobrescu, G. Fera (Italy), D. Fleischman, K. Gatesman, O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, N. Hodges (UK), W. Janous (Austria), M. Getz & D. Jones, A. M. Karparvar (Iran), K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), E. Mika & I. Adams & L. Loprieno & R. McMullen & D. Schmitz, J. Minkus, D. Pinchon (France), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Zhou, T. Zvonaru (Romania), Davis Problem Solving Group, and the proposer.

Sum of Squares Modulo 6

12255 [2021, 467]. *Proposed by Besfort Shala, student, University of Primorska, Koper, Slovenia.* Given a positive integer a_0, define $a_1, \dots, a_n$ recursively by $a_i = 1^2 + 2^2 + \cdots + a_{i-1}^2$ for $i \geq 1$. Is it true that, given any subset A of $\{1, \dots, n\}$, there is a positive integer a_0 such that, for $1 \leq i \leq n$, 6 divides a_i if and only if $i \in A$?

Solution by Nigel Hodges, Cheltenham, UK. The answer is yes. We prove the following more general result: Given a list $b_1, \dots, b_n$ of integers, there is a positive integer a_0 such that $a_i \equiv b_i \pmod 6$ for $1 \leq i \leq n$. We may assume that each b_i lies in $\{0, 1, \dots, 5\}$. Since $a_i = a_{i-1}(a_{i-1}+1)(2a_{i-1}+1)/6$ for $i \geq 1$, it is reasonable to extend the definition by letting the sequence be identically 0 when $a_0 = 0$. The identity

$$\frac{(a_{i-1}+6^r)(a_{i-1}+6^r+1)(2a_{i-1}+2\cdot 6^r+1)}{6} - \frac{a_{i-1}(a_{i-1}+1)(2a_{i-1}+1)}{6}$$
$$= 2\cdot 6^{3r-1} + 6^{2r-1}(6a_{i-1}+3) + 6^{r-1}(6a_{i-1}^2 + 6a_{i-1} + 1)$$

describes the change in a_i when a_{i-1} increases by 6^r. Modulo 6^r, the change is 6^{r-1}. This allows the following inductive algorithm to find a_0 to satisfy the given conditions.

Start with $a_0 = 0$, so $a_1 = 0$ as well. Add 6 to a_0 exactly b_1 times, adding $6b_1$ overall. Since $6^0 = 1$, applying the identity with $r = 1$ yields $a_1 \equiv b_1 \pmod 6$.

Recalculate a_2 from the revised a_0. Choose δ_2 nonnegative so that $\delta_2 \equiv b_2 - a_2 \pmod 6$, and then add $6^2 \cdot \delta_2$ to a_0. This increases a_1 by a multiple of 6, so still $a_1 \equiv b_1 \pmod 6$. Also, a_2 increases by δ_2 modulo 6, so $a_2 \equiv b_2 \pmod 6$.

Continue in this manner. At stage j, recalculate $a_1, \dots, a_j$ from the revised a_0. Choose δ_j nonnegative so that $\delta_j \equiv b_j - a_j \pmod 6$, and add $6^j \cdot \delta_j$ to a_0. This increases each of $a_1, \dots, a_{j-1}$ by a multiple of 6, so still $a_i \equiv b_i \pmod 6$ for $1 \le i \le j-1$. Also, a_j increases by δ_j, so $a_j \equiv b_j \pmod 6$.

Repeat this process until $j = n$. If the resulting value of a_0 is still 0, set $a_0 = 6^{n+1}$ to make it positive, as required. This does not affect any of $a_1, \dots, a_n$ modulo 6, so each required congruence is still satisfied, finishing the proof.

Also solved by Y. J. Ionin, O. P. Lossers (Netherlands), D. Pinchon (France), M. A. Prasad (India), K. Sarma (India), R. Stong, R. Tauraso (Italy), and the proposer.

An Integral Formula for Apéry's Constant

12256 [2021, 468]. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Prove

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx = -\frac{5}{8}\zeta(3),$$

where $\zeta(3)$ is Apéry's constant $\sum_{n=1}^{\infty} 1/n^3$.

Solution by Giuseppe Fera, Vicenza, Italy. With $A = \log(1-x)$ and $B = \log(1+x)$, the algebraic identity $AB = (1/4)((A+B)^2 - (A-B)^2)$ yields

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx = \frac{1}{4}\left(\int_0^1 \frac{\log^2(1-x^2)}{x}\,dx - \int_0^1 \frac{1}{x}\log^2\left(\frac{1-x}{1+x}\right)dx\right).$$

To evaluate the first integral on the right side, we use the substitution $y = 1 - x^2$, obtaining

$$\begin{aligned}\int_0^1 \frac{\log^2(1-x^2)}{x}\,dx &= \frac{1}{2}\int_0^1 \frac{\log^2(y)}{1-y}\,dy = \frac{1}{2}\int_0^1 \log^2(y)\sum_{n=1}^{\infty} y^{n-1}\,dy \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 y^{n-1}\log^2 y\,dy = \sum_{n=1}^{\infty}\frac{1}{n^3} = \zeta(3),\end{aligned}$$

where the last integral is computed using integration by parts twice. Similarly, the substitution $y = (1-x)/(1+x)$ in the second integral yields

$$\begin{aligned}\int_0^1 \frac{1}{x}\log^2\left(\frac{1-x}{1+x}\right)dx &= 2\int_0^1 \frac{\log^2(y)}{1-y^2}\,dy = 2\int_0^1 \log^2(y)\sum_{n=1}^{\infty} y^{2(n-1)}\,dy \\ &= 2\sum_{n=1}^{\infty}\int_0^1 y^{2n-2}\log^2 y\,dy = 4\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} \\ &= 4\left(\sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{\infty}\frac{1}{(2n)^3}\right) = 4\left(\zeta(3) - \frac{1}{8}\zeta(3)\right) = \frac{7}{2}\zeta(3).\end{aligned}$$

Thus

$$\int_0^1 \frac{\log(1+x)\log(1-x)}{x}\,dx = \frac{1}{4}\left(\zeta(3) - \frac{7}{2}\zeta(3)\right) = -\frac{5}{8}\zeta(3).$$

Editorial comment. Several solvers pointed out that this integral appears in C. I. Vălean (2019), *(Almost) Impossible Integrals, Sums, and Series*, Cham, Switzerland: Springer. This integral played a role in some submitted solutions to problem 12206 [2020, 722; 2022, 492] from this MONTHLY.

Also solved by T. Amdeberhan & A. Tefera, F. R. Ataev (Uzbekistan), M. Bataille (France), A. Berkane (Algeria), N. Bhandari (Nepal), B. Bradie, V. Brunetti & D. B. Malesani & A. Aurigemma (Denmark), H. Chen, N. S. Dasireddy (India), B. E. Davis, J. Fu (China), A. Garcia (France), S. Gayen (India), M. L. Glasser, R. Gordon, H. Grandmontagne (France), G. C. Greubel, J.-P. Grivaux (France), R. Guadalupe (Philippines), L. Han (US) & X. Tang (China), D. Henderson, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), A. M. Karparvar (Iran), O. Kouba (Syria), O. P. Lossers (Netherlands), R. Mortini (France) & R. Rupp (Germany), M. Omarjee (France), D. Pinchon (France), M. A. Prasad (India), C. Sanford, K. Sarma (India), V. Schindler (Germany), S. Sharma (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), J. Van Casteren & L. Kempeneers (Belgium), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), T. Wilde (UK), M. Wildon (UK), FAU Problem Solving Group, The Logic Coffee Circle (Switzerland), UM6P Math Club (Morocco), Westchester Area Math Circle, and the proposer.

A Saturated Arrangement of Equilateral Triangles

12257 [2021, 468]. *Proposed by Erich Friedman, Stetson University, DeLand, FL, and James Tilley, Bedford Corners, NY.* An arrangement of equilateral triangles in the plane is called *saturated* if the intersection of any two is either empty or is a common vertex and every vertex is shared by exactly two triangles. What is the smallest positive integer n such that there exists a saturated arrangement of n equilateral triangles with integer length sides?

Solution by the Davis Problem Solving Group, Davis, CA. The smallest such n is 10, with an example given by Figure 1.

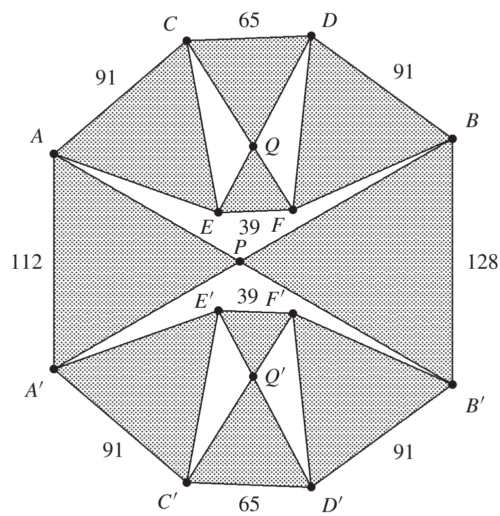


Figure 1

First, we show that $n \geq 10$ for a saturated arrangement of n equilateral triangles, whether or not the sides have integer lengths. The total number of vertices is $3n/2$, so n must be even. Consider the simple polygon consisting of the edges of the triangles bordering the unbounded region outside the arrangement. Because the triangles intersect in at most a vertex, each interior angle of this polygon is greater than 120 degrees. Thus the polygon has at least seven edges, corresponding to distinct boundary triangles in the arrangement.

If no two boundary triangles share a vertex inside the polygon, then we have at least seven interior vertices and hence at least three additional triangles. In this case $n \geq 7+3 = 10$. If two boundary triangles have a common interior vertex, then they cannot be adjacent on the polygon, so there must be an interior vertex on each side of their union. Hence there must also be an interior triangle on each side of their union. Therefore, $n \geq 7+2 = 9$ and n is even, so $n \geq 10$.

Returning to our example, we establish that such an arrangement does indeed exist. We begin with two equilateral triangles as in Figure 2, where $\angle APB = 120°$ and $a \leq b$. Applying the law of cosines to $\triangle ABP$, we find $AB^2 = a^2 + b^2 + ab$.

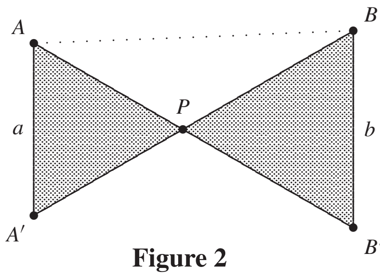


Figure 2

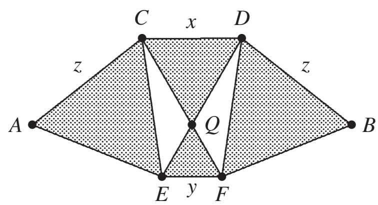


Figure 3

To obtain a saturated arrangement, we combine two copies of the configuration of four equilateral triangles in Figure 3 (one upside down) with that in Figure 2 to obtain the saturated configuration in Figure 1. Here $\angle DQF = 120°$ and $y \leq x$. There are three conditions on the integers a, b, x, y, z that together are necessary and sufficient for the construction to yield a saturated configuration. Applying the law of cosines to $\triangle DFQ$ yields the first: $x^2 + y^2 + xy = z^2$.

The second is that AB has the same length in both figures. To compute AB in Figure 3, observe that the quadrilateral $BDQF$ has opposite angles summing to 180°, so these four points lie on a circle. Angles BDF and BQF subtend the same arc of the circle, so $\angle BQF = \angle BDF = 60°$. Similarly $\angle AQE = 60°$, so A, Q, and B are collinear. Applying the law of sines to $\triangle DFQ$, we find

$$\sin \angle FDQ = \frac{\sqrt{3}y}{2z}, \qquad \text{and so} \qquad \cos \angle FDQ = \sqrt{1 - \frac{3y^2}{4z^2}} = \frac{2x+y}{2z}.$$

Applying the law of sines and the addition formula for sines to $\angle BDQ$, we find that BQ and AQ have length $x + y$ and AB has length $2x + 2y$. Therefore, the second condition is $a^2 + b^2 + ab = 4(x + y)^2$.

The third and final condition is that the triangles do not overlap when we combine the pieces. Because $a \leq b$, it follows that $\angle ABP \leq \angle BAP$. Thus, the requirement becomes $\angle FBQ < \angle ABP$, which, because both angles are acute, is equivalent to $\sin \angle FBQ < \sin \angle ABP$. Because $BDQF$ is cyclic, also $\sin \angle FBQ = \sqrt{3}y/(2z) = \sqrt{3}/(2\sqrt{(x/y)^2 + 1 + x/y})$. Applying the law of sines to $\triangle ABP$ yields $\sin \angle ABP = \sqrt{3}/(2\sqrt{(1 + (b/a)^2 + b/a})$. We conclude that the third condition is equivalent to $x/y > b/a$.

It is easy to check that the necessary and sufficient set of two equalities and one inequality holds when $a = 112$, $b = 128$, $x = 65$, $y = 39$, and $z = 91$.

Editorial comment. In the above solution, one can also prove that $BQ = x + y$ geometrically. Rotate $\triangle DFQ$ by 60° counterclockwise about Q. The image of F is a point R on

BQ with $QR = y$. The image of D is C and $CR = z$. Therefore, $BDCR$ is a parallelogram and $BR = CD = x$.

Solvers presented several other constructions, including some with seven boundary triangles and three interior triangles.

Also solved by T. Fujita & S. Kim, S. M. Gagola Jr., O. P. Lossers (Netherlands), A. Martin & R. Martin (Germany), R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

Factorials That Are Not the Sum of Three Squares

12258 [2021, 563]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Let S be the set of positive integers n such that $n!$ is not the sum of three squares. Show that S has bounded gaps, i.e., there is a positive constant C such that for every positive integer n, there is an element of S between n and $n + C$.

Solution by Michael Reid, University of Central Florida, Orlando, FL. We prove that the difference between any two consecutive elements of S is at most 77.

Legendre proved that a positive integer is not a sum of three squares if and only if it has the form $4^c(8q + 7)$ for some nonnegative integers c and q. We claim that for every nonnegative integer m, there is an integer t with $1 \leq t \leq 14$ such that $64m + t \in S$. Write $(64m)!$ uniquely as $2^a(8q + r)$, where a is a nonnegative integer and $r \in \{1, 3, 5, 7\}$. When $r = 5$ and a is odd, with $a = 2b + 1$, we take $t = 3$. This yields

$$(64m + 3)! = 2^{2b+2}(8q + 5)(64m + 1)(32m + 1)(64m + 3) = 4^{b+1}(8k + 7)$$

for some positive integer k. Hence in this case $(64m + 3)!$ is not a sum of three squares. When $r = 1$ and a is odd, with $a = 2b + 1$, we take $t = 5$. This yields

$$\begin{aligned}(64m + 5)! &= 2^{2b+4}(8q + 1)(64m + 1)(32m + 1)(64m + 3)(16m + 1)(64m + 5)\\ &= 4^{b+2}(8k + 7)\end{aligned}$$

for some positive integer k. Hence in this case $(64m + 5)!$ is not a sum of three squares.

Similar computations for the other six cases of the parity of a and the value of r yield the following table of values of t such that $64m + t \in S$.

	$r = 1$	$r = 3$	$r = 5$	$r = 7$
a odd	5	14	3	2
a even	10	6	7	1

This establishes that consecutive elements of S differ by at most $64 + 13$.

Editorial comment. Michael Reid and John Robertson independently showed that the difference between consecutive elements of S never exceeds 42. To prove this, one can consider 15 consecutive values of n, having the form $64m + 16j + t$ with $1 \leq t \leq 15$ for fixed m and fixed $j \in \{1, 2, 3\}$. Like $(64m)!$ as discussed above, one writes $(64m + 16j)!$ as $2^a(8q + r)$ with eight cases for a and r. For each j in $\{1, 2, 3\}$, there is thus a table like that above in whose cells are listed the values of t such that $(64m + 16j + t)!$ can be expressed in the form $4^c(8k + 7)$. If all cells were nonempty, then consecutive members of S would differ by at most $16 + 14$.

In fact, there are two empty cells, for $64m + 32$ with $(64m + 32)! = 2^{2b+1}(8q + 3)$ and for $64m + 48$ with $(64m + 48)! = 2^{2b}(8q + 5)$. For these cases one must go farther than $t = 15$. In the first case, $64m + 49$ or $64m + 58$ is in S, depending on the parity of m. Since also $64m + 16 + t \in S$ for some $t \geq 5$, these consecutive members of S differ by at most $58 - 21$, which equals 37. In the second case, $64m + 36$ or $64m + 47$ is in S, depending on the parity of m. The chart above shows $64(m + 1) + t \in S$ for some t

with $1 \le t \le 14$. Thus in this case $78 - 36$ is an upper bound on the difference between consecutive members of S, and hence in all cases the bound is at most 42.

Furthermore, differences of 42 occur infinitely often. A computer search shows that the first such difference occurs for the 2932nd and 2933rd elements in S, which are 23268 and 23310. Using $(2n)! = 2^n n! \prod_{i=1}^{n}(2i-1)$, it is easy to show by induction that $(9 \cdot 2^t)!$ has the form $4^b(8q+1)$ for $t \ge 2$. When t is sufficiently large and $1 \le j \le 23310$, the factors j and $9 \cdot 2^t + j$ are divisible by the same power of 2 and have odd parts that are congruent modulo 8 (in fact, $t \ge \lfloor \log_2 23310 \rfloor + 3 = 17$ is sufficient). More precisely, for j in this range, $j! = 2^b(8q+r)$ and $(9 \cdot 2^t + j)! = 2^B(8Q+R)$ with $b \equiv B \bmod 2$ and $r \equiv R \bmod 8$. Thus $j \in S$ if and only if $9 \cdot 2^t + j \in S$. Therefore, $9 \cdot 2^t + 23268$ and $9 \cdot 2^t + 23310$ are consecutive elements of S when $t \ge 17$, differing by 42.

A proof that the density of S is $1/8$ can be found in J.-M. Deshouillers and F. Luca, How often is $n!$ the sum of three squares?, K. Alladi, J. R. Klauder, and C. R. Rao, Eds. (2010), *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, Springer, 243–251.

Also solved by R. Dietmann (UK), A. Goel, N. Hodges (UK), O. P. Lossers (Netherlands), R. Martin (Germany), J. P. Robertson, C. Schacht, A. Stadler (Switzerland), R. Stong, M. Tang, R. Tauraso (Italy), L. Zhou, and the proposer.

Supplementary Pairs of Heronian Triangles

12259 [2021, 563]. *Proposed by Giuseppe Fera, Vicenza, Italy.* A triangle is *Heronian* if it has integer sides and integer area. A pair of noncongruent Heronian triangles is called a *supplementary pair* if the triangles have the same perimeter and the same area and some interior angle of one is the supplement of some interior angle of the other. Prove that there are infinitely many supplementary pairs of Heronian triangles.

Solution by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA, and Savannah, GA. We claim that for each integer $n \ge 2$, the triangles with side lengths (a_1, b_1, c_1) and (a_2, b_2, c_2) given by

$$(a_1, b_1, c_1) = (n^4 + n^2 + 1, n^6 + n^4 + 2n^2 + 1, n^6 + 2n^4 + n^2)$$

and

$$(a_2, b_2, c_2) = (n^4 + 2n^2 + 1, n^6 + n^4 + n^2, n^6 + 2n^4 + n^2 + 1)$$

form a supplementary pair of Heronian triangles. Note that $a_i < b_i < c_i$ for $i \in \{1, 2\}$. Also, $a_i + b_i > c_i$, so there is indeed a triangle for each triple. Since $c_2 = c_1 + 1$, the two triangles are not congruent.

Since $a_1 + b_1 + c_1 = 2(n^6 + 2n^4 + 2n^2 + 1) = a_2 + b_2 + c_2$, the two triangles have the same perimeter. Let s be the common semiperimeter; note that $s = (n^2+1)(n^4+n^2+1)$. By Heron's formula, the area of the ith triangle is $\sqrt{s(s-a_i)(s-b_i)(s-c_i)}$. Thus the area of the first triangle is

$$\sqrt{(n^2+1)(n^4+n^2+1) \cdot n^2(n^4+n^2+1) \cdot n^4 \cdot (n^2+1)},$$

and the area of the second triangle is

$$\sqrt{(n^2+1)(n^4+n^2+1) \cdot n^4(n^2+1) \cdot (n^4+n^2+1) \cdot n^2}.$$

Therefore, each triangle has area $n^3(n^2+1)(n^4+n^2+1)$.

Finally, let B_1 be the angle opposite the side of length b_1, and let C_2 be the angle opposite the side of length c_2. By the law of cosines, after some calculation, we find

$$\cos B_1 = \frac{a_1^2 + c_1^2 - b_1^2}{2a_1c_1} = \frac{n^2-1}{n^2+1},$$

and

$$\cos C_2 = \frac{a_2^2 + b_2^2 - c_2^2}{2a_2b_2} = -\frac{n^2-1}{n^2+1} = -\cos B_1.$$

Thus B_1 and C_2 are supplementary, and for $n \geq 2$, we have a supplementary pair of Heronian triangles. As n runs through the integers greater than 1, we obtain infinitely many distinct values for $\cos B_1$, so this method produces infinitely many such pairs.

Also solved by J. Keadey & J. Boltz & S. Kompella & S. Vemuru, P. Lalonde (Canada), C. R. Pranesachar (India), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), and the proposer.

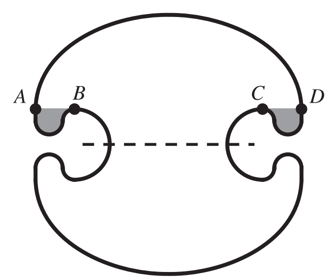
CLASSICS

C13. *Due to Leo Moser; suggested by the editors.* Let n be a multiple of 4, and consider an arrangement of n great circles on the sphere, no three concurrent, dividing the sphere into regions. Show that there is no path on the sphere that visits each region once and only once and never passes through an intersection point of two of the great circles.

The Unilluminable Room

C12. *Due to Lionel Penrose and Roger Penrose; suggested by the editors.* Is there a plane region bounded by a differentiable Jordan curve with the property that no matter where a light source is placed inside it, some part of the region remains unilluminated? Assume that the curve acts as a perfect mirror.

Solution. An unilluminable region is shown below. It has a horizontal line of symmetry. The arc AD is the upper half of the ellipse with foci B and C. The remaining portion of the boundary curve may be constructed from circular arcs, although any differentiable curve with the approximate shape of the diagram will suffice. Any light ray that starts below the segment BC might visit the part of the region above BC, but to do so it will have to pass through BC and strike the elliptical arc AD. By a well-known property of the ellipse, a light ray from B that strikes the elliptical arc AD will reflect back to C. It follows that a ray that passes through BC and strikes the elliptical arc will be reflected back between B and C, and therefore it cannot visit the two shaded parts of the region. Similarly, a light ray that starts above the reflection of BC across the line of symmetry will never visit the reflections of the shaded regions across the line of symmetry.

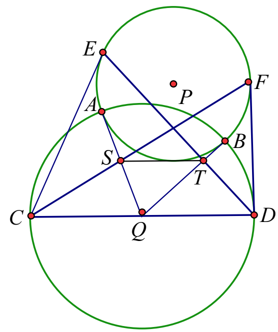


Editorial comment. The question was raised by E. G. Straus in the early 1950s and solved in L. S. Penrose and R. Penrose (1958), Puzzles for Christmas, *The New Scientist*, 1580–1581, 1597. Victor Klee, in V. Klee (1979), Some unsolved problems in plane geometry, *Math. Mag.* 52, 131–145, asked if a polygonal solution was possible and, somewhat surprisingly, the answer is yes. In G. W. Tokarsky (1995), Polygonal rooms not illuminable from every point, this MONTHLY, 102, 867–879, a 26-gon is constructed that cannot be illuminated from a point. This was later improved by D. Castro to a 24-gon. In the polygonal examples, only a single point stays dark.

SOLUTIONS

Constructing a Tangent to a Circle

12245 [2021, 376]. *Proposed by Jiahao Chen, Tsinghua University, Beijing, China.* Suppose that two circles α and β, with centers P and Q, respectively, intersect orthogonally at A and B. Let CD be a diameter of β that is exterior to α. Let E and F be points on α such that CE and DF are tangent to α, with C and E on one side of PQ and D and F on the other side of PQ. Let S be the intersection of CF and QA, and let T be the intersection of DE and QB. Prove that ST is parallel to CD and is tangent to α.



Solution by Davis Problem Solving Group, Davis, CA. Let Y be the intersection point of lines BC and AD. We claim that Y lies on circle α and that the tangent line ℓ to α at Y is parallel to CD. To prove the claim, we assume for ease of exposition that A and C are on the same side of PQ, with B and D on the other side, as in the figure that accompanies the problem statement; however, the argument also works if the roles of A and B are switched, as long as we view all angles as directed. Note that $\angle BYA = \angle CYD = 180^\circ - \angle DCB - \angle ADC$, while $\angle APB = 180^\circ - \angle BQA = 2\angle DCB + 2\angle ADC$. Thus $\angle BYA$ is inscribed in circle α and Y lies on α. Now let Z denote the intersection of AQ and ℓ. Since ZA and ZY are both tangent to α, $\angle ZYA = \angle YAZ = \angle DAQ = \angle QDA$, and therefore ℓ is parallel to CD. This proves the claim.

Now let D' denote the second intersection point of line PD and circle β. Since inversion in circle α preserves circle β, this inversion sends D to D'. Since $\angle DD'C = 90°$, it follows that C is on the polar line d of point D with respect to circle α. The circumcircle of $\triangle PDF$ has diameter PD and thus maps to d under inversion in α. Thus line FC is the polar line d of point D. Similarly, line ED is the polar line of point C with respect to α.

The polar lines of points A and B with respect to α are QA and QB, respectively, so S is the intersection of the polar lines of A and D, and T is the intersection of the polar lines of B and C. By duality, the polar lines of S and T are lines AD and BC, respectively. By our initial claim, these polar lines intersect in Y. It follows that line ST is the polar line of point Y, which is just the tangent line ℓ to α at Y. Thus ST is parallel to CD and tangent to α, as desired.

Also solved by M. Bataille (France), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), J. Cade, G. Fera (Italy), D. Fleischman, K. Gatesman, N. Hodges (UK), A. M. Karparvar (Iran), K.-W. Lau (China), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

An Integral over the Sphere

12247 [2021, 377]. *Proposed by Prathap Kasina Reddy, Bhabha Atomic Research Centre, Mumbai, India.* For positive real constants a, b, and c, prove

$$\int_0^\pi \int_0^\infty \frac{a}{\pi(x^2+a^2)^{3/2}} \frac{x}{\sqrt{x^2+b^2+c^2-2cx\cos\theta}}\, dx\, d\theta = \frac{1}{\sqrt{(a+b)^2+c^2}}.$$

Solution by Giuseppe Fera, Vicenza, Italy. Let $f(a,b,c)$ be the left side of the desired equation. With the substitution $x = a\tan(\varphi/2)$, we obtain

$$f(a,b,c) = \frac{\sqrt{2}}{4\pi} \int_0^\pi \int_0^\pi \frac{\sin\varphi\, d\varphi\, d\theta}{\sqrt{a^2+b^2+c^2+(b^2+c^2-a^2)\cos\varphi - 2ac\cos\theta\sin\varphi}}.$$

Since the integrand is invariant under the substitution $\theta \mapsto 2\pi - \theta$, we can write

$$f(a,b,c) = \frac{\sqrt{2}}{8\pi} \int_0^{2\pi} \int_0^\pi \frac{\sin\varphi\, d\varphi\, d\theta}{\sqrt{a^2+b^2+c^2+(b^2+c^2-a^2)\cos\varphi - 2ac\cos\theta\sin\varphi}}.$$

Interpret φ and θ as the spherical coordinates for a point

$$\mathbf{r} = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$$

on the unit sphere S, and let $\mathbf{v} = (-2ac, 0, b^2+c^2-a^2)$. We see that

$$f(a,b,c) = \frac{\sqrt{2}}{8\pi} \iint_S \frac{1}{\sqrt{a^2+b^2+c^2+\mathbf{v}\cdot\mathbf{r}}}\, dS.$$

To evaluate this integral, we write it in cylindrical coordinates z and θ, with the positive z-axis aligned with the vector $\mathbf{v}$. Setting $t = a^2+b^2+c^2$ and

$$v = \|\mathbf{v}\| = \sqrt{4a^2c^2+(b^2+c^2-a^2)^2} = \sqrt{(a^2+b^2+c^2)^2-4a^2b^2} = \sqrt{t^2-4a^2b^2},$$

this yields

$$\begin{aligned} f(a,b,c) &= \frac{\sqrt{2}}{8\pi} \int_{-1}^1 \int_0^{2\pi} \frac{d\theta\, dz}{\sqrt{t+vz}} = \frac{\sqrt{2}}{2v}\left(\sqrt{t+v}-\sqrt{t-v}\right) \\ &= \frac{\sqrt{2}}{2v}\sqrt{\left(\sqrt{t+v}-\sqrt{t-v}\right)^2} = \frac{\sqrt{t-\sqrt{t^2-v^2}}}{v} = \frac{\sqrt{t-2ab}}{\sqrt{t^2-4a^2b^2}} \\ &= \frac{1}{\sqrt{t+2ab}} = \frac{1}{\sqrt{a^2+b^2+c^2+2ab}} = \frac{1}{\sqrt{(a+b)^2+c^2}}. \end{aligned}$$

Also solved by M. L. Glasser, O. Kouba (Syria), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), R. Tauraso (Italy), and the proposer.

An Identity from the Pfaffian

12248 [2021, 377]. *Proposed by Askar Dzhumadil'daev, Almaty, Kazakhstan.* Let n be a positive integer, and let x_k be a real number for $1 \le k \le 2n$. Let C be the $2n$-by-$2n$ skew-symmetric matrix with i, j-entry $\cos(x_i - x_j)$ when $1 \le i < j \le 2n$. Prove

$$\det(C) = \cos^2(x_1 - x_2 + x_3 - x_4 + \cdots + x_{2n-1} - x_{2n}).$$

Solution by Richard Ehrenborg, University of Kentucky, Lexington, KY. The determinant of a skew-symmetric matrix A is equal to the square of the Pfaffian of the matrix A. The *Pfaffian* $\mathrm{Pf}(A)$ of a $2n$-by-$2n$ skew-symmetric matrix A with entries $a_{i,j}$ for $1 \le i, j \le 2n$ is defined by

$$\mathrm{Pf}(A) = \sum_M (-1)^{c(M)} \cdot \prod_{(i,j)\in M} a_{ij}.$$

Here the sum is over all perfect matchings M on the set $\{1, \ldots, 2n\}$, where an edge (i, j) is written with $i < j$. Also $c(M)$ is the number of pairs of crossing edges in M, where two edges (i', j') and (i'', j'') in M form a *crossing* if $i' < i'' < j' < j''$. The *sign* of a matching M is $(-1)^{c(M)}$. Our goal is to prove

$$\mathrm{Pf}(C) = \cos(x_1 - x_2 + x_3 - \cdots - x_{2n}).$$

Using the identity $2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta)$ and the fact that cosine is an even function, a straightforward induction yields

$$2^n \cdot \prod_{i=1}^{n} \cos(\alpha_i) = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm 1\}^n} \cos(\varepsilon_1\alpha_1 + \cdots + \varepsilon_n\alpha_n).$$

Thus we express $\mathrm{Pf}(C)$ as follows, where we denote the edges of a matching M by $(i_1, j_1), \ldots, (i_n, j_n)$.

$$2^n \cdot \mathrm{Pf}(C) = \sum_M \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm 1\}^n} (-1)^{c(M)} \cos(\varepsilon_1(x_{i_1} - x_{j_1}) + \cdots + \varepsilon_n(x_{i_n} - x_{j_n})).$$

By reordering the terms in the argument to cos, we can express each term on the right side in the form $\cos(\pm x_1 \pm \cdots \pm x_{2n})$, with n numbers weighted positively and n numbers weighted negatively.

Consider a term for a matching M in which x_k and x_{k+1} have the same coefficients, that is, $\cos(\cdots + \varepsilon x_k + \varepsilon x_{k+1} \cdots)$, where $\varepsilon \in \{\pm 1\}$. Since any two indices forming an edge of M are given different signs, k and $k+1$ do not form an edge in M.

Hence we can obtain another matching M' by switching the mates of k and $k+1$ in M. Always $|c(M') - c(M)| = 1$, and hence this mapping τ_k is a sign-reversing involution on the set of matchings. The fixed points of τ_k are exactly those matchings that pair k and $k+1$. Hence the contributions of M and M' to the coefficient of any term of the form $\cos(\cdots + \epsilon x_k + \epsilon x_{k+1} + \cdots)$ cancel.

Thus for each M the only terms that remain uncanceled under all τ_k are the two terms with alternating signs: $\cos(x_1 - x_2 + x_3 - \cdots - x_{2n})$ and $\cos(-x_1 + x_2 - x_3 + \cdots + x_{2n})$. Since cosine is an even function, these two terms are equal. We conclude

$$\mathrm{Pf}(C) = c_n \cdot \cos(x_1 - x_2 + x_3 - \cdots - x_{2n}), \qquad (*)$$

where c_n is a constant depending on n. To determine c_n, set $x_1 = \cdots = x_{2n} = 0$. The left side of $(*)$ is now the Pfaffian of a skew-symmetric matrix having all entries above the diagonal equal to 1. Expressing it in terms of matchings reduces it to $\sum_M (-1)^{c(M)}$, where the sum is over all matchings on $\{1, \ldots, 2n\}$.

We prove $\sum_M (-1)^{c(M)} = 1$ by induction on n. The base case $n = 1$ is easy: there is exactly one matching on $\{1, 2\}$, with no crossings. For the induction step, define an involution on the set of matchings on $\{1, \ldots, 2n\}$ by switching the elements $2n-1$ and $2n$. If the result is a new matching, then the numbers of crossings in these two matchings differ by 1, and the terms for these two matchings cancel in the sum. What remains are the matchings where $2n-1$ and $2n$ form an edge. This edge crosses no other, so the sum for these matchings is the same as the sum for all matchings on $\{1, \ldots, 2n-2\}$, which by the induction hypothesis is 1.

Also solved by F. R. Ataev (Uzbekistan), H. Chen, N. Hodges (UK), P. Lalonde (Canada), O. P. Lossers (Netherlands), M. Omarjee (France), C. R. Pranesachar (India), K. Sarma (India), A. Stadler (Switzerland), M. Tang, R. Tauraso (Italy), J. Van hamme (Belgium), T. Wiandt, M. Wildon (UK), and the proposer.

Simplifying a Sum

12249 [2021, 377]. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.* Prove

$$\sum_{k=\lfloor n/2\rfloor}^{n-1} \sum_{m=1}^{n-k} (-1)^{m-1} \frac{k+m}{k+1} \binom{k+1}{m-1} 2^{k-m} = \frac{n}{2}$$

for any positive integer n.

Solution by Rory Molinari, Beverly Hills, MI. Call the desired sum $T(n)$, and let $S(n) = 2T(n)/n$. We prove $S(n) = 1$ for $n > 0$. For $n > 0$ and $\lfloor n/2\rfloor \le k \le n-1$, set

$$t_m = (-1)^{m-1} \frac{k+m}{k+1} \binom{k+1}{m-1} 2^{k-m}.$$

Letting

$$s_m = -\frac{2(m-1)}{k+m} t_m = (-1)^m \binom{k}{m-2} 2^{k-m+1},$$

it can easily be verified that $t_m = s_{m+1} - s_m$. Note that $S(n) = \sum_{k=\lfloor n/2\rfloor}^{n-1} f(n,k)$, where $f(n,k) = (2/n) \sum_{m=1}^{n-k} t_m$ for $n > 0$ and $\lfloor n/2\rfloor \le k \le n-1$. Using $s_1 = 0$, we have

$$f(n,k) = \frac{2}{n}(s_{n-k+1} - s_1) = \frac{2s_{n-k+1}}{n} = \frac{(-1)^{n-k+1}}{n} \binom{k}{n-k-1} 2^{2k-n+1}.$$

Noting that $\binom{k}{n-k-1}$ is taken to be 0 unless $\lfloor n/2\rfloor \le k \le n-1$, it is natural to extend $f(n,k)$ by letting it be 0 unless $\lfloor n/2\rfloor \le k \le n-1$. Now

$$S(n) = \sum_{k=\lfloor n/2\rfloor}^{n-1} f(n,k) = \sum_k f(n,k),$$

where k ranges over all integers. Let

$$R(n,k) = \frac{(2k-n+1)(2k-n)}{2(n-k)(n+1)},$$

and put $g(n,k) = R(n,k) f(n,k)$. Direct manipulation yields

$$f(n+1,k) - f(n,k) = g(n,k+1) - g(n,k)$$

for all k and positive n. When summed over k, the right side telescopes to 0, so

$$S(n+1) - S(n) = \sum_k f(n+1,k) - \sum_k f(n,k) = 0.$$

Thus $S(n)$ is constant, and $S(1) = f(1,0) = 1$, as required.

Editorial comment. The factors $-2(m-1)/(k+m)$ and $R(n,k)$ come from Gosper's algorithm and the WZ algorithm, respectively (see M. Petkovšek, H. S. Wilf, and D. Zeilberger (1997), *A=B*, A K Peters). In particular, $R(n,k)$ is the certificate showing that (f, g) is a Wilf-Zeilberger pair, meaning that f and g satisfy the properties needed to ensure that the sum of f over k telescopes.

Also solved by J. Boswell & C. Curtis, P. Bracken, G. Fera (Italy), K. Gatesman, G. C. Greubel, D. Henderson, N. Hodges (UK), O. Kouba (Syria), O. P. Lossers (Netherlands), E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), and the proposer.

A Polygon Inequality

12250 [2021, 377]. *Proposed by Dorin Mărghidanu, Colegiul National A. I. Cuza, Corabia, Romania.* With $n \geq 4$, let $a_1, \ldots, a_n$ be the lengths of the sides of a polygon. Prove

$$\sqrt{\frac{a_1}{-a_1 + a_2 + \cdots + a_n}} + \sqrt{\frac{a_2}{a_1 - a_2 + \cdots + a_n}} + \cdots + \sqrt{\frac{a_n}{a_1 + a_2 + \cdots - a_n}} > \frac{2n}{n-1}.$$

Solution by UM6P Math Club, Mohammed VI Polytechnic University, Ben Guerir, Morocco. Since the left side is unaffected when the a_i are scaled by a constant factor, we may assume that the perimeter of the polygon is 1. Therefore, we need to show

$$\sum_{k=1}^{n} \sqrt{\frac{a_k}{1-2a_k}} > \frac{2n}{n-1}.$$

By the triangle inequality, each a_k belongs to the interval $(0, 1/2)$, so by the AM–GM inequality,

$$\sqrt{\frac{a_k}{1-2a_k}} = \sqrt{\frac{a_k^2}{a_k(1-2a_k)}} \geq \sqrt{\frac{a_k^2}{(1-a_k)^2/4}} = \frac{2a_k}{1-a_k}.$$

Note that this inequality is strict unless $a_k = 1/3$. Since $n \geq 4$, the inequality is strict for some k, and therefore it suffices to show

$$\sum_{k=1}^{n} \frac{2a_k}{1-a_k} \geq \frac{2n}{n-1}.$$

Let $g(x) = 2x/(1-x)$. Since g is convex on $(0, 1/2)$, by Jensen's inequality

$$\sum_{k=1}^{n} \frac{2a_k}{1-a_k} = n \cdot \frac{\sum_{k=1}^{n} g(a_k)}{n} \geq n \cdot g\left(\frac{\sum_{k=1}^{n} a_k}{n}\right) = n \cdot g(1/n) = \frac{2n}{n-1},$$

as required.

Solution II by Nigel Hodges, Cheltenham, UK. Denote the left side of the inequality by $T(a_1, \ldots, a_n)$. Since $n \geq 4$, we have $4(n-1) \geq 3n$, so $2n/(n-1) \leq 8/3 < 2\sqrt{2}$. We prove the stronger result $T(a_1, \ldots, a_n) \geq 2\sqrt{2}$.

As in the first solution above, we may assume $\sum_{j=1}^n a_j = 1$, and hence $0 < a_j < 1/2$ for all j. Set $a_j = (1/2)\sin^2\theta_j$ with $\theta_j \in (0, \pi/2)$. This yields

$$\sqrt{\frac{a_j}{-2a_j + \sum_{t=1}^n a_t}} = \sqrt{\frac{a_j}{1-2a_j}} = \frac{\tan\theta_j}{\sqrt{2}} = \frac{\sqrt{2}\sin^2\theta_j}{\sin(2\theta_j)} \geq \sqrt{2}\sin^2\theta_j = 2\sqrt{2}a_j.$$

Therefore

$$T(a_1, \ldots, a_n) = \sum_{j=1}^n \sqrt{\frac{a_j}{-2a_j + \sum_{t=1}^n a_t}} \geq 2\sqrt{2}\sum_{j=1}^n a_j = 2\sqrt{2}.$$

It is easy to see that this result is the best possible in that no larger constant can replace $2\sqrt{2}$. Set $a_1 = a_2 = a_3 = a_4 = 1$ and $a_j = \epsilon$ for $5 \leq j \leq n$, where ϵ is a small positive constant. We have $T(a_1, \ldots, a_n) = 2\sqrt{2} + O(\sqrt{\epsilon})$, and so $T(a_1, \ldots, a_n)$ can be made arbitrarily close to $2\sqrt{2}$ by choosing ϵ small enough.

Also solved by K. F. Andersen (Canada), M. Bataille (France), M. V. Channakeshava (India), H. Chen (China), H. Chen (US), C. Chiser (Romania), K. Gatesman, C. Geon (Korea), W. Janous (Austria), O. Kouba (Syria), S. S. Kumar, J. H. Lindsey II, O. P. Lossers (Netherlands), M. Lukarevski (North Macedonia), M. Omarjee (France), E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. Văcaru (Romania), F. Visescu (Romania), L. Zhou, Westchester Area Math Circle, and the proposer.

Forcing Monochromatic Convex Pentagons with Fixed Area

12251 [2021, 467]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Each point in the plane is colored either red or blue. Show that for any positive real number S, there is a proper convex pentagon of area S all five of whose vertices have the same color. (By a *proper* convex pentagon we mean a convex pentagon whose internal angles are less than π.)

Solution by Michael Tang, University of Washington, Seattle, WA. Denote the area of a polygon by placing brackets around a list of its vertices, and let XY denote both the segment with endpoints X and Y and its length. Let B and R be the sets of blue and red points, respectively. We begin with three observations that follow from assuming that the coloring yields no such pentagon.

(i) *Both B and R are unbounded.* If B is bounded, then we find five acceptable red vertices; similarly for bounded R.

(ii) *Both R and B are dense in the plane.* If R is not dense in the plane, then B contains a disk D of some radius r centered at some point O. Also, B contains a point P with $OP > 2S/r$. Choose $X, Y \in D$ such that XY contains O and $XY \perp OP$ and $OX = OY = (S-\epsilon)/OP$. Thus $OX = OY < r/2$ and $[PXY] = S - \epsilon$. We choose ϵ small enough to guarantee the existence of a chord WZ on the circumcircle of PXY close and parallel to XY (but farther from P than XY is) so that the isosceles trapezoid $XYZW$ has area ϵ. Now $XZWYP$ is a proper convex pentagon with area S.

(iii) *Every line segment contains points of both colors.* If segment X_1X_2 is all red, then we construct such a pentagon. Choose Y, W, Z, V so that $XYWZV$ is proper convex for all $X \in X_1X_2$ and $[X_1YZWV] < S < [X_2YZWV]$. Since R is dense in the plane, we may choose Y', W', Z', V' in R arbitrarily close to Y, W, Z, V preserving convexity and the inequality $[X_1Y'Z'W'V'] < S < [X_2Y'Z'W'V']$. By continuity of the area function, $[XYZWV] = S$ for some $X \in R$, and this is our desired pentagon.

Given $X, Y \in B$, take $Z \in B$ from a parallel line segment on one side of XY at a distance $(2S - 4\epsilon)/XY$ from it. Thus $[XYZ] = S - 2\epsilon$. From the other side of XY choose $W \in B$. For sufficiently small ϵ, we can chose W inside the circumcircle of XYZ so that

$[XYW] = \epsilon$. Similarly chose V inside the circumcircle of XYZ (but outside the triangle XYZ near the edge XZ) so that $[XVZ] = \epsilon$. Now $[XVZYW] = S$, and the construction guarantees that $XVZYW$ is proper convex.

Editorial comment. Most solvers constructed a class of monochromatic quadrilaterals and used casework to obtain a pentagon. The proposer started with a monochromatic rectangle (similar to Problem 8.5 of the 1991 Colorado Math Olympiad). Many extended the result to proper convex n-gons.

Also solved by J. Barát (Hungary), H. Chen (China), K. Gatesman, N. Hodges (UK), Y. J. Ionin, M. Reid, C. Schacht, R. Stong, and the proposer.

Some Floors and Ceilings

12252 [2021, 467]. *Proposed by Nguyen Quang Minh, Saint Joseph's Institution, Singapore.* Let k, q, and n be positive integers with $k \geq 2$, and let P be the set of positive integers less than k^n that are not divisible by k. Prove

$$\sum_{p\in P} \left\lceil \frac{\lfloor n - \log_k p \rfloor}{q} \right\rceil = \left\lfloor \frac{k^{q-1}(k^{n-1}-1)(k-1)}{k^q - 1} \right\rfloor + 1.$$

Solution by M. A. Prasad, Navi Mumbai, India. Write $\sum_{p\in P} \left\lceil \frac{\lfloor n - \log_k(p) \rfloor}{q} \right\rceil = T_1 - T_2$, where

$$\begin{aligned} T_1 = \sum_{0<p<k^n} \left\lceil \frac{\lfloor n - \log_k(p) \rfloor}{q} \right\rceil &= \left\lceil \frac{n}{q} \right\rceil + \sum_{j=0}^{n-1} \sum_{p=k^j+1}^{k^{j+1}} \left\lceil \frac{\lfloor n - \log_k(p) \rfloor}{q} \right\rceil \\ &= \left\lceil \frac{n}{q} \right\rceil + \sum_{j=0}^{n-1} k^j(k-1) \left\lceil \frac{n-j-1}{q} \right\rceil \end{aligned}$$

and

$$\begin{aligned} T_2 = \sum_{0<j\leq k^{n-1}} \left\lceil \frac{\lfloor n - \log_k(jk) \rfloor}{q} \right\rceil &= \sum_{0<j\leq k^{n-1}} \left\lceil \frac{\lfloor n - 1 - \log_k(j) \rfloor}{q} \right\rceil \\ &= \left\lceil \frac{n-1}{q} \right\rceil + \sum_{j=0}^{n-2} k^j(k-1) \left\lceil \frac{n-j-2}{q} \right\rceil. \end{aligned}$$

Combining these yields

$$T_1 - T_2 = \left\lceil \frac{n}{q} \right\rceil - \left\lceil \frac{n-1}{q} \right\rceil + \sum_{j=0}^{n-2} k^j(k-1) \left(\left\lceil \frac{n-j-1}{q} \right\rceil - \left\lceil \frac{n-j-2}{q} \right\rceil \right).$$

Let $n - 2 = \ell q + r$ with $0 \leq r < q$. If $r \neq q - 1$, then the only terms that contribute to the right side are those with $j \equiv r \pmod{q}$, so we obtain

$$\begin{aligned} T_1 - T_2 = \sum_{i=0}^{\ell} k^{iq+r}(k-1) &= \frac{(k-1)k^r(k^{(\ell+1)q} - 1)}{k^q - 1} \\ &= \frac{(k-1)k^{q-1}(k^{n-1}-1)}{k^q - 1} + \frac{(k-1)(k^{q-1} - k^r)}{k^q - 1}. \end{aligned}$$

Since $0 < (k-1)(k^{q-1} - k^r)/(k^q - 1) < 1$, the result follows. If $r = q - 1$, we similarly obtain

$$T_1 - T_2 = 1 + \sum_{i=0}^{\ell} k^{iq+r}(k-1) = 1 + \frac{(k-1)k^{q-1}(k^{n-1}-1)}{k^q - 1}.$$

Since the sum is an integer, the right side is an integer, and again we have the desired value.

Also solved by N. Hodges (UK), Y. J. Ionin, O. P. Lossers (Netherlands), K. Sarma (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), and the proposer.

An Arctangent Integral Solves a Summation

12254 [2021, 467]. *Proposed by Cezar Lupu, Texas Tech University, Lubbock, TX, and Tudorel Lupu, Constanţa, Romania.* Prove

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}\sum_{k=1}^{n}\frac{1}{n+k}\right)=\frac{3\pi}{8}\log 2-G,$$

where G is Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

Composite solution by Michel Bataille, Rouen, France, and Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let S denote the requested sum. We first compute

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{n+k}&=\sum_{k=1}^{2n+1}\frac{1}{k}-2\sum_{k=1}^{n}\frac{1}{2k}-\frac{1}{2n+1}=\sum_{k=1}^{2n+1}\frac{(-1)^{k-1}}{k}-\frac{1}{2n+1}\\&=\int_0^1\sum_{k=1}^{2n+1}(-x)^{k-1}\,dx-\frac{1}{2n+1}=\int_0^1\frac{1+x^{2n+1}}{1+x}\,dx-\frac{1}{2n+1}\\&=\int_0^1\frac{x^{2n+1}}{1+x}\,dx+\log 2-\frac{1}{2n+1}.\end{aligned}$$

It follows that

$$\begin{aligned}S&=\sum_{n=0}^{\infty}\int_0^1\frac{(-1)^n x^{2n+1}}{(2n+1)(1+x)}\,dx+\log 2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\\&=\sum_{n=0}^{\infty}\int_0^1\frac{(-1)^n x^{2n+1}}{(2n+1)(1+x)}\,dx+\frac{\pi}{4}\log 2-G.\end{aligned}\tag{$*$}$$

To evaluate the last sum, first note that

$$\begin{aligned}\sum_{n=0}^{\infty}\int_0^1\left|\frac{(-1)^n x^{2n+1}}{(2n+1)(1+x)}\right|dx&=\sum_{n=0}^{\infty}\int_0^1\frac{x^{2n+1}}{(2n+1)(1+x)}\,dx\\&\le\sum_{n=0}^{\infty}\int_0^1\frac{x^{2n+1}}{2n+1}\,dx=\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)}<\infty.\end{aligned}$$

Hence we can reverse the order of the summation and integration to obtain

$$\sum_{n=0}^{\infty}\int_0^1\frac{(-1)^n x^{2n+1}}{(2n+1)(1+x)}\,dx=\int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(1+x)}\,dx=\int_0^1\frac{\arctan x}{1+x}\,dx.$$

Using the change of variables $x=(1-t)/(1+t)$ and the fact that for $0\le t\le 1$, $\arctan((1-t)/(1+t))=\pi/4-\arctan t$ we get

$$\int_0^1\frac{\arctan x}{1+x}\,dx=\int_0^1\frac{\pi/4-\arctan t}{1+t}\,dt,$$

and therefore

$$2\int_0^1\frac{\arctan x}{1+x}=\frac{\pi}{4}\int_0^1\frac{dt}{1+t}=\frac{\pi}{4}\log 2.$$

We conclude that the sum in $(*)$ equals $(\pi/8)\log 2$, and therefore $S = (3\pi/8)\log 2 - G$, as required.

Also solved by A. Berkane (Algeria), N. Bhandari (Nepal), P. Bracken, B. Bradie, A. C. Castrillón (Colombia), H. Chen, B. E. Davis, G. Fera (Italy), M. L. Glasser, R. Gordon, H. Grandmontagne (France), G. C. Greubel, N. Grivaux (France), N. Hodges (UK), L. Kempeneers & J. Van Casteren (Belgium), O. P. Lossers (Netherlands), J. R. McCrorie (Scotland), M. Omarjee (France), D. Pinchon (France), M. A. Prasad (India), J. Song (China), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, M. Wildon (UK), FAU Problem Solving Group, and the proposer.

CLASSICS

C12. *Due to Lionel Penrose and Roger Penrose; suggested by the editors.* Is there a plane region bounded by a differentiable Jordan curve with the property that no matter where a light source is placed inside it, some part of the region remains unilluminated? Assume that the curve acts as a perfect mirror.

Guessing When a Playing Card is Red

C11. *Suggested by Richard Stanley, University of Miami, Coral Gables, FL.* A standard deck of cards has 26 red cards and 26 black cards. Deal out the cards in a shuffled standard deck, one card at a time. At any point before the last card is dealt, you can guess that the next card is red. For example, you may guess that the very first card is red, and your guess will be correct with probability $1/2$. Or you may watch some cards go by, noting their color in order to decide when to guess. What strategy maximizes the probability that your guess is correct?

Solution I. It is not possible to improve on 1/2. In fact, all stopping strategies have success probability exactly 1/2. To see this, compare the game to a variant in which, after the guess is made, the revealed card is the bottom card in the deck rather than the next card. When any strategy is applied to this variant, the chance of success is clearly 1/2, since the bottom card in a shuffled deck is red with probability 1/2. The key observation is that, no matter when the guess is made, the next card has the same probability of being red as does the bottom card. The probabilities are $r/(r+b)$, where r and b are the number of red cards and the number of black cards, respectively, in the deck following the specified position. Since these probabilities determine the probability of success, the original game and the variant have the same probability of success, independent of the strategy that is applied.

Solution II. We use induction on the size of the deck, proving the more general result that any strategy wins with probability $r/(b+r)$ when the deck starts with r red cards and b black cards. If you guess that the first card is red, your probability of success is $r/(b+r)$. If you don't, then consider two cases depending on the color of the first card. With probability $b/(b+r)$, the first card is black, and you are facing $b-1$ black cards and r red cards in the remaining deck. With probability $r/(b+r)$, the first card is red, and you are facing b black cards and $r-1$ red cards. By the induction hypothesis, the probability of success, independent of how the strategy continues, is

$$\frac{b}{b+r}\frac{r}{b+r-1} + \frac{r}{b+r}\frac{r-1}{b+r-1},$$

which equals $r/(b+r)$.

Editorial comment. The problem is folklore, and appears on p. 67 of P. Winkler (2003), *Mathematical Puzzles, A Connoisseur's Collection*, A K Peters/CRC Press.

SOLUTIONS

Golden Eigenvalues of Special Matrices

12240 [2021, 276]. *Proposed by Yue Liu, Fuzhou University, Fuzhou, China, and Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, FL.* We denote by A^* the conjugate transpose of the matrix A.

(a) Let $x \in \mathbb{C}^m$ be a unit column vector. Find the eigenvalues of the $(m+1)$-by-$(m+1)$ matrices

$$\begin{bmatrix} x^*x & x^* \\ x & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} xx^* & x \\ x^* & 0 \end{bmatrix}.$$

(b) More generally, let X be an m-by-n complex matrix, and let ρ be any real number. Find the eigenvalues of the $(m+n)$-by-$(m+n)$ matrices

$$\begin{bmatrix} X^*X & X^* \\ X & \rho I_m \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} XX^* & X \\ X^* & \rho I_n \end{bmatrix}.$$

Solution to part **(a)** *by Jean-Pierre Grivaux, Paris, France.* Let M and N be the two specified matrices. Since x is a unit vector, $x^*x = 1$. The rank of M is two. Thus it has two nonzero eigenvalues λ_1 and λ_2, plus 0 with multiplicity $m-1$. Note $\lambda_1 + \lambda_2 = \mathrm{tr}(M) = 1$. We calculate M^2:

$$M^2 = \begin{bmatrix} 2 & x^* \\ x & xx^* \end{bmatrix}.$$

With the entries of x indexed as $x_1, \dots, x_m$, the m-by-m matrix xx^* has diagonal entries $|x_1|^2, \dots, |x_m|^2$. Thus $\operatorname{tr}(M^2) = 2 + \sum |x_i|^2 = 3$, so $\lambda_1^2 + \lambda_2^2 = 3$. Substituting $\lambda_2 = 1 - \lambda_1$ yields a quadratic equation, and we obtain $\{\lambda_1, \lambda_2\} = \{(1-\sqrt{5})/2, (1+\sqrt{5})/2\}$.

The argument for N is similar; it also has rank 2 and trace 1. Now

$$N^2 = \begin{bmatrix} 2xx^* & xx^*x \\ x^*xx^* & 1 \end{bmatrix},$$

so $\operatorname{tr}(N^2) = 3$. Again the two nonzero eigenvalues are $(1-\sqrt{5})/2$ and $(1+\sqrt{5})/2$.

Solution to part **(b)** *by Kuldeep Sarma, Tezpur University, Tezpur, India.* Again let M and N be the two specified matrices. We use the singular value decomposition (SVD). The SVD factors the m-by-n complex matrix X as $U\Sigma V^*$, where U is an m-by-m complex unitary matrix, V is an n-by-n complex unitary matrix, and Σ is an m-by-n rectangular diagonal matrix with nonnegative real numbers $\sigma_1, \dots, \sigma_s$ on the diagonal, where $s = \min\{m, n\}$. We can then write

$$M = \begin{bmatrix} V\Sigma^*\Sigma V^* & V\Sigma^* U^* \\ U\Sigma V^* & U[\rho I_m]U^* \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \Sigma^*\Sigma & \Sigma^* \\ \Sigma & \rho I_m \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}.$$

Since multiplication by a unitary matrix does not change eigenvalues, it suffices to find the eigenvalues of the matrix S given by

$$S = \begin{bmatrix} \Sigma^*\Sigma & \Sigma^* \\ \Sigma & \rho I_m \end{bmatrix}.$$

We consider a simultaneous permutation of the rows and columns of S, which does not change the eigenvalues. Since Σ is nonzero only on its diagonal, many entries in S are 0. Index the first n rows (and columns) of S as 1 through n, and index the last m rows (and columns) as $1'$ through m'. Let $s = \min\{m, n\}$. Reorder the rows (and columns) in the order $(1, 1', 2, 2', \dots, s, s')$, followed by the remaining $m + n - 2s$ rows (and columns) in their original order. This converts S to a block-diagonal matrix S' in which the ith block, for $1 \le i \le s$, is the 2-by-2 matrix

$$\begin{bmatrix} \sigma_i^2 & \sigma_i \\ \sigma_i & \rho \end{bmatrix},$$

and the final $m + n - 2s$ blocks are 1-by-1 blocks that are all $[\rho]$ if $m > n$ and are all $[0]$ if $m < n$ (there are none of these 1-by-1 blocks if $m = n$). Note that $m + n - 2s = |m - n|$.

The eigenvalues are the eigenvalues of the blocks: 0 or ρ with the stated multiplicity $|m - n|$, plus

$$\frac{\rho + \sigma_i^2 \pm \sqrt{\left(\rho - \sigma_i^2\right)^2 + 4\sigma_i^2}}{2}$$

from the block for σ_i, where $1 \le i \le s$. Note that if $\sigma_i = 0$, then the block for σ_i reduces to two extra 1-by-1 blocks $[0]$ and $[\rho]$, but this is in fact described by the formula given above for the eigenvalues of the block for σ_i.

The matrix N is generated in the same way as the matrix M, using X^* instead of X. It follows that the spectrum of N is the same as the spectrum of M, except that the multiplicities of 0 and ρ generated by the 1-by-1 blocks are, respectively, $\max\{m - n, 0\}$ and $\max\{n - m, 0\}$, obtained by interchanging the roles of m and n.

Also solved by D. Fleischman, K. Gatesman, L. Han (US) & X. Tang (China), E. A. Herman, C. P. A. Kumar (India), O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Stong, E. I. Verriest, T. Wiandt, and the proposer.

An Integral Limit for This Year—Or, As It Turns Out, Any Year

12242 [2021, 277]. *Proposed by Elena Corobea, Technical College Carol I, Constanţa, Romania.* For $n \geq 1$, let

$$I_n = \int_0^1 \frac{\left(\sum_{k=0}^{n} x^k/(2k+1)\right)^{2022}}{\left(\sum_{k=0}^{n+1} x^k/(2k+1)\right)^{2021}}\,dx.$$

Let $L = \lim_{n\to\infty} I_n$. Compute L and $\lim_{n\to\infty} n(I_n - L)$.

Solution by Kyle Gatesman (student), Johns Hopkins University, Baltimore, MD. We show that $L = 2\ln 2$ and $\lim_{n\to\infty} n(I_n - L) = -1/2$.

For integers $n \geq 1$ and $p \geq 0$, let

$$S_n(x) = \sum_{k=0}^{n} \frac{x^k}{2k+1} \qquad \text{and} \qquad I_n(p) = \int_0^1 \frac{(S_n(x))^{p+1}}{(S_{n+1}(x))^p}\,dx.$$

For $p \geq 1$,

$$\begin{aligned} I_n(p) &= \int_0^1 \frac{(S_n(x))^p}{(S_{n+1}(x))^{p-1}} \cdot \frac{S_n(x)}{S_{n+1}(x)}\,dx \\ &= \int_0^1 \frac{(S_n(x))^p}{(S_{n+1}(x))^{p-1}} \cdot \left(1 - \frac{x^{n+1}}{(2n+3)S_{n+1}(x)}\right) dx \\ &= I_n(p-1) - \int_0^1 \left(\frac{S_n(x)}{S_{n+1}(x)}\right)^p \cdot \frac{x^{n+1}}{2n+3}\,dx. \end{aligned}$$

For $x \in [0,1]$, we have

$$0 \leq \left(\frac{S_n(x)}{S_{n+1}(x)}\right)^p \cdot \frac{x^{n+1}}{2n+3} \leq \frac{x^{n+1}}{2n+3},$$

so

$$0 \leq I_n(p-1) - I_n(p) \leq \int_0^1 \frac{x^{n+1}}{2n+3}\,dx = \frac{1}{(n+2)(2n+3)}.$$

Therefore $\lim_{n\to\infty}(I_n(p-1) - I_n(p)) = 0$, and by a straightforward induction on p we conclude that $\lim_{n\to\infty}(I_n(0) - I_n(p)) = 0$ for all $p \in \mathbb{Z}^+$. Moreover, for any constant $c \in \mathbb{R}$,

$$0 \leq n(I_n(p-1) - c) - n(I_n(p) - c) \leq \frac{n}{(n+2)(2n+3)},$$

and so $\lim\limits_{n\to\infty}(n(I_n(p-1) - c) - n(I_n(p) - c)) = \lim\limits_{n\to\infty}(n(I_n(0) - c) - n(I_n(p) - c)) = 0$.

Because

$$I_n(0) = \int_0^1 S_n(x)\,dx = \int_0^1 \sum_{k=0}^{n} \frac{x^k}{2k+1}\,dx = \sum_{k=0}^{n} \frac{1}{(k+1)(2k+1)},$$

we conclude

$$\begin{aligned} \lim_{n\to\infty} I_n(p) &= \lim_{n\to\infty} I_n(0) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{(k+1)(2k+1)} = \sum_{k=0}^{\infty} \frac{1}{(k+1)(2k+1)} \\ &= 2\sum_{k=0}^{\infty} \frac{1}{(2k+2)(2k+1)} = 2\sum_{k=0}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 2\ln 2. \end{aligned}$$

In particular, in the case $p = 2021$, we obtain $L = 2\ln 2$.

Similarly, observe that

$$\begin{aligned}
\lim_{n\to\infty} n(I_n(p) - L) &= \lim_{n\to\infty} n(I_n(0) - L) \\
&= \lim_{n\to\infty} n\left(\sum_{k=0}^{n}\frac{1}{(k+1)(2k+1)} - \sum_{k=0}^{\infty}\frac{1}{(k+1)(2k+1)}\right) \\
&= \lim_{n\to\infty} n\left(-\sum_{k=n+1}^{\infty}\frac{1}{(k+1)(2k+1)}\right).
\end{aligned}$$

For every $n \in \mathbb{Z}^+$ we have

$$\sum_{k=n+1}^{\infty}\frac{1}{(k+1)(2k+4)} \le \sum_{k=n+1}^{\infty}\frac{1}{(k+1)(2k+1)} \le \sum_{k=n+1}^{\infty}\frac{1}{(k+1)2k}.$$

Since

$$\sum_{k=n+1}^{\infty}\frac{1}{(k+1)(2k+4)} = \frac{1}{2}\sum_{k=n+1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2(n+2)}$$

and

$$\sum_{k=n+1}^{\infty}\frac{1}{(k+1)2k} = \frac{1}{2}\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2(n+1)},$$

we conclude

$$-\frac{n}{2(n+1)} \le n\left(-\sum_{k=n+1}^{\infty}\frac{1}{(k+1)(2k+1)}\right) \le -\frac{n}{2(n+2)}.$$

Thus, by the squeeze theorem,

$$\lim_{n\to\infty} n(I_n(p) - L) = \lim_{n\to\infty} n\left(-\sum_{k=n+1}^{\infty}\frac{1}{(k+1)(2k+1)}\right) = -\frac{1}{2},$$

and setting $p = 2021$ completes the solution of the stated problem.

Editorial comment. The solution shows that the answers are the same if 2021 and 2022 are replaced by p and $p+1$ for any nonnegative integer p. Indeed, since $I_n(p)$ is a decreasing function of p, the answers are the same if 2021 and 2022 are replaced by x and $x+1$ for any nonnegative real number x.

Also solved by K. F. Andersen (Canada), P. Bracken, H. Chen, G. Fera (Italy), D. Fleischman, L. Han (USA) & X. Tang (China), E. A. Herman, N. Hodges (UK), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, J. Yan (China), and the proposer.

A Hyperbolic Integral

12243 [2021, 277]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* For $a > 0$, evaluate

$$\int_0^a \frac{t}{\sinh t\sqrt{1 - \operatorname{csch}^2 a \cdot \sinh^2 t}}\,dt.$$

Solution by Kuldeep Sarma, Tezpur University, Tezpur, India. Let $I(a)$ be the desired value. First, we observe that

$$1 - \operatorname{csch}^2 a \sinh^2 t = \cosh^2 t(1 - \coth^2 a \tanh^2 t).$$

Using this, we obtain

$$I(a) = \int_0^a \frac{t\,dt}{\sinh t\sqrt{1-\operatorname{csch}^2 a\cdot\sinh^2 t}} = \int_0^a \frac{t\,dt}{\sinh t\cosh t\sqrt{1-\coth^2 a\cdot\tanh^2 t}}.$$

Now using the substitution $\cos x = \coth a\tanh t$, we have

$$I(a) = \int_0^{\pi/2} \frac{\tanh^{-1}(\tanh a\cos x)}{\cos x}\,dx$$

and hence

$$I'(a) = \int_0^{\pi/2} \frac{\operatorname{sech}^2 a}{1-\tanh^2 a\cos^2 x}\,dx = \operatorname{sech} a\tan^{-1}(\cosh a\tan x)\Big|_0^{\pi/2} = \frac{\pi}{2}\operatorname{sech} a.$$

Thus

$$I(a) = \int_0^a I'(s)\,ds = \frac{\pi}{2}\int_0^a \operatorname{sech} s\,ds = \frac{\pi}{2}\tan^{-1}(\sinh a).$$

Editorial comment. Several solvers noted that the requested integral can be reduced to integral (3.535) from I. S. Gradshteyn, I. M. Ryzhik, et al. (2014), *Table of Integrals, Series, and Products*, 8th edition, Cambridge, MA: Academic Press.

Also solved by U. Abel & V. Kushnirevych (Germany), P. Bracken, H. Chen, G. Fera (Italy), L. Han (US) & X. Tang (China), N. Hodges (UK), O. P. Lossers (Netherlands), T. M. Mazzoli (Austria), M. Omarjee (France), A. Stadler (Switzerland), S. M. Stewart (Saudi Arabia), R. Tauraso (Italy), UM6P Math Club (Morocco), and the proposer.

Equitable Polyominos in a Box

12244 [2021, 376]. *Proposed by Rob Pratt, SAS Institute Inc., Cary, NC, Stan Wagon, Macalester College, St. Paul, MN, Douglas B. West, University of Illinois, Urbana, IL, and Piotr Zielinski, Cambridge, MA.* A *polyomino* is a region in the plane with connected interior that is the union of a finite number of squares from a grid of unit squares. For which integers k and n with $4 \le k \le n$ does there exist a polyomino P contained entirely within an n-by-n grid such that P contains exactly k unit squares in every row and every column of the grid? Clearly such polyominos do not exist when $k = 1$ and $n \ge 2$. Nikolai Beluhov noticed that they do not exist when $k = 2$ and $n \ge 3$, and his Problem 12137 [2019, 756; 2021, 381] shows that they do not exist when $k = 3$ and $n \ge 5$.

Solution by Jacob Boswell, Missouri Southern State University, Joplin, MO. Polyominos with the desired properties, which we call (k, n)-*equitable polyominos*, exist whenever $4 \le k \le n$.

Denote the n-by-n grid by $\mathcal{G}_n$. We call its unit squares *cells* and specify their positions in matrix notation. We call the three cells (1,1), (1,2), and (2,1) the top left *guard*. Similarly, we define top right, bottom left, and bottom right guards.

We argue by induction on k that in $\mathcal{G}_n$ there is a (k, n)-equitable polynomino that contains two diagonally opposite guards such that removing the corner square from one of those guards leaves the remainder connected. Let $\mathcal{C}_{k,n}$ denote the class of such polyominos. We postpone the discussion of the base cases.

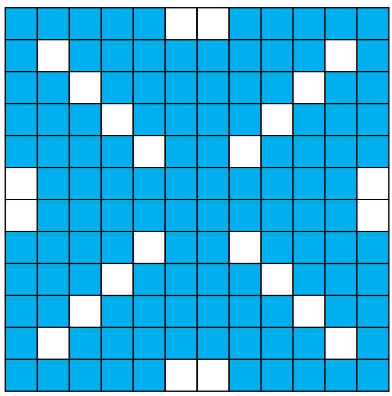
For the induction step, consider (k, n) with $n \ge k \ge 9$. Cover $\mathcal{G}_n$ using two diagonally opposite copies of $\mathcal{G}_{\lceil n/2\rceil}$ and two diagonally opposite copies of $\mathcal{G}_{\lfloor n/2\rfloor}$. When n is odd, the two larger subgrids share one cell in the center, but other than that the subgrids share no cells.

We describe a uniform construction for all cases except when n is odd and k is even. In the two opposite copies of $\mathcal{G}_{\lceil n/2\rceil}$, place members of $\mathcal{C}_{\lceil k/2\rceil,\lceil n/2\rceil}$, with one of the guards that are inductively guaranteed to exist placed in the center of $\mathcal{G}_n$. In the two opposite copies of $\mathcal{G}_{\lfloor n/2 \rfloor}$, similarly place members of $\mathcal{C}_{\lfloor k/2 \rfloor, \lfloor n/2 \rfloor}$ with their guaranteed guards in the center of $\mathcal{G}_n$.

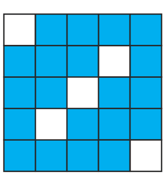
When n is odd and k is even, use members of $\mathcal{C}_{k/2+1, \lceil n/2 \rceil}$ in the larger subgrids and $\mathcal{C}_{k/2-1, \lfloor n/2 \rfloor}$ in the smaller subgrids, and (in this case) delete the central cell from the resulting polyomino. The use of $\mathcal{C}_{k/2-1, \lfloor n/2 \rfloor}$ here is the reason we need $k = 8$ in the basis.

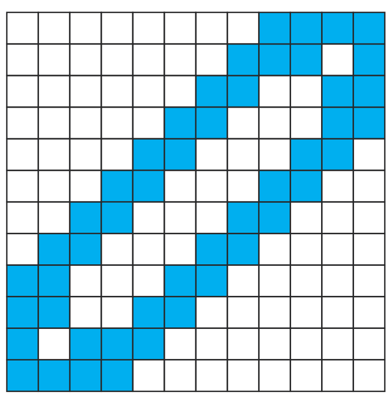
In each case, the guards from each subpolyomino retain a cell adjacent to a cell retained from the guard in a neighboring subpolyomino, so the resulting full polyomino is connected. The polyomino also retains diagonally opposite complete guards, and deleting the corner cell from one of those guards does not disconnect the polyomino, because it does not disconnect the subpolyomino (even when the central cell is deleted, the two neighbors of the central cell are connected through the other subpolyominos).

When n is even, the number of cells in each row and column of the final polyomino is $\lceil k/2 \rceil + \lfloor k/2 \rfloor$. When n is odd and k is odd, the computation is the same except for the central row and column, where it is $\lceil k/2 \rceil + \lceil k/2 \rceil - 1$ as desired, since the central cell contributes only once. When n is odd and k is even, we have $k/2 + 1 + k/2 - 1$ cells in each noncentral row and column, and in the central row and column we have $k/2 + 1 + k/2 + 1 - 2$ cells, since the central cell was deleted. (Keeping the larger subgrid connected in this case is the reason for the special condition on the subgrid.) Below we show the construction of a member of $\mathcal{C}_{10,12}$ from four members of $\mathcal{C}_{5,6}$.



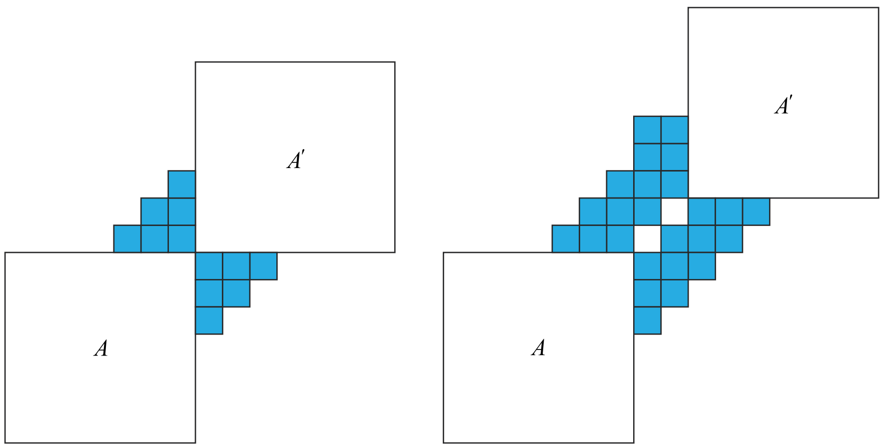
Now we return to the base cases. Because the induction step for k needs the induction hypothesis for $\lfloor (k-1)/2 \rfloor$ and (k, n)-equitable polyominos do not generally exist when $k \le 3$, we need base cases for $4 \le k \le 8$. Below we show members of $\mathcal{C}_{4,5}$ and $\mathcal{C}_{4,12}$. The general construction shown for $(k, n) = (4, 12)$ is valid when $n \ge 6$, which completes the proof for $k = 4$.



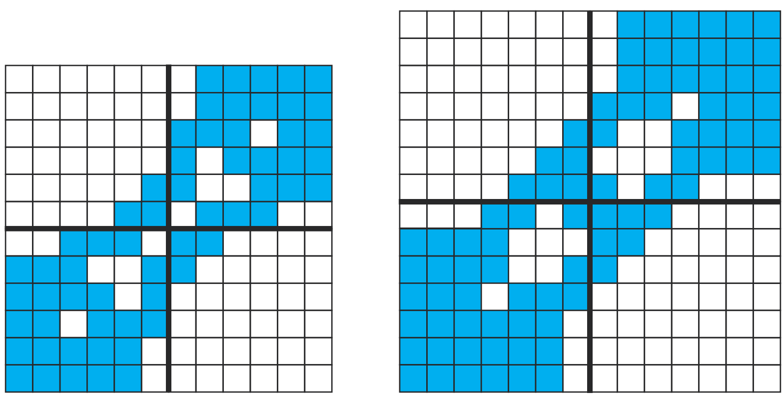


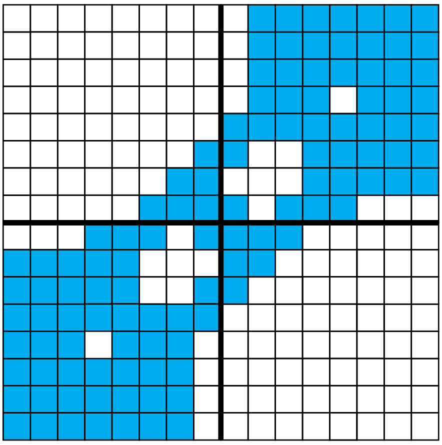
For $k \geq 5$, we show first that a special construction for $n = 2k + 2$ yields constructions for all larger n. Say that a member of $\mathcal{C}_{k,2k+2}$ is a *butterfly* if its portion in the upper left and lower right quadrants consists precisely of triangular arrays of cells with side-length $\lfloor k/2 \rfloor$ touching the center of $\mathcal{G}_{2k+2}$, as indicated on the left below. Suppose that $\mathcal{C}_{k,2k+2}$ contains a butterfly B_k. Note that the polyomino A' in the upper right quadrant of B_k can be assumed to be the transpose of A.

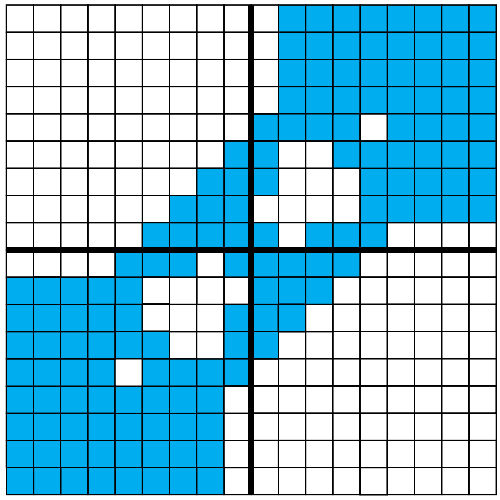
From B_k one can obtain a member of $\mathcal{C}_{k,n}$ whenever $n > 2k + 2$ by enlarging the central portion of the butterfly and spreading A and A' farther apart, as shown on the right below. When k is even, the central diagonal of the added portion is omitted, but when k is odd it is present. The correct counts in the rows and columns occupied by A and A' are inherited from B_k.



Below we show butterflies for $5 \leq k \leq 8$. One issue in these constructions is ensuring that the polyomino is connected; this is the reason we provided a different construction for $k = 4$.







At this point the proof is completed by exhibiting explicit examples for $k \le n \le 2k+1$ when $5 \le k \le 8$. General constructions for $n = k$ and $n = k+1$ are trivial. What remains is a finite problem, exhibiting 26 polynominos. We leave the constructions to the reader.

Editorial comment. The constructions are far from unique. For example, there is a construction similar to the butterfly that exists when $n = 2k$ and expands like the butterfly, reducing the finite problem to 18 polyominos.

Also solved by K. Gatesman, R. Stong, and the proposer.

CLASSICS

C11. *Suggested by Richard Stanley, University of Miami, Coral Gables, FL.* A standard deck of cards has 26 red cards and 26 black cards. Deal out the cards in a shuffled standard deck, one card at a time. At any point before the last card is dealt, you can guess that the next card is red. For example, you may guess that the very first card is red, and your guess will be correct with probability $1/2$. Or you may watch some cards go by, noting their color in order to decide when to guess. What strategy maximizes the probability that your guess is correct?

Repetitions in the Interior of Pascal's Triangle

C10. *Due to Douglas Lind, suggested by the editors.* Show that there are infinitely many numbers that appear at least six times in Pascal's triangle.

Solution. For $m \ge 3$, m occurs twice as $\binom{m}{1}$ and $\binom{m}{m-1}$. By symmetry, it will suffice to find infinitely many values of m with at least two more occurrences in the left half of the triangle.

There are several small examples of such pairs of occurrences: $120 = \binom{10}{3} = \binom{16}{2}$, $210 = \binom{10}{4} = \binom{21}{2}$, $1540 = \binom{22}{3} = \binom{56}{2}$, and $3003 = \binom{15}{5} = \binom{14}{6}$. The last of these exhibits the intriguing relationship $\binom{n}{k} = \binom{n-1}{k+1}$. To solve the problem, we will find infinitely many solutions of this equation with $k > 1$ and $k + 1 < (n-1)/2$.

The equation $\binom{n}{k} = \binom{n-1}{k+1}$ is equivalent to $n(k+1) - (n-k)(n-k-1) = 0$. We claim that for every positive integer j, this equation is satisfied by the values $n = F_{2j+2}F_{2j+3}$

and $k = F_{2j}F_{2j+3}$, where F_i is the ith Fibonacci number. To see why, note that with these values we have $n - k = (F_{2j+2} - F_{2j})F_{2j+3} = F_{2j+1}F_{2j+3}$, and therefore

$$\begin{aligned} n(k+1) - (n-k)(n-k-1) &= F_{2j+2}F_{2j+3}(F_{2j}F_{2j+3} + 1) - F_{2j+1}F_{2j+3}(F_{2j+1}F_{2j+3} - 1) \\ &= F_{2j+3}(F_{2j+2}F_{2j}F_{2j+3} + F_{2j+2} - F_{2j+1}^2F_{2j+3} + F_{2j+1}) \\ &= F_{2j+3}(F_{2j+2}F_{2j}F_{2j+3} - F_{2j+1}^2F_{2j+3} + F_{2j+3}) \\ &= F_{2j+3}^2(F_{2j+2}F_{2j} - F_{2j+1}^2 + 1) = 0, \end{aligned}$$

where the last step uses the well-known identity $F_{i+1}F_{i-1} - F_i^2 = (-1)^i$.

The case $j = 1$ yields $n = 15$ and $k = 5$, the example we found earlier. When $j = 2$ we get $n = 104$ and $k = 39$, and indeed $\binom{104}{39} = \binom{103}{40} = 61218182743304701891431482520$.

Editorial comments. The appearance of the Fibonacci numbers in this solution can be explained by reference to classic problem C2 (this MONTHLY, Feb. 2022, p. 194). Viewing the equation $n(k+1) - (n-k)(n-k-1) = 0$ as a quadratic in n and applying the quadratic formula yields

$$n = \frac{3k + 2 \pm \sqrt{5k^2 + 8k + 4}}{2}.$$

For n to be an integer, we need $5k^2 + 8k + 4$ to be a perfect square. Setting $5k^2 + 8k + 4 = t^2$ and solving for k by the quadratic formula, we get

$$k = \frac{-4 \pm \sqrt{5t^2 - 4}}{5}.$$

For k to be an integer, $5t^2 - 4$ must be a perfect square, and the solution to classic problem C2 (March 2022, pp. 293–294) shows that this happens if and only if t is an odd-indexed Fibonacci number. Setting $t = F_{2i+1}$ and applying Fibonacci identities leads to the values

$$n = F_{i+1}F_{i+2} + \frac{(-1)^{i+1} - 1}{5}, \quad k = F_{i-1}F_{i+2} + \frac{4((-1)^{i+1} - 1)}{5}.$$

These are integers when i is odd, and setting $i = 2j + 1$ leads to the values used in the solution.

This result is due to Lind (D. Lind, The quadratic field $Q(\sqrt{5})$ and a certain Diophantine equation, *Fib. Quart.* **6** (1968) 86–94, fq.math.ca/Scanned/6-3/lind.pdf). See also C. A. Tovey, Multiple occurrences of binomial coefficients, *Fib. Quart.* **23** (1985) 356–358. It is related to a 1971 conjecture of Singmaster (D. Singmaster, How often does an integer occur as a binomial coefficient?, this MONTHLY **78** (1971) 385–386). For an integer m with $m \geq 2$, let S_m be the number of times m appears in Pascal's triangle. Singmaster conjectured that S_m is bounded, and suggested that 10 or 12 might be a bound. The problem shows that 5 cannot be an asymptotic bound. It turns out that $S_{3003} = 8$; there are no other known values of m for which $S_m \geq 8$. The sequence of binomial coefficients for which $S_m \geq 6$ starts 120, 210, 1540, 3003, 7140, 11628, 24310, 61218182743304701891431482520 (see the OEIS sequences: oeis.org/A003015, oeis.org/A003016, and oeis.org/A090162). See also K. Matomäki, M. Radziwiłł, X. Shao, T. Tao, and J. Teräväinen, Singmaster's conjecture in the interior of Pascal's triangle, arxiv.org/abs/2106.03335.

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SOLUTIONS

Counting Sets Without Consecutive Elements

12233 [2021, 178]. *Proposed by C. R. Pranesachar, Indian Institute of Science, Bengaluru, India.* Let n and k be positive integers with $1 \le k \le (n+1)/2$. For $1 \le r \le n$, let $h(r)$ be the number of k-element subsets of $\{1, \dots, n\}$ that do not contain consecutive elements but that do contain r. For example, with $n = 7$ and $k = 3$, the string $h(1), \dots, h(7)$ is 6, 3, 4, 4, 4, 3, 6. Prove
(a) $h(r) = h(r+1)$ when $r \in \{k, \dots, n-k\}$.
(b) $h(k-1) = h(k) \pm 1$.
(c) $h(r) > h(r+2)$ when $r \in \{1, \dots, k-2\}$ and r is odd.
(d) $h(r) < h(r+2)$ when $r \in \{1, \dots, k-2\}$ and r is even.

Composite solution by Kyle Gatesman, Johns Hopkins University, Baltimore, MD, and Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy. The problem statement requires correction in parts **(c)** and **(d)**, where in the special case $k = (n+1)/2$ we have $h(r) = h(r+2)$ for all r.

For a proof by induction, we make the dependence on n and k explicit. Let $h_{n,k}(r) = h(r)$, and extend the definition to give 0 when n, k, or r is outside its natural domain. For $1 \le r \le n-1$, partition the k-element subsets containing r by whether they contain n, obtaining

$$h_{n,k}(r) = h_{n-1,k}(r) + h_{n-2,k-1}(r). \tag{1}$$

Similarly, for $1 < r \le n$, partition the k-element subsets containing r by whether they contain 1. After shifting indices to start at 2 or 3, this yields

$$h_{n,k}(r) = h_{n-1,k}(r-1) + h_{n-2,k-1}(r-2). \tag{2}$$

(a) We use induction on n. Note that $h_{n,1}(r) = 1$ for all r and n, from which **(a)** follows for $k = 1$, including all cases with $n \le 3$. Now suppose $n > 3$ and $k > 1$. By symmetry,

$h_{n,k}(r) = h_{n,k}(n+1-r)$, so we need only consider $k \le r \le (n-1)/2$. In that case, $r \le (n-1)-k = (n-2)-(k-1)$. Now (1) and the induction hypothesis imply

$$h_{n,k}(r) = h_{n-1,k}(r) + h_{n-2,k-1}(r) = h_{n-1,k}(r+1) + h_{n-2,k-1}(r+1) = h_{n,k}(r+1).$$

(b) We use induction on k to prove that $h_{n,k}(k-1) - h_{n,k}(k) = (-1)^k$, for all positive integers n beginning with $h_{n,1}(0) = 0$ and $h_{n,1}(1) = 1$. By (1) and (2),

$$\begin{aligned} h_{n,k}(r) - h_{n,k}(r+1) &= \big(h_{n-1,k}(r) + h_{n-2,k-1}(r)\big) - \big(h_{n-1,k}(r) + h_{n-2,k-1}(r-1)\big) \\ &= -\big(h_{n-2,k-1}(r-1) - h_{n-2,k-1}(r)\big). \end{aligned} \tag{3}$$

With $r = k-1 \le ((n-2)+1)/2$, the induction hypothesis completes the proof.

(c, d) We use induction on r. The number of k-element subsets of $\{1, \dots, n\}$ having no consecutive elements is $\binom{n-k+1}{k}$, corresponding to insertions of k balls in distinct positions between or outside $n-k$ markers in a row. Thus $h_{n,k}(1) = \binom{n-k}{k-1}$, $h_{n,k}(2) = \binom{n-k-1}{k-1}$, and, by (2), $h_{n,k}(3) = \binom{n-k-2}{k-1} + \binom{n-k-1}{k-2}$. Using Pascal's formula for binomial coefficients twice, $h_{n,k}(1) - h_{n,k}(3) = \binom{n-k-2}{k-2}$. Thus $h_{n,k}(1) - h_{n,k}(3) > 0$ unless $k = (n+1)/2$, in which case the difference is 0. This completes the proof for $r = 1$.

Now suppose $r \ge 2$. If $k = (n+1)/2$, then n is odd, and $h_{n,k}(r)$ is 1 when r is odd and 0 when r is even, so the desired difference is 0. Hence we may restrict our attention to $k \le n/2$, which yields $k-1 \le (n-3+1)/2$. Using (1) and (2), then (3), and finally (1) and (2) again, we find

$$\begin{aligned} h_{n,k}(r) - h_{n,k}(r+2) &= h_{n-1,k}(r) + h_{n-2,k-1}(r) - h_{n-1,k}(r+1) - h_{n-2,k-1}(r) \\ &= -\big(h_{n-3,k-1}(r-1) - h_{n-3,k-1}(r)\big) \\ &= -\big(h_{n-2,k-1}(r-1) - h_{n-2,k-1}(r+1)\big). \end{aligned}$$

Now the induction hypothesis completes the proof.

Editorial comment. Nigel Hodges conditioned on the number j of selected elements preceding r to prove

$$h(r) = \sum_{j=0}^{k-1} \binom{r-1-j}{j}\binom{n-r-k+1+j}{k-1-j}.$$

He then used induction and Pascal's formula to prove for $r \le n-k+1$ that this expression equals $\sum_{j=0}^{r-1}(-1)^j\binom{n-k-j}{k-1-j}$, from which **(a)–(d)** all follow quickly.

Also solved by H. Chen (China), C. Curtis & J. Boswell, N. Hodges (UK), Y. J. Ionin, O. P. Lossers (Netherlands), L. J. Peterson, R. Stong, and the proposer.

A Congruence for a Product of Quadratic Forms

12234 [2021, 179]. *Proposed by Nicolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Let p be an odd prime, and let $Ax^2 + Bxy + Cy^2$ be a quadratic form with A, B, and C in $\mathbb{Z}$ such that $B^2 - 4AC$ is neither a multiple of p nor a perfect square modulo p. Prove that

$$\prod_{0<x<y<p} (Ax^2 + Bxy + Cy^2)$$

is 1 modulo p if exactly one or all three of A, C, and $A+B+C$ are perfect squares modulo p and is -1 modulo p otherwise.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. All expressions below involving x and y take place in the finite field $\mathbb{F}_p$ with p elements. We first study the desired product in general, leaving until later a consideration of how many elements of $\{A, C, A+B+C\}$ are squares. For convenience, define

$$Q(x, y) = Ax^2 + Bxy + Cy^2.$$

Since we are given that $B^2 - 4AC$ is a nonsquare, A and C must be nonzero, and it follows that $Q(x, y) \neq 0$ when $(x, y) \neq (0, 0)$. In order to evaluate the product $\prod_{0<x<y<p} Q(x, y)$, we want to group the factors by the value of $Q(x, y)$. That is, for each D we seek the number of solutions of $Q(x, y) = D$ such that $0 < x < y < p$.

For $D \neq 0$, since $Q(x, y) - Dz^2 = 0$ determines a nondegenerate quadric, there are altogether $p^2 - 1$ solution triples (x, y, z) to $Q(x, y) - Dz^2 = 0$. (See Lemma 7.23 on p. 142 of J. W. P. Hirschfeld (1979), *Projective Geometries over Finite Fields*, Clarendon Press.) The set of solution triples is invariant under multiplication by any nonzero element of $\mathbb{F}_p$. Hence the solutions come in $p + 1$ multiplicative classes of size $p - 1$, each containing one triple of the form $(x, y, 1)$, yielding $p + 1$ solutions to $Q(x, y) = D$.

This partitions the set of nonzero pairs (x, y) by the value of $Q(x, y)$, with each value D occurring exactly $p + 1$ times. Note that $Q(x, y) = Q(p - x, p - y)$, so for fixed D the number of pairs satisfying $Q(x, y) = D$ with $x < y$ equals the number of pairs with $x > y$. Hence we will need to divide the number of occurrences of D by 2.

Since we require $0 < x < y < p$ in the stated product, we must also exclude occurrences of D that arise when $x = 0$, $y = 0$, or $x = y$. Two nonzero elements of $\mathbb{F}_p$ have the same *quadratic character* if they are both squares or both nonsquares, equivalent to their ratio being a square. Occurrences of D on the line $x = 0$ have $Cy^2 - D = 0$, or $y^2 = D/C$, so there are two such pairs yielding D when D and C have the same quadratic character; otherwise none. Similarly, there are two occurrences of D on $y = 0$ if and only if A and D have the same quadratic character (satisfying $x^2 = D/A$), and two occurrences of D on $x = y$ if and only if $A + B + C$ and D have the same quadratic character (satisfying $x^2 = D/(A + B + C)$). Also, such occurrences on the three lines are distinct.

Let the number of squares among $\{A, C, A + B + C\}$ be s. Starting with the $p + 1$ pairs $(x, y) \in \mathbb{F}_p^2 - (0, 0)$ that generate D, we subtract the occurrences with $x = 0$, $y = 0$, or $x = y$ and then divide the remaining occurrences by 2, as discussed above. We thus compute that each square D occurs in the product $(p + 1 - 2s)/2$ times, while each nonsquare D occurs in the product $\big(p + 1 - 2(3 - s)\big)/2$ times.

This tells us how many times we have the product of all the squares and how many times we have the product of all the nonsquares. It is well known that the product of all the squares is $(-1)^{(p+1)/2}$, and the product of all the nonsquares is $(-1)^{(p-1)/2}$, because an element and its reciprocal have the same quadratic character. After canceling reciprocal pairs and ignoring 1, we are left with -1, which is a square if and only if $p \equiv 1 \bmod 4$.

We thus compute

$$\begin{aligned}
\prod_{0<x<y<p} Q(x, y) &= (-1)^{\frac{1}{2}(p+1)\frac{1}{2}(p+1-2s)}(-1)^{\frac{1}{2}(p-1)\frac{1}{2}(p+1+2s-6)} \\
&= (-1)^{\frac{1}{4}\left((p+1)^2+(p^2-1)-4s-6(p-1)\right)} \\
&= (-1)^{\frac{1}{2}(p^2-2p+3-2s)} = (-1)^{\frac{1}{2}\left((p-1)^2+2-2s\right)} = (-1)^{1-s}.
\end{aligned}$$

This equals 1 or -1 when the number s of squares in $\{A, C, A + B + C\}$ is odd or even, respectively, as desired.

Also solved by C. Curtis & J. Boswell, Y. J. Ionin, R. Tauraso (Italy), and the proposer.

An Application of Liouville's Theorem

12235 [2021, 179]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $a_0, a_1, \ldots$ be a sequence of real numbers tending to infinity, and let $f : \mathbb{C} \to \mathbb{C}$ be an entire function satisfying

$$|f^{(n)}(a_k)| \le e^{-a_k}$$

for all nonnegative integers k and n. Prove $f(z) = ce^{-z}$ for some constant $c \in \mathbb{C}$ with $|c| \le 1$.

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. We prove that the entire function $g(z) = e^z f(z)$ satisfies

$$|g(z)| \le 1 \tag{$*$}$$

for all z. From this, Liouville's theorem yields $g(z) = c$ for some constant c, and then $(*)$ yields $|c| \le 1$. Hence, $f(z) = ce^{-z}$ with $|c| \le 1$, as claimed.

Since $f(z)$ is entire, for $z = x + iy$ and $k \ge 0$ we have

$$\begin{aligned} |g(z)| = |e^z| \left| \sum_{n=0}^{\infty} \frac{f^{(n)}(a_k)}{n!}(z - a_k)^n \right| &\le e^x \sum_{n=0}^{\infty} \frac{|f^{(n)}(a_k)|}{n!}|z - a_k|^n \\ &\le e^x e^{-a_k} \sum_{n=0}^{\infty} \frac{|z - a_k|^n}{n!} = e^{x - a_k + |z - a_k|}. \end{aligned}$$

Since $\lim_{k\to\infty} a_k = \infty$, we have $x < a_k$ for sufficiently large k. Thus, for such k,

$$|g(z)| \le e^{|z-a_k|-|x-a_k|} = \exp\left(\frac{y^2}{|z - a_k| + |x - a_k|} \right).$$

Taking the limit as $k \to \infty$, we obtain $(*)$, which completes the proof.

Also solved by P. Bracken, L. Han (USA) & X. Tang (China), E. A. Herman, K. T. L. Koo (China), O. Kouba (Syria), K. Sarma (India), A. Sasane (UK), A. Stadler (Switzerland), J. Yan (China), and the proposer.

The Googolth Term of a Sequence

12237 [2021, 276]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Let $x_0 = 1$ and $x_{n+1} = x_n + \lfloor x_n^{3/10} \rfloor$ for $n \ge 0$. What are the first 40 decimal digits of x_n when $n = 10^{100}$?

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The first 40 digits are 43236 87954 44259 51263 21573 91617 78825 77073.

Let $f(x) = (10/7)x^{7/10}$, and let $a_k = f(x_k)$ for all k. Applying the mean value theorem to f yields $c_n \in (x_n, x_{n+1})$ such that

$$a_{n+1} - a_n = c_n^{-3/10}(x_{n+1} - x_n) = c_n^{-3/10}\lfloor x_n^{3/10} \rfloor.$$

Since $c_n > x_n$, this implies $a_{n+1} - a_n < 1$. Computing $x_6 = 7$ and $a_6 = 10 \cdot 7^{-3/10} < 6$, we obtain $a_n < n$ and hence $x_n < (7n/10)^{10/7}$ for $n \ge 6$. Putting $n = 10^{100}$, we obtain an upper bound for x_n less than

$$4.3236\ 87954\ 44259\ 51263\ 21573\ 91617\ 78825\ 77073\ 38123 \times 10^{142}.$$

We now provide a lower bound for x_n. Applying the mean value theorem to $g(x) = x^{3/10}$ yields $b_n \in (x_n, x_{n+1})$ such that

$$c_n^{3/10} - x_n^{3/10} < x_{n+1}^{3/10} - x_n^{3/10} = \frac{3}{10} b_n^{-7/10}(x_{n+1} - x_n) = \frac{3}{10} b_n^{-7/10}\lfloor x_n^{3/10} \rfloor < 1.$$

Hence

$$a_{n+1} - a_n = 1 - \frac{c_n^{3/10} - \lfloor x_n^{3/10} \rfloor}{c_n^{3/10}} > 1 - \frac{2}{x_n^{3/10}}. \tag{$*$}$$

By direct iteration, $x_{45} = 102 > 4^{10/3}$. Since $\langle x_n \rangle$ is increasing, $a_{n+1} \geq a_n + 1/2$ whenever $n \geq 45$. From $a_{45} > 45/2$, for $n \geq 45$ we conclude that $a_n > n/2$, hence $x_n > (7n/20)^{10/7}$. Explicit computation shows that this lower bound for x_n also holds for $n < 45$. Therefore, summing ($*$) from 1 through $n-1$ gives

$$a_n > a_1 + (n-1) - \sum_{k=1}^{n-1} \frac{2}{x_k^{3/10}} > n - \sum_{k=1}^{n-1} \frac{2}{(7k/20)^{3/7}} > n - \frac{7}{2(7/20)^{3/7}}\, n^{4/7},$$

where at the last step we used the standard integral bound

$$\sum_{k=1}^{n-1} \frac{1}{k^{3/7}} \leq \int_0^n \frac{1}{t^{3/7}}\, dt = \frac{7}{4}\, n^{4/7}.$$

For $n = 10^{100}$, this yields a lower bound for x_n greater than

$$4.3236\ 87954\ 44259\ 51263\ 21573\ 91617\ 78825\ 77073\ 37651 \times 10^{142}.$$

Therefore, the first 40 digits of x_n when $n = 10^{100}$ are as claimed.

Also solved by O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Tauraso (Italy), E. Treviño, T. Wilde (UK), The Logic Coffee Circle (Switzerland), and the proposer.

Collinear Midpoints from a Glide Reflection

12238 [2021, 276]. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a convex quadrilateral with $AD = BC$. Let P be the intersection of the diagonals AC and BD, and let K and L be the circumcenters of triangles PAD and PBC, respectively. Show that the midpoints of segments AB, CD, and KL are collinear.

Solution by Michel Bataille, Rouen, France. Let E and F be the midpoints of AB and CD, respectively. Let m be the line through D that is parallel to EF, and let m' be the image of m under reflection through EF. Since F is the midpoint of CD, the point C must lie on m'. Let Γ be the circle centered at B with radius AD. Since $AD = BC$, the point C also lies on Γ.

Consider the $180°$ rotation of the plane centered at E. This rotation sends A to B and D to some point D'. The rotation sends m to m', so D' lies on m', and since $BD' = AD$, the point D' also lies on Γ. However, D' cannot be C, because the midpoint of $D'D$ is E, whereas the midpoint of CD is F. Thus Γ and m' intersect at two points, and those two points are C and D'. It follows that if n is the line through B that is perpendicular to EF, then C is the reflection of D' through n.

Let g be the transformation of the plane consisting of rotation by $180°$ centered at E followed by reflection through n. One sees easily that g is an orientation-reversing isometry that sends A to B and D to C. (The transformation g can also be described as a glide reflection with axis EF.)

For any lines ℓ and ℓ', let $\angle(\ell, \ell')$ denote the directed angle from ℓ to ℓ'. Let Γ_{AD} and Γ_{BC} be the circumcircles of $\triangle PAD$ and $\triangle PBC$, respectively, and let $Q = g(P)$.

Since g is orientation-reversing, $\angle(QB, QC) = \angle(PD, PA) = \angle(PB, PC)$. Therefore Q lies on Γ_{BC}. However, also Q, B, and C lie on $g(\Gamma_{AD})$, so $g(\Gamma_{AD}) = \Gamma_{BC}$. It follows that $g(K) = L$, and therefore the midpoint of KL lies on EF.

Editorial comment. This solution shows that the quadrilateral need not be convex. Indeed, it need not even be simple, as long as the lines AC and BD intersect.

Also solved by A. Ali (India), J. Cade, H. Chen (China), P. De (India), G. Fera (Italy), D. Fleischman, K. Gatesman, O. Geupel (Germany), J.-P. Grivaux (France), W. Janous (Austria), D. Jones & M. Getz, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), T. Wiandt, L. Wimmer (Germany), L. Zhou, Davis Problem Solving Group, and the proposer.

Factorials and Powers of 2

12239 [2021, 276]. *Proposed by David Altizio, University of Illinois, Urbana, IL.* Determine all positive integers r such that there exist at least two pairs of positive integers (m, n) satisfying the equation $2^m = n! + r$.

Solution by Celia Schacht, North Carolina State University, Raleigh, NC. There are two such values of r. They are $r = 2$, with $2^3 = 3! + 2$ and $2^2 = 2! + 2$, and $r = 8$, with $2^7 = 5! + 8$ and $2^5 = 4! + 8$. We show that there are no other values.

If $2^{m_1} = n_1! + r$ and $2^{m_2} = n_2! + r$, then $2^{m_1} - n_1! = 2^{m_2} - n_2!$. For $x \in \mathbb{N}$, let $2^{v(x)}$ be the highest power of 2 dividing x. Note that x can be uniquely written as $2^{v(x)}$ times an odd number, which we call the odd part of x. Since $r > 0$, we have $2^{m_i} > n_i!$, so $m_i > v(n_i!)$ for $i \in \{1, 2\}$. Therefore,

$$v(n_1!) = v(2^{m_1} - n_1!) = v(2^{m_2} - n_2!) = v(n_2!).$$

Given that $(m_1, n_1) \neq (m_2, n_2)$, we may assume $m_1 > m_2$ and $n_1 > n_2$. If there are any even numbers from $n_2 + 1$ to n_1, then $v(n_1!) > v(n_2!)$, so $v(n_1!) = v(n_2!)$ implies that n_2 is even and $n_1 = n_2 + 1$. Let $n_2 = 2k$. Thus

$$2^{m_1} - 2^{m_2} = n_1! - n_2! = (2k) \cdot (2k)!. \tag{4}$$

The odd part of the left side is $2^{m_1 - m_2} - 1$. It equals the product of the odd parts of $2k$ and $(2k)!$, so it is at least the odd part of $(2k)!$, which we write as $2q + 1$. That is, $2^{m_1 - m_2} - 1 \geq 2q + 1$.

By dividing out all the factors of 2 from $(2k)!$, we obtain

$$v((2k)!) = \sum_{i=1}^{\infty} \left\lfloor \frac{2k}{2^i} \right\rfloor < \sum_{i=1}^{\infty} \frac{2k}{2^i} = 2k.$$

First consider the case $k \geq 5$. By induction, $(2k)! > 2^{4k}$ for $k \geq 5$. Therefore,

$$2^{4k} < (2k)! = 2^{v((2k)!)}(2q + 1) < 2^{2k}(2q + 1),$$

so $2^{2k} - 1 < 2^{2k} < 2q + 1 \leq 2^{m_1 - m_2} - 1$. Also $(2k)! = n_2! < n_2! + r = 2^{m_2}$, which yields

$$(2k)! \left(2^{2k} - 1\right) < 2^{m_2} \left(2^{2k} - 1\right) < 2^{m_1} - 2^{m_2} = (2k) \cdot (2k)!.$$

Dividing by $(2k)!$ yields $2^{2k} - 1 < 2k$, which is false for all positive k. This contradiction eliminates the possibility $k \geq 5$.

It remains to check the cases of the form $(n_1, n_2) = (2k + 1, 2k)$ for $k \in \{1, 2, 3, 4\}$. According to (4), we need powers of 2 differing by $2k(2k)!$. For $1 \leq k \leq 4$, the values of $2k(2k)!$ are 4, 96, 4320, and 322560, respectively. Examining powers of 2 yields the solutions for $k \in \{1, 2\}$ listed at the start, but no solution for $k \in \{3, 4\}$.

Also solved by A. Ali (India), F. R. Ataev (Uzbekistan), C. Curtis & J. Boswell, S. M. Gagola Jr., K. Gatesman, M. Ghelichkhani (Iran), N. Hodges (UK), P. Komjáth (Hungary), O. P. Lossers (Netherlands), S. Omar (Morocco), J. Polo-Gómez (Canada), K. Sarma (India), A. Stadler (Switzerland), R. Stong, M. Tang, R. Tauraso (Italy), E. Treviño, T. Wilde (UK), L. Zhou, and the proposer.

Harmonic Sums: Euler Once, Abel Twice

12241 [2021, 276]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Prove

$$\sum_{n=1}^{\infty}(-1)^n\, n\left(\frac{1}{4n}-\ln 2+\sum_{k=n+1}^{2n}\frac{1}{k}\right)=\frac{\ln 2-1}{8}.$$

Solution by Kee-Wai Lau, Hong Kong, China. We first address the partial sum of the series on the left side and show

$$8\sum_{n=1}^{N}(-1)^n n\left(\frac{1}{4n}-\ln 2+\sum_{k=n+1}^{2n}\frac{1}{k}\right) \tag{1}$$
$$=2(-1)^N(2N+1)\left(\sum_{k=N+1}^{2N}\frac{1}{k}-\ln 2\right)+\sum_{n=1}^{N}\frac{(-1)^n}{n}+(-1)^N-1+2\ln 2.$$

Since $\ln 2$ is irrational, it must have the same coefficient on both sides, requiring

$$8\sum_{n=1}^{N}(-1)^n n=2(-1)^N(2N+1)-2.$$

This equality is easily verified by considering odd and even N separately. Let $K(N)$ denote the quantity on both sides. In addition, since $8\sum_{n=1}^{N}(-1)^n(1/4)=(-1)^N-1$, the sum of the N initial terms on the left in (1) equals the sum of two terms on the right. It remains to prove

$$\sum_{n=1}^{N}8(-1)^n n\sum_{k=n+1}^{2n}\frac{1}{k}=2(-1)^N(2N+1)\sum_{k=N+1}^{2N}\frac{1}{k}+\sum_{n=1}^{N}\frac{(-1)^n}{n}.$$

Let $L(N)$ denote the left side in this equation. Rewrite that double sum as

$$L(N)=\sum_{n=1}^{N}(K(n)-K(n-1))J(n),$$

where $J(n)=\sum_{k=n+1}^{2n}1/k$ and $K(0)=0$. By partial summation,

$$L(N)=K(N)J(N)+\sum_{n=1}^{N-1}K(n)(J(n)-J(n+1)).$$

Now

$$J(n)-J(n+1)=\frac{1}{n+1}-\frac{1}{2n+1}-\frac{1}{2n+2}=\frac{-1}{2(n+1)(2n+1)}.$$

Hence

$$L(N)=\big(2(-1)^N(2N+1)-2\big)\,J(N)+\sum_{n=1}^{N-1}\big((-1)^{n+1}(2n+1)+1\big)\,\frac{1}{(n+1)(2n+1)}$$
$$=2(-1)^N(2N+1)J(N)+\sum_{n=1}^{N-1}\frac{(-1)^{n+1}}{n+1}-2J(N)+\sum_{n=1}^{N-1}\frac{1}{(n+1)(2n+1)}. \tag{2}$$

Restoring the expression involving J in the last summand, the last two terms in (2) simplify by telescoping as

$$-2J(N)-2\sum_{n=1}^{N-1}(J(n)-J(n+1))=-2J(N)-2(J(1)-J(N))=-1.$$

Now the expression for $L(N)$ reduces to the right side of (1), completing the proof of the identity.

Let H_N denote the harmonic number $\sum_{n=1}^{N} 1/n$. By Euler–Maclaurin summation,

$$H_N = \ln N + \gamma + \frac{1}{2N} + O(N^{-2}),$$

where γ is Euler's constant. Thus

$$\sum_{n=N+1}^{2N} \frac{1}{n} = H_{2N} - H_N = \ln 2 - \frac{1}{4N} + O(N^{-2}).$$

Hence the first term on the right side of (1) simplifies as

$$2(-1)^N(2N+1)\left(\frac{-1}{4N} + O(N^{-2})\right) = -(-1)^N + O(N^{-1}).$$

Also,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2.$$

Thus the right side of $(*)$ converges to $-1 + \ln 2$, which completes the proof.

Editorial comment. Another approach to evaluating the left side is to introduce the factor x^n for $0 < x < 1$ into the sum, expand, and let x approach 1. This is an application of Abel's limit theorem, known as Abel summation. Ulrich Abel (fittingly) and Vitaliy Kushnirevych used this method. With

$$a_n = \frac{1}{4n} - \ln 2 + H_{2n} - H_n \qquad \text{and} \qquad g(x) = \sum_{n=1}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x},$$

let

$$f(x) = \sum_{n=1}^{\infty} a_n(-x)^n = \frac{-\ln(1+x)}{4} - \frac{x\ln 2}{1+x} + \frac{g(i\sqrt{x}) + g(-i\sqrt{x})}{2} - g(-x).$$

Upon differentiating $f(x)$, we obtain a power series for $(-1)^n n a_n$, and Abel summation yields the result.

Many solvers used a method somewhat akin to Abel summation, that of integral representation. For example, Richard Stong used

$$a_n = \frac{1}{2}\int_0^1 \frac{1-x}{1+x} x^{2n-1}\, dx.$$

Upon interchange of summation and integration (justified by dominated convergence), the desired sum then becomes the readily evaluated integral

$$-\frac{1}{2}\int_0^1 \frac{1-x}{1+x} \frac{x}{(1+x^2)^2}\, dx.$$

Also solved by U. Abel & V. Kushnirevych (Germany), A. Berkane (Algeria), P. Bracken, B. Bradie, H. Chen, G. Fera (Italy), K. Gatesman, M. L. Glasser, G. C. Greubel, L. Han (US) & X. Tang (China), E. A. Herman, N. Hodges (UK), S. Kaczkowski, O. Kouba (Syria), P. W. Lindstrom, O. P. Lossers (Netherlands), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, and the proposer.

CLASSICS

C10. *Due to Douglas Lind, suggested by the editors.* Show that there are infinitely many numbers that appear at least six times in Pascal's triangle.

How Much of a Parabolic Arc Can Fit in a Unit Disk?

C9. *From the 2001 Putnam Competition.* Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Solution. The answer is yes. For a positive real number A, the parabola $y = Ax^2$ intersects the circle $x^2 + (y-1)^2 = 1$ at the origin and at the points $(\sqrt{2A-1}/A, 2 - 1/A)$ and $(-\sqrt{2A-1}/A, 2-1/A)$. The length $L(A)$ of the parabolic arc between these points consists of two congruent parts, one in each quadrant. Expressing the length of one of these parts as an integral with respect to the variable y and then letting $u = Ay$, we obtain

$$L(A) = 2\int_0^{2-1/A} \sqrt{1 + \frac{1}{4Ay}}\, dy = \frac{2}{A}\int_0^{2A-1} \sqrt{1+\frac{1}{4u}}\, du.$$

It suffices to find a value of A so that $L(A)$ is greater than 4. This occurs when

$$\int_0^{2A-1} \left(\sqrt{1+\frac{1}{4u}} - 1\right) du \geq 1.$$

Since

$$\left(\sqrt{1+\frac{1}{4u}} - 1\right)\left(\sqrt{1+\frac{1}{4u}} + 1\right) = \frac{1}{4u},$$

when $u > 1/12$ we have

$$\sqrt{1+\frac{1}{4u}} - 1 \geq \frac{1}{12u}.$$

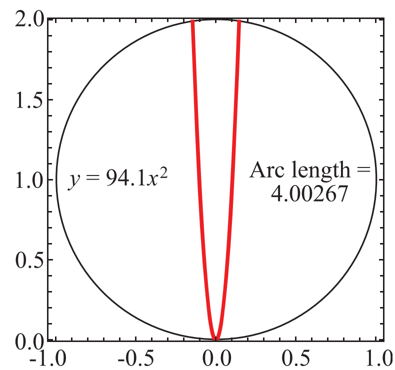
Therefore

$$\int_0^{2A-1} \left(\sqrt{1+\frac{1}{4u}} - 1\right) du \geq \int_1^{2A-1} \left(\sqrt{1+\frac{1}{4u}} - 1\right) du \geq \int_1^{2A-1} \frac{1}{12u}\, du.$$

Because $\int_1^{\infty} (1/x)\, dx$ diverges, we may choose A so large that this last integral exceeds 1.

Editorial comments. Numerical calculation shows that the longest arc is achieved when A is approximately 94.1, at which point the length is approximately 4.00267. The figure shows this longest parabolic arc. Not until A is approximately 37 does the arc length exceed 4.

In the 2001 Putnam Competition, just one participant (out of approximately 3000) earned full credit for solving this problem.



SOLUTIONS

Making Equality Improbable with Two Dice

12223 [2021, 88]. *Proposed by Michael Elgersma, Plymouth, MN, and James R. Roche, Ellicott City, MD.* Two weighted m-sided dice have faces labeled with the integers 1 to m. The first die shows the integer i with probability p_i, while the second die shows the integer i with probability r_i. Alice rolls the two dice and sums the resulting integers; Bob then independently does the same.
(a) For each m with $m \geq 2$, find the probability vectors $(p_1, \ldots, p_m)$ and $(r_1, \ldots, r_m)$ that minimize the probability that Alice's sum equals Bob's sum.
(b)* Generalize to n dice, with $n \geq 3$.

Composite solution to part **(a)** *by the proposers and Shuyang Gao, George Washington University, Washington, DC.* The minimum probability is $3/(6m-4)$, achieved only by the two distributions

$$\left(\frac{1}{2}, 0, 0, \ldots, 0, 0, \frac{1}{2}\right) \qquad \text{and} \qquad \frac{1}{3m-2}(2, 3, 3, \ldots, 3, 3, 2).$$

We start with some notation. We write $\mathbf{v}$ for a probability (row) vector $(v_1, \ldots, v_m)$ associated with the faces of an m-sided die; that is, the probability that a toss of such a die turns up value i is v_i (similarly with other letters). The reverse $R(\mathbf{v})$ of $\mathbf{v}$ is $(v_m, \ldots, v_1)$. We say that $\mathbf{v}$ is *symmetric* if $\mathbf{v} = R(\mathbf{v})$. For symmetrization and antisymmetrization, let $S_{\mathbf{v}} = (\mathbf{v} + R(\mathbf{v}))/2$ and $A_{\mathbf{v}} = (\mathbf{v} - R(\mathbf{v}))/2$. Thus $\mathbf{v} = S_{\mathbf{v}} + A_{\mathbf{v}}$, $R(S_{\mathbf{v}}) = S_{\mathbf{v}}$, and $R(A_{\mathbf{v}}) = -A_{\mathbf{v}}$.

Let $\mathbf{p}$ and $\mathbf{r}$ denote the probability vectors for the two dice. Let X and Y be the sums rolled by Alice and Bob, respectively. Note that X and Y have the same distribution. Let $\mathbf{s} = (s_2, \ldots, s_{2m})$, where

$$s_k = \mathbb{P}(X = k) = \mathbb{P}(Y = k) = \sum_{i=1}^{m} p_i r_{k-i},$$

with the understanding that $r_j = 0$ unless $1 \leq j \leq m$. With $*$ denoting convolution of vectors, we write $\mathbf{s}$ as $\mathbf{p} * \mathbf{r}$.

Our first task is to show that the probability is minimized only when $\mathbf{p}$ and $\mathbf{r}$ are symmetric. The tool for this is the claim

$$\mathbb{P}(X = Y) \geq (S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}}),$$

with equality holding if and only if $\mathbf{p}$ and $\mathbf{r}$ are both symmetric probability vectors. Given this, let $\mathbf{p}$ and $\mathbf{r}$ be minimizing probability vectors. If we replace $\mathbf{p}$ and $\mathbf{r}$ by their symmetrizations $S_{\mathbf{p}}$ and $S_{\mathbf{r}}$, then the new resulting probability $\mathbb{P}(X = Y)$ will be equal to $(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}})$, which will be strictly smaller than the original probability unless $\mathbf{p} = S_{\mathbf{p}}$ and $\mathbf{r} = S_{\mathbf{r}}$.

Hence we proceed to the claim. Since the players' rolls are independent,

$$\mathbb{P}(X = Y) = \sum_{k=2}^{2m} \mathbb{P}(X = k)\,\mathbb{P}(Y = k) = \sum_{k=2}^{2m} \Big(\sum_{i=1}^{m} p_i r_{k-i} \Big)^2.$$

We write this using convolution and inner product as

$$\mathbb{P}(X = Y) = (\mathbf{p} * \mathbf{r}) \cdot (\mathbf{p} * \mathbf{r}) = \big((S_{\mathbf{p}}+A_{\mathbf{p}}) * (S_{\mathbf{r}}+A_{\mathbf{r}})\big) \cdot \big((S_{\mathbf{p}}+A_{\mathbf{p}}) * (S_{\mathbf{r}}+A_{\mathbf{r}})\big).$$

By linearity of convolution and inner product, this expression expands into sixteen terms of the form $(f_{\mathbf{p}} * g_{\mathbf{r}}) \cdot (h_{\mathbf{p}} * i_{\mathbf{r}})$ with $f, g, h, i \in \{S, A\}$. We show that the contribution from the terms other than $(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}})$ is nonnegative and is 0 if and only if $\mathbf{p}$ and $\mathbf{r}$ are symmetric.

Since $S_{\mathbf{p}} * S_{\mathbf{r}}$ and $A_{\mathbf{p}} * A_{\mathbf{r}}$ are symmetric and $S_{\mathbf{p}} * A_{\mathbf{r}}$ and $A_{\mathbf{p}} * S_{\mathbf{r}}$ are antisymmetric, each of the eight terms having one or three factors in $\{A_{\mathbf{p}}, A_{\mathbf{r}}\}$ is the dot product of a symmetric and an antisymmetric vector and hence vanishes.

With $f, g \in \{S, A\}$, we find four terms of the form $(f_{\mathbf{p}} * g_{\mathbf{r}}) \cdot (f_{\mathbf{p}} * g_{\mathbf{r}})$. Each is nonnegative, since it is the dot product of a vector with itself, and it equals 0 if and only if $f_{\mathbf{p}} * g_{\mathbf{r}} = \mathbf{0}$. The convolution is $\mathbf{0}$ when $f = A$ and $\mathbf{p}$ is symmetric, since then $A_{\mathbf{p}} = \mathbf{0}$. However, if $\mathbf{p}$ is not symmetric, then $A_{\mathbf{p}} * S_{\mathbf{r}} \neq \mathbf{0}$. The corresponding statements hold also for g. Hence the contribution from these four terms is at least $(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (S_{\mathbf{p}} * S_{\mathbf{r}})$, with equality if and only if both $\mathbf{p}$ and $\mathbf{r}$ are symmetric.

The remaining four terms use each factor in $\{S_{\mathbf{p}}, S_{\mathbf{r}}, A_{\mathbf{p}}, A_{\mathbf{r}}\}$. They sum to

$$2\big((S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * A_{\mathbf{r}}) + (S_{\mathbf{p}} * A_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * S_{\mathbf{r}})\big). \tag{1}$$

We claim that this sum is 0. We have

$$(S_{\mathbf{p}} * S_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * A_{\mathbf{r}}) = \sum S_{\mathbf{p}}(k) S_{\mathbf{r}}(\ell) A_{\mathbf{p}}(k') A_{\mathbf{r}}(\ell') \tag{2}$$

and

$$(S_{\mathbf{p}} * A_{\mathbf{r}}) \cdot (A_{\mathbf{p}} * S_{\mathbf{r}}) = \sum S_{\mathbf{p}}(k) A_{\mathbf{r}}(\ell') A_{\mathbf{p}}(k') S_{\mathbf{r}}(\ell), \tag{3}$$

where the sum in (2) is over choices of k, ℓ, k', ℓ' in $\{1, \dots, m\}$ such that $k + \ell = k' + \ell'$, and the sum in (3) is over choices such that $k + \ell' = k' + \ell$. Note that $k + \ell = k' + \ell'$ if and only if $k - k' = \ell' - \ell$ and that $k + \ell' = k' + \ell$ if and only if $k - k' = \ell - \ell'$. By symmetry and antisymmetry,

$$S_{\mathbf{r}}(\ell) = S_{\mathbf{r}}(m - \ell + 1) \qquad \text{and} \qquad A_{\mathbf{r}}(\ell') = -A_{\mathbf{r}}(m - \ell' + 1).$$

Thus $S_{\mathbf{p}}(k) S_{\mathbf{r}}(\ell) A_{\mathbf{p}}(k') A_{\mathbf{r}}(\ell') = -S_{\mathbf{p}}(k) S_{\mathbf{r}}(m - \ell + 1) A_{\mathbf{p}}(k') A_{\mathbf{r}}(m - \ell' + 1)$. When we require $k - k' = \ell' - \ell$, at the same time we have $k - k' = (m - \ell + 1) - (m - \ell' + 1)$. Hence terms in the sum in (3) negate corresponding terms in the sum in (2), and the expression in (1) is 0. This completes the proof of the claim.

The claim implies the desired result in the case $m = 2$, giving $\mathbf{p} = \mathbf{r} = (1/2, 1/2)$. For the remainder of the argument, we assume $m \geq 3$. With $\mathbf{p}$ and $\mathbf{r}$ symmetric, the convolution $\mathbf{s}$ is also a symmetric probability vector, and the desired probability is $\sum_{k=2}^{2m} s_k^2$. By symmetry,

$$s_{m+1} = \sum_{i=1}^{m} p_i r_{m-i+1} \geq p_1 r_m + p_m r_1 = 2 p_1 r_1 = 2 s_2. \tag{4}$$

This suggests that we consider the following nonlinear optimization problem:

$$\text{minimize} \qquad 2(s_2^2 + \cdots + s_m^2) + s_{m+1}^2$$

subject to the constraints

$$2(s_2 + s_3 + \cdots + s_m) + s_{m+1} = 1, \quad 2s_2 \le s_{m+1}, \quad \text{and} \quad s_i \ge 0 \text{ for } 2 \le i \le m+1.$$

Extending $(s_2, \ldots, s_{m+1})$ by letting $s_{2m-i} = s_{2+i}$ for $0 \le i \le m-2$ relates this optimization problem to the symmetric probability vector $\mathbf{s}$ considered earlier. This problem incorporates the constraint (4), but it ignores the requirement in the original problem that $\mathbf{s}$ be realizable as the convolution of two probability vectors. It then suffices to show that we can realize the resulting optimum by such a convolution.

Such constrained optimization problems can be solved using the Karush-Kuhn-Tucker (KKT) conditions (see for example S. Boyd and L. Vandenberghe (2004), *Convex Optimization*, Cambridge University Press). Satisfying the conditions is sufficient for a global optimum. The method starts with a generalized Lagrangian incorporating the objective function, the inequality constraints, and the equality constraints:

$$L = 2(s_2^2 + \cdots + s_m^2) + s_{m+1}^2 + \mu(2s_2 - s_{m+1}) + \lambda\big(2(s_2 + \cdots + s_m) + s_{m+1} - 1\big).$$

The KKT conditions require partial derivatives with respect to the original variables and the multipliers for equality constraints to be 0, while for the multipliers of the inequality constraints we must have nonnegativity (see (9)) and "complementary slackness" (see (10)). That is,

$$\frac{\partial L}{\partial s_2} = 4s_2 + 2\mu + 2\lambda = 0; \tag{5}$$

$$\frac{\partial L}{\partial s_i} = 4s_i + 2\lambda = 0 \qquad \text{for } 3 \le i \le m; \tag{6}$$

$$\frac{\partial L}{\partial s_{m+1}} = 2s_{m+1} - \mu + \lambda = 0; \tag{7}$$

$$2(s_2 + \cdots + s_m) + s_{m+1} - 1 = 0. \tag{8}$$

$$\mu \ge 0; \quad \text{and} \tag{9}$$

$$\mu(2s_2 - s_{m+1}) = 0. \tag{10}$$

We also require $s_i \ge 0$ for all i in $\{2, \ldots, m+1\}$.

We show first that λ must be negative. If $\lambda > 0$, then by (6) each s_i with $i \ge 3$ is negative, which is forbidden. If $\lambda = 0$, then (6) requires $s_3 = \cdots = s_m = 0$. Since (5) now reads $4s_2 + 2\mu = 0$, it forbids $\mu > 0$, so $\mu = 0$ by (9). Now $s_2 = 0$ by (5) and $s_{m+1} = 0$ by (7), but that contradicts (8).

Hence $\lambda < 0$. Note that subtracting (5) from (7) gives $2s_{m+1} - 4s_2 = 3\mu + \lambda$. Since we require $2s_2 \le s_{m+1}$ and have $\lambda < 0$, we must have $\mu > 0$. Now (10) requires $2s_2 = s_{m+1}$.

With these restrictions, (5)–(7) reduce to

$$\lambda = -3\mu, \qquad s_2 = \mu, \qquad s_{m+1} = 2\mu, \qquad \text{and} \qquad s_i = \frac{3}{2}\mu \quad \text{for } 3 \le i \le m.$$

Using $s_{m+1} + 2\sum_{i=2}^m s_i = 1$, we obtain $\mu = 1/(3m-2)$, and consequently

$$s_2 = \frac{1}{3m-2}, \qquad s_{m+1} = \frac{2}{3m-2}, \qquad \text{and} \qquad s_i = \frac{3}{6m-4} \quad \text{for } 3 \le i \le m.$$

Extending back to the probability vector $\mathbf{s}$ with indices 2 through $2m$, we obtain

$$\mathbf{s} = \frac{1}{6m-4}(2, 3, 3, \ldots, 3, 3, 4, 3, 3, \ldots, 3, 3, 2), \tag{11}$$

yielding the minimum probability $\sum_{k=2}^{2m} s_k^2 = 3/(6m-4)$.

This solution to the optimization problem is achievable as the convolution of the two probability vectors

$$\left(\frac{1}{2}, 0, 0, \ldots, 0, 0, \frac{1}{2}\right) \qquad \text{and} \qquad \frac{1}{3m-2}(2, 3, 3, \ldots, 3, 3, 2).$$

Our final task is to show that these are the only probability vectors whose convolution is (11). To achieve $s_2 = s_{2m} > 0$, we have $p_1 = p_m > 0$ and $r_1 = r_m > 0$. Since we must satisfy

$$2s_2 = s_{m+1} = p_1 r_m + p_m r_1 + \sum_{i=2}^{m-1} p_i r_{m+1-i},$$

we obtain $p_i r_{m+1-i} = 0$ for $2 \le i \le m-1$. Consequently, for each i with $2 \le i \le m-1$,

$$p_i = p_{m+1-i} = 0 \quad \text{or} \quad r_{m+1-i} = r_i = 0.$$

By symmetry, we may take $p_2 = 0$. Now let k be the least integer in $\{2, \ldots, m\}$ such that $p_k > 0$. It suffices to show that $k = m$, which yields $\mathbf{p} = (1/2, 0, \ldots, 0, 1/2)$, whereupon the known convolution (11) yields $\mathbf{r}$ as claimed.

Suppose $k < m$. By (11),

$$\frac{3}{6m-4} = s_i = p_1 r_{i-1} + 0 + 0 + \cdots + 0 \qquad \text{for } 3 \le i \le k.$$

Since $p_1 r_1 = 2/(6m-4)$, we obtain $r_{i-1} = 3r_1/2 > 0$ for $3 \le i \le k$.

Next, $s_{k+1} = p_1 r_k + p_k r_1$. Since $p_k r_k = p_k r_{m+1-k} = 0$ and $p_k > 0$, we have $r_k = 0$. Now $p_k r_1 = s_{k+1} = 3/(6m-4)$ and $p_1 r_1 = s_2 = 2/(6m-4)$. Thus, $p_k = 3p_1/2$. Finally,

$$s_{k+2} \ge p_k r_2 = \left(\frac{3}{2}p_1\right)\left(\frac{3}{2}r_1\right) > 2s_2 = \frac{4}{6m-4},$$

contradicting $s_{k+2} \le 4/(6m-4)$. Thus $k = m$, completing the proof.

Editorial comment. The problem arose as an extension of Problem 1290 in Stan Wagon's Problem of the Week, which in turn was inspired by a problem on Tanya Khovanova's blog: blog.tanyakhovanova.com/2018/12/two-dice.

No solutions to part (b) or other correct solutions to part (a) were received.

A Lower Bound on Average Squared Acceleration

12229 [2021, 89]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $f : [0, 1] \to \mathbb{R}$ be a function that has a continuous second derivative and that satisfies $f(0) = f(1)$ and $\int_0^1 f(x)\,dx = 0$. Prove

$$30240 \left(\int_0^1 x f(x)\,dx\right)^2 \le \int_0^1 \left(f''(x)\right)^2\,dx.$$

Solution by Rory Molinari, Beverly Hills, MI. Applying integration by parts twice, and using $\int_0^1 f(x)\,dx = 0$ and $\int_0^1 f'(x)\,dx = f(1) - f(0) = 0$, we get

$$\begin{aligned}\int_0^1 x f(x)\,dx &= \int_0^1 \left(x - \frac{1}{2}\right) f(x)\,dx = -\int_0^1 \left(\frac{x^2}{2} - \frac{x}{2}\right) f'(x)\,dx \\ &= -\int_0^1 \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}\right) f'(x)\,dx = \int_0^1 \left(\frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}\right) f''(x)\,dx.\end{aligned}$$

Thus, by the Cauchy–Schwarz inequality,

$$\left(\int_0^1 xf(x)\,dx\right)^2 = \left(\int_0^1 \left(\frac{x^3}{6}-\frac{x^2}{4}+\frac{x}{12}\right) f''(x)\,dx\right)^2$$
$$\leq \left(\int_0^1 \left(\frac{x^3}{6}-\frac{x^2}{4}+\frac{x}{12}\right)^2 dx\right)\cdot\left(\int_0^1 (f''(x))^2\,dx\right) = \frac{1}{30240}\int_0^1 (f''(x))^2\,dx,$$

and the desired conclusion follows.

Editorial comment. Justin Freeman generalized the problem by proving

$$\frac{(2n+2)!}{|B_{2n+2}|}\left(\int_0^1 xf(x)\,dx\right)^2 \leq \int_0^1 (f^{(n)}(x))^2\,dx,$$

where B_k is the kth Bernoulli number.

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), P. Bracken, B. Bradie, H. Chen, G. Fera (Italy), J. Freeman (Netherlands), K. Gatesman, G. Góral (Poland), N. Grivaux (France), L. Han, E. A. Herman, L. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), Z. Lin (China), J. H. Lindsey II, O. P. Lossers (Netherlands), I. Manzur (UK) & M. Graczyk (France), T. M. Mazzoli (Austria), A. Natian (UK), A. Pathak (India), B. Shala (Slovenia), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, M. Vowe (Switzerland), J. Vukmirović (Serbia), T. Wiandt, J. Yan (China), L. Zhou, U. M. 6. P. MathClub (Morocco), and the proposer.

Families of Permutations with Equal Size

12230 [2021, 178]. *Proposed by David Callan, University of Wisconsin, Madison, WI.* Let $[n]=\{1,\dots,n\}$. Given a permutation $(\pi_1,\dots,\pi_n)$ of $[n]$, a *right-left minimum* occurs at position i if $\pi_j > \pi_i$ whenever $j > i$, and a *small ascent* occurs at position i if $\pi_{i+1} = \pi_i + 1$. Let $A_{n,k}$ denote the set of permutations π of $[n]$ with $\pi_1 = k$ that do not have right-left minima at consecutive positions, and let $B_{n,k}$ denote the set of permutations π of $[n]$ with $\pi_1 = k$ that have no small ascents.
(a) Prove $|A_{n,k}| = |B_{n,k}|$ for $1 \leq k \leq n$.
(b) Prove $|A_{n,j}| = |A_{n,k}|$ for $2 \leq j < k \leq n$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. For $n = 1$, we have $|A_{1,1}| = |B_{1,1}| = 1$. Hence it suffices to show that both $c_{n,k} = |A_{n,k}|$ and $c_{n,k} = |B_{n,k}|$ satisfy the recurrence

$$c_{n,k} = \begin{cases} \sum_{j=2}^{n-1} c_{n-1,j} & \text{if } k = 1, \\ \sum_{j=1}^{n-1} c_{n-1,j} & \text{if } k > 1. \end{cases}$$

The common recurrence then shows (**a**), and its form implies (**b**).

To a permutation π of $[n]$, associate the permutation σ of $[n-1]$ obtained by deleting π_1 and decreasing all entries exceeding π_1 by 1. From π_1 and σ, we can reconstruct π uniquely. In addition, σ has a right-left minimum at position i if and only if π has a right-left minimum at position $i+1$.

For $k > 1$, any permutation σ of $[n-1]$ with no right-left minima in consecutive positions arises from a permutation $\pi \in A_{n,k}$, and permutations in $A_{n,k}$ generate such σ, since position 1 in π is not a right-left minimum. Thus, the recursive formula holds for $|A_{n,k}|$ when $k > 1$. When $k = 1$, π has a right-left minimum in position 1, so we must ensure that the corresponding σ has no right-left minimum in position 1, which is equivalent to $\sigma_1 \neq 1$. Thus, the formula holds also for $|A_{n,1}|$.

We show that this recurrence also holds for $B_{n,k}$. Again consider the same map, with $\pi \in B_{n,k}$. If σ has no small ascents, then also π has none, unless $\sigma_1 = k$. On the other hand, if π has no small ascents, then σ has at most one small ascent, with equality exactly when $\pi_j = k-1$ and $\pi_{j+1} = k+1$ for some j. Let $E_{n-1,k}$ be the set of permutations of $[n-1]$ with a small ascent involving entries $k-1$ and k and no other small ascents. We obtain

$$|B_{n,k}| = \begin{cases} \sum_{j=2}^{n-1} |B_{n-1,j}| & \text{if } k=1, \\ |E_{n-1,k}| + \sum_{j\neq k} |B_{n-1,j}| & \text{if } 2 \le k \le n-1, \\ \sum_{j=1}^{n-1} |B_{n-1,j}| & \text{if } k=n. \end{cases}$$

We now prove $|E_{n-1,k}| = |B_{n-1,k}|$ when $n \ge 3$, which reduces this expression to the desired recurrence. Suppose $\sigma \in E_{n-1,k}$. Since σ has only one small ascent, the value $k+1$ does not follow k in σ. Hence collapsing the pair $(k-1,k)$ of consecutive values to $k-1$ and decreasing larger values by 1 gives a permutation of $[n-2]$ with no small ascent, and the map is reversible. Hence $|E_{n-1,k}| = \sum_{j=1}^{n-2} |B_{n-2,j}|$. We now have a proof of the desired recurrence by induction on n, since the induction hypothesis yields $|E_{n-1,k}| = |B_{n-1,k}|$.

Editorial comment. The proposer constructed a bijection from $A_{n,k}$ to $B_{n,k}$ iteratively as follows. If the current permutation has a small ascent, choose the left-most small ascent and move the larger value $j+1$ so that it immediately follows the largest right-left minimum m that it exceeds. For example, $\pi = (10, 11, 12, 2, 3, 1, 6, 7, 4, 8, 9, 5)$ has right-left minima at values 5, 4, and 1 (no two consecutive), and it has small ascents ending in the values 11, 12, 3, 7, and 9. The first iteration moves 11 to immediately after 5 and the fourth and final iteration yields $(10, 12, 2, 1, 3, 6, 4, 8, 5, 7, 9, 11)$.

Yury Ionin observed that exchanging the values k and $k+1$ in $\pi \in A_{n,k}$ yields a bijection between $A_{n,k}$ and $A_{n,k+1}$ for $k > 1$. This is implicit in the featured solution.

Also solved by K. Gatesman, A. Goel, Y. J. Ionin, and the proposer. Part (**b**) also solved by N. Hodges (UK).

Complete Elliptic Integrals and Watson's Integrals

12232 [2021, 178]. *Proposed by Seán Stewart, Bomaderry, Australia.* Prove

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{x(1-x)}\sqrt{y(1-y)}\sqrt{1-xy}}\,dx\,dy = \frac{1}{4\pi}\left(\int_0^\infty e^{-t} t^{-3/4}\,dt\right)^4.$$

Solution I by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. Let I denote the integral on the left side of the desired equation. Substituting $x = k^2$ and $y = \sin^2 t$, we get

$$I = 4\int_0^1 \frac{1}{\sqrt{1-k^2}} \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2 t}}\,dt\,dk = 4\int_0^1 \frac{K(k)\,dk}{\sqrt{1-k^2}}, \tag{1}$$

where $K(k)$ is the complete elliptic integral of the first kind given by the formula

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

The last integral in (1) is given by equation 6.143 on page 632 of I. S. Gradshteyn and I. M. Ryzhik (2007), *Table of Integrals, Series, and Products*, 7th ed., Burlington, MA: Academic Press. Filling in its value, we obtain

$$I = 4\big(K(\sqrt{2}/2)\big)^2 = \frac{\big(\Gamma(1/4)\big)^4}{4\pi} = \frac{1}{4\pi}\left(\int_0^\infty e^{-t}t^{-3/4}\,dt\right)^4.$$

Solution II by Lixing Han, University of Michigan, Flint, MI, and Xinjia Tang, Changzhou University, Changzhou, China. Let I be as in Solution I. Substituting $x = \cos^2 u$, $y = \cos^2 v$, we get

$$I = 4\int_0^{\pi/2}\int_0^{\pi/2}\frac{du\,dv}{\sqrt{1-\cos^2 u\cos^2 v}} = \int_0^{\pi}\int_0^{\pi}\frac{du\,dv}{\sqrt{1-\cos^2 u\cos^2 v}}. \tag{2}$$

For $|a| < 1$, the substitution $s = \tan(t/2)$ yields

$$\int_0^\pi \frac{dt}{1-a\cos t} = \frac{2}{1-a}\int_0^\infty \frac{ds}{1+\frac{1+a}{1-a}s^2} = \frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\,s\right)\Bigg|_0^\infty = \frac{\pi}{\sqrt{1-a^2}}.$$

Setting $a = \cos u \cos v$ leads to

$$\int_0^\pi \frac{dt}{1-\cos u\cos v\cos t} = \frac{\pi}{\sqrt{1-\cos^2 u\cos^2 v}}.$$

Substituting into (2), we obtain

$$I = \frac{1}{\pi}\int_0^\pi\int_0^\pi\int_0^\pi \frac{dt\,du\,dv}{1-\cos u\cos v\cos t} = \pi^2 I_1,$$

where I_1 is one of Watson's triple integrals (see I. J. Zucker (2011), 70+ years of the Watson Integrals, *J. Stat. Phys.* 145: 591–612, inp.nsk.su/~silagadz/Watson_Integral.pdf). Filling in the known value of I_1 gives the desired result.

Also solved by U. Abel & V. Kushnirevych (Germany), A. Berkane (Algeria), N. Bhandari (India), P. Bracken, H. Chen, B. E. Davis, G. Fera (Italy), M. L. Glasser, J.-P. Grivaux (France), J. A. Grzesik, N. Hodges (UK), Z. Lin (China), O. P. Lossers (Netherlands), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), M. Wildon (UK), and the proposer.

Squarefree Sums

12236 [2021, 179]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Let p_k be the kth prime number, and let $a_n = \prod_{k=1}^n p_k$. Prove that for $n \in \mathbb{N}$ every positive integer less than a_n can be expressed as a sum of at most $2n$ distinct divisors of a_n.

Solution by Rory Molinari, Beverly Hills, MI. The divisors of a_n are exactly the positive squarefree integers whose largest prime factor is no bigger than p_n. We need the claim that every positive integer r can be written as the sum of at most two distinct positive squarefree integers.

It is easy to verify the claim for $r \le 9$, so assume $r \ge 10$. Let $A(r)$ be the set of positive squarefree integers not greater than r. If $r \in A(r)$, we are done. Otherwise, it is known that $|A(r)| \ge 53r/88$ for all r (see K. Rogers (1964), The Schnirelmann density of the squarefree integers, *Proc. Am. Math. Soc.* 15(4): 515–516). Thus $|A(r)| > 1 + r/2$ for $r \ge 10$, and the pigeonhole principle implies that $A(r)$ and $\{r - k : k \in A(r)\}$ share at least two elements. At least one of them is not $r/2$, yielding an expression of r as the sum of two elements of $A(r)$.

To prove the problem statement, we use induction on n. The claim holds trivially for $n = 1$. For $n > 1$, consider m such that $1 \le m < a_n$. Write m as $q \cdot p_n + r$ with $0 \le q < a_{n-1}$ and $0 \le r < p_n$. By the claim, r is the sum of at most two positive squarefree numbers. These numbers cannot have p_n as a factor since $r < p_n$, so they are factors of a_{n-1}. By the induction hypothesis, q is the sum of at most $2(n-1)$ distinct factors of a_{n-1}. Hence, $q \cdot p_n + r$ is the sum of at most $2(n-1)$ distinct divisors of a_n, all of which are multiples of p_n, plus at most two distinct divisors of a_{n-1}. It follows that m is the sum of at most $2n$ distinct divisors of a_n.

Editorial comment. The problem statement above corrects a typo in the original printing. All solvers used similar proofs. Some used bounds such as

$$|A(r)| \ge r - r \sum_{k=1}^{\infty} p_k^{-2} > .54r$$

in the proof of the initial claim.

Also solved by O. Geupel (Germany), N. Hodges (UK), M. Hulse (India), Y. J. Ionin, O. P. Lossers (Netherlands), C. Schacht, A. Stadler (Switzerland), M. Tang, R. Tauraso (Italy), and the proposer.

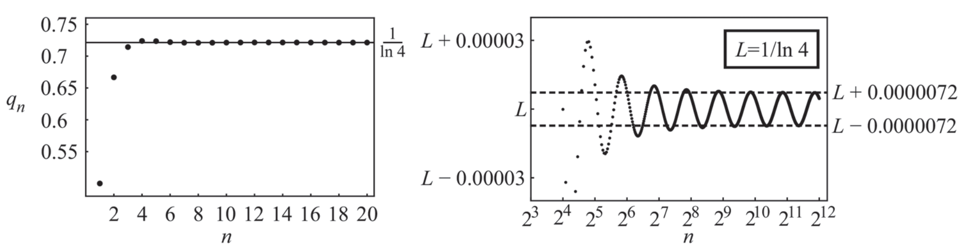
CLASSICS

C9. *From the 2001 Putnam Competition, suggested by the editors.* Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Flipping Coins Until They are All Heads

C8. *Due to Leonard Räde, suggested by the editors*. Start with n fair coins. Flip all of them. After this first flip, take all coins that show tails and flip them again. After the second flip, take all coins that still show tails and flip them again. Repeat until all coins show heads. Let q_n be the probability that the last flip involved only a single coin. What is $\lim_{n\to\infty} q_n$?

Solution. Let $L = 1/\ln 4$. Rough computation suggests that q_n converges to L, but we show that q_n oscillates around L with an asymptotic amplitude of about 10^{-5}, and so the limit does not exist. Here at left we display the graph of q_n for $1 \le n \le 20$, illustrating the apparent convergence. At right we graph the same sequence, zooming in and using a logarithmic horizontal axis. That view reveals what appears to be a persistent asymptotic oscillation.



To prove that the limit does not exist, take $n \ge 2$, let C be one of the coins, and let k be a positive integer. Consider the event that C shows heads for the first time on flip $k+1$, and all other coins show heads earlier. This occurs only if C shows tails for each of the first k flips and then heads on flip $k+1$. This has probability $2^{-(k+1)}$. For each of the other $n-1$ coins, it must not be the case that all of the first k flips show tails. This has probability $1 - 2^{-k}$. So the probability of the event is $2^{-(k+1)}(1-2^{-k})^{n-1}$.

Because there are n possibilities for C, and because k can be any positive integer,

$$q_n = \sum_{k=1}^{\infty} \frac{n}{2^{k+1}} \left(1 - \frac{1}{2^k}\right)^{n-1}. \tag{$*$}$$

We show that the sequence $q_1, q_2, \ldots$ does not converge by showing that it has different subsequences that converge but to different limits.

Let $c_k = (1 - 2^{-k})^{2^k}$. It is well known and easy to show that $c_1, c_2, \ldots$ is an increasing sequence and $\lim_{k\to\infty} c_k = 1/e$.

We have

$$q_n = \sum_{k=1}^{\infty} \frac{n}{2^{k+1}} \left(\left(1 - \frac{1}{2^k}\right)^{2^k}\right)^{n/2^k} \left(1 - \frac{1}{2^k}\right)^{-1} = \sum_{k=1}^{\infty} \frac{n}{2^{k+1}} c_k^{n/2^k} \left(\frac{2^k}{2^k - 1}\right).$$

Now fix an odd integer m, and let $a_j = q_{m2^j}$ for $j \geq 1$. We have

$$a_j = \sum_{k=1}^{\infty} \frac{m2^j}{2^{k+1}} c_k^{m2^j/2^k} \left(\frac{2^k}{2^k - 1}\right) = \sum_{k=1-j}^{\infty} \frac{m}{2^{k+1}} c_{k+j}^{m/2^k} \left(\frac{2^{k+j}}{2^{k+j} - 1}\right).$$

The kth term of this series is bounded above by $(m/2^k)e^{-m/2^k}$, whose sum over k from $-\infty$ to ∞ is finite. Hence, by the dominated convergence theorem,

$$\lim_{j\to\infty} a_j = \sum_{k=-\infty}^{\infty} \lim_{j\to\infty} \frac{m}{2^{k+1}} c_{k+j}^{m/2^k} \left(\frac{2^{k+j}}{2^{k+j} - 1}\right) = \sum_{k=-\infty}^{\infty} \frac{m}{2^{k+1}} e^{-m/2^k}.$$

With $m = 1$, this last sum can be approximated by letting k run from -5 to 27, giving an approximation of $L + 4.58 \cdot 10^{-6}$ for the sum, and the error in this approximation is seen by a simple integration to be less than 10^{-8}. Similarly, when $m = 3$, the last sum is approximately $L - 1.17 \cdot 10^{-6}$, again with an error of less than 10^{-8}. The distinct limits prove that $\lim_{n\to\infty} q_n$ does not exist.

Editorial comment. One can approximate the sum in $(*)$ by

$$\int_0^{\infty} n2^{-(x+1)}(1 - 2^{-x})^{n-1}\, dx,$$

which is L, independent of n. The error in this approximation does not vanish with n, however.

The problem appeared in this Monthly [1991, 366; 1994, 78]. A version of the same problem appeared almost a decade earlier in the 1982 *Can. Math. Bull.* as Problem P322 by George Szekeres, who asked whether

$$\lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{i-1} \frac{i}{2^i - 1} \binom{n}{i}$$

equals $1/\ln 2$. It turns out that the nth term here is just $2q_n$ in disguise, so the answer to the Szekeres problem is negative.

In N. J. Calkin, E. R. Canfield, and H. S. Wilf (2000), Averaging sequences, deranged mappings, and a problem of Lambert and Slater, *J. Comb. Th., Ser. A* 91(1–2): 171–190, a general class of sequences is found to exhibit the oscillating sequence phenomenon. In particular, they answer an open question in D. E. Lampert and P. J. Slater (1998), Parallel knockouts in the complete graph, this Monthly 105: 556–558.

SOLUTIONS

A Double Sum for Apéry's Constant

12222 [2020, 945]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Prove

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sum_{n=k}^{\infty} \frac{1}{n2^n} = -\frac{13\,\zeta(3)}{24},$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$.

Composite solution by Brian Bradie and Hongwei Chen, Christopher Newport University, Newport News, VA. In general, $\zeta(m) = \sum_{k=1}^{\infty} 1/k^m$. In working with expressions involving reciprocal powers, it is useful to have the gamma function integral and its logarithmic version

$$\frac{n!}{k^{n+1}} = \int_0^{\infty} \mathrm{e}^{-kt} t^n dt = (-1)^n \int_0^1 x^{k-1} (\ln x)^n dx, \tag{1}$$

where the latter integral is obtained from the former by setting $t = -\ln x$.

Let S be the desired double sum. After interchanging the order of summation, we invoke (1) with $n = 1$ to obtain

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{k=1}^{n} \frac{(-1)^k}{k^2} = \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{k=1}^{n} (-1)^{k+1} \int_0^1 x^{k-1} \ln x\, dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n2^n} \int_0^1 \left(\sum_{k=1}^{n} (-1)^{k+1} x^{k-1} \right) \ln x\, dx = \sum_{n=1}^{\infty} \frac{1}{n2^n} \int_0^1 \frac{1-(-x)^n}{1+x} \ln x\, dx. \end{aligned}$$

Because the integrand in this last expression is nonpositive for every x in $[0, 1]$ and every n, one can interchange the summation and integration to obtain

$$S = \int_0^1 \frac{-\ln(1-1/2) + \ln(1+x/2)}{1+x} \ln x\, dx = \int_0^1 \frac{\ln(2+x)\ln x}{x+1}\, dx.$$

We break the integral for S into three integrals by applying the polarization identity $ab = \frac{1}{2}(a^2 + b^2 - (a-b)^2)$ to the numerator of the integrand, using $a = \ln x$ and $b = \ln(2+x)$. Letting

$$J(f(x)) = \int_0^1 \frac{(\ln f(x))^2}{1+x}\,dx,$$

we obtain

$$2S = J(x) + J(x+2) - J(x/(2+x)). \tag{2}$$

Expanding $1/(1+x)$ into a geometric series and applying (1) with $n=2$ yields

$$J(x) = \sum_{k=0}^{\infty}(-1)^k \int_0^1 x^k (\ln x)^2 dx = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^3}.$$

To evaluate $J(x+2)$, we substitute $t = 1/(x+2)$. Since $1/(x+1) = t/(1-t)$, we obtain $dx/(1+x) = dt/(t(t-1))$. Using partial fraction expansion and then another geometric series,

$$J(x+2) = \int_{1/3}^{1/2} \left(\frac{1}{t} + \frac{1}{1-t}\right)(\ln t)^2 dt = \frac{(\ln 3)^3 - (\ln 2)^3}{3} + \sum_{k=0}^{\infty} \int_{1/3}^{1/2} t^k (\ln t)^2 dt.$$

Integrating by parts twice yields

$$\int_{1/3}^{1/2} t^k (\ln t)^2 dt = t^{k+1}\left(\frac{(\ln t)^2}{(k+1)} - \frac{2\ln t}{(k+1)^2} + \frac{2}{(k+1)^3}\right)\Bigg|_{1/3}^{1/2}. \tag{3}$$

Summing over k, we now have $J(x+2)$ expressed in terms of polylogarithms, where the *polylogarithm* $\mathrm{Li}_s(z)$ is defined by $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$. Note that $\mathrm{Li}_1(z) = -\ln(1-z)$. The function Li_2 is called the *dilogarithm*, and Li_3 is called the *trilogarithm*. In particular, $J(x) = -2\,\mathrm{Li}_3(-1)$ and

$$\begin{aligned}
J(x+2) &= \frac{(\ln 3)^3 - (\ln 2)^3}{3} + \sum_{k=1}^{\infty}(1/2)^k\left(\frac{(\ln(1/2))^2}{k} - \frac{2\ln(1/2)}{k^2} + \frac{2}{k^3}\right)\\
&\quad - \sum_{k=1}^{\infty}(1/3)^k\left(\frac{(\ln(1/3))^2}{k} - \frac{2\ln(1/3)}{k^2} + \frac{2}{k^3}\right)\\
&= \frac{(\ln 3)^3 - (\ln 2)^3}{3} + (\ln 2)^2\,\mathrm{Li}_1(1/2) + 2\ln 2\,\mathrm{Li}_2(1/2) + 2\,\mathrm{Li}_3(1/2)\\
&\quad - (\ln 3)^2\,\mathrm{Li}_1(1/3) - 2\ln 3\,\mathrm{Li}_2(1/3) - 2\,\mathrm{Li}_3(1/3)\\
&= \frac{(\ln 2)^3 - (\ln 3)^3}{3/2} + 2\ln 2\,\mathrm{Li}_2(1/2) + 2\,\mathrm{Li}_3(1/2)\\
&\quad - 2\ln 3\,\mathrm{Li}_2(1/3) - 2\,\mathrm{Li}_3(1/3) + (\ln 3)^2\ln 2,
\end{aligned}$$

where the last step uses $\mathrm{Li}_1(z) = -\ln(1-z)$.

To evaluate $J(x/(2+x))$, we substitute $t = x/(2+x)$, which yields $x = 2t/(1-t)$, $1+x = (1+t)/(1-t)$, $dx = 2\,dt/(1-t)^2$, and $dx/(1+x) = 2\,dt/(1-t^2)$. Integrating as we did in (3) after expanding a geometric sum yields

$$\begin{aligned}
J(x/(2+x)) &= 2\int_0^{1/3} \frac{1}{1-t^2}(\ln t)^2 dt\\
&= 2\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{2k+1}\left(\frac{(\ln 3)^2}{2k+1} + \frac{2\ln 3}{(2k+1)^2} + \frac{2}{(2k+1)^3}\right).
\end{aligned}$$

The odd terms in a Taylor series $T(x)$ at 0 sum to $\big(T(x) - T(-x)\big)/2$, so

$$J(x/(2+x)) = (\ln 3)^2 \ln 2 + 2\ln 3\big(\mathrm{Li}_2(1/3) - \mathrm{Li}_2(-1/3)\big) + 2\big(\mathrm{Li}_3(1/3) - \mathrm{Li}_3(-1/3)\big).$$

Substituting these expressions for $J(x)$, $J(x+2)$, and $J(x/(2+x))$ into (2) and combining like terms yields

$$S = \frac{(\ln 2)^3 - (\ln 3)^3}{3} - \ln 3\big(2\,\mathrm{Li}_2(1/3) - \mathrm{Li}_2(-1/3)\big) - \big(2\,\mathrm{Li}_3(1/3) - \mathrm{Li}_3(-1/3)\big) \\ + \ln 2\,\mathrm{Li}_2(1/2) - \mathrm{Li}_3(-1) + \mathrm{Li}_3(1/2).$$

The following are known evaluations of dilogarithms and trilogarithms at -1, $1/2$, and $\pm 1/3$:

$$\begin{aligned}
\mathrm{Li}_3(-1) &= -\frac{3}{4}\zeta(3) \\
\mathrm{Li}_2(1/2) &= \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2} \\
\mathrm{Li}_3(1/2) &= \frac{-\pi^2 \ln 2}{12} + \frac{(\ln 2)^3}{6} + \frac{7}{8}\zeta(3) \\
2\,\mathrm{Li}_2(1/3) - \mathrm{Li}_2(-1/3) &= \frac{\pi^2}{6} - \frac{(\ln 3)^2}{2} \\
2\,\mathrm{Li}_3(1/3) - \mathrm{Li}_3(-1/3) &= -\frac{\pi^2 \ln 3}{6} + \frac{(\ln 3)^3}{6} + \frac{13}{6}\zeta(3).
\end{aligned}$$

After substituting these evaluations into the last expression for S, remarkably all terms not involving $\zeta(3)$ cancel, leaving

$$S = \frac{3}{4}\zeta(3) + \frac{7}{8}\zeta(3) - \frac{13}{6}\zeta(3) = -\frac{13}{24}\zeta(3).$$

Editorial comment. The generation of many terms not involving $\zeta(3)$, which then cancel, suggests that there should be a shorter solution not involving polylogarithms, but no solver was able to contribute such a solution. Some solvers replaced the original 2 by $1/x$, differentiated, summed, integrated, and thereby reduced the desired sum to

$$\int_0^{1/2} \frac{\mathrm{Li}_2(-x)}{x(1-x)}\,dx.$$

However, this also does not seem to lead to a shorter solution.

A standard reference for polylogarithms and their evaluations is L. Lewin (1981), *Polylogarithms and Associated Functions*, Amsterdam: North-Holland. For further examples of series summing to $\zeta(3)$ and historical background, see A. van der Poorten (1979), A proof that Euler missed, *Math. Intelligencer* 1: 195–203, and W. Dunham (2021), Euler and the cubic Basel problem, this MONTHLY 128: 291–301.

Also solved by N. Bhandari (Nepal), R. Boukharfane (Morocco), G. Fera (Italy), M. L. Glasser, P. W. Lindstrom, M. Omarjee (France), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, and the proposer.

Collinear Intersection Points

12224 [2021, 88]. *Proposed by Cherng-tiao Perng, Norfolk State University, Norfolk, VA.* Let ABC be a triangle, with D and E on AB and AC, respectively. For a point F in the plane, let DF intersect BC at G and let EF intersect BC at H. Furthermore, let AF

intersect BC at I, let DH intersect EG at J, and let BE intersect CD at K. Prove that I, J, and K are collinear.

Solution I by Nigel Hodges, Cheltenham, UK. We use $XY.ZW$ to denote the intersection of lines XY and ZW. Let $L = AG.DI$, $M = AH.EI$, and $N = BC.DE$. Lines EH, AI, and GD concur at F. Therefore, by the theorem of Desargues, the points $EA.HI$, $EG.HD$, and $AG.ID$ are collinear. Since E lies on AC, and since H and I lie on BC, we have $EA.HI = C$, and by definition, $EG.HD = J$ and $AG.ID = L$. Thus, we have

$$C,\ J,\ \text{and}\ L\ \text{are collinear.} \tag{1}$$

Similarly, applying the theorem of Desargues to EH, IA, and GD we conclude that

$$M,\ J,\ \text{and}\ B\ \text{are collinear,} \tag{2}$$

and using EH, IA, and DG we get

$$M,\ N,\ \text{and}\ L\ \text{are collinear.} \tag{3}$$

Statement (3) implies that lines LM, DE, and CB concur at N, so by one more application of the theorem of Desargues we conclude that $LD.ME$, $LC.MB$, and $DC.EB$ are collinear. But L lies on DI and M lies on EI, so $LD.ME = I$, (1) and (2) imply that $LC.MB = J$, and $DC.EB = K$ by definition. Thus I, J, and K are collinear.

Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. We use homogeneous coordinates with $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$, and $K = (1 : 1 : 1)$. This gives $D = (1 : 1 : 0)$ and $E = (1 : 0 : 1)$. Let $F = (a : b : c)$. Since G lies on BC and DF, we have $G = (0 : b - a : c)$. Similarly,

$$H = (0 : b : c - a),\ I = (0 : b : c),\ \text{ and }\ J = (a : a - b : a - c),$$

so it follows that I, J, and K are collinear.

Also solved by M. Bataille (France), J. Cade, C. Curtis, I. Dimitrić, G. Fera (Italy), R. Frank (Germany), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, W. Janous (Austria), J. H. Lindsey II, C. R. Pranesachar (India), C. Schacht, V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, L. Zhou, Davis Problem Solving Group, The Zurich Logic-Coffee (Switzerland), and the proposer.

Gamma at Reciprocals of Positive Integers

12225 [2021, 88]. *Proposed by Pakawut Jiradilok, Massachusetts Institute of Technology, Cambridge, MA, and Wijit Yangjit, University of Michigan, Ann Arbor, MI.* Let Γ denote the gamma function, defined by $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ for $x > 0$.
(a) Prove that $\lceil \Gamma(1/n) \rceil = n$ for every positive integer n, where $\lceil y \rceil$ denotes the smallest integer greater than or equal to y.
(b) Find the smallest constant c such that $\Gamma(1/n) \geq n - c$ for every positive integer n.

Solution by Missouri State University Problem Solving Group, Springfield, MO. We use three facts about the gamma function: (i) $\Gamma(x + 1) = x\Gamma(x)$, (ii) $\Gamma'(1) = -\gamma$, where γ is the Euler–Mascheroni constant, and (iii) the gamma function is convex on $(0, \infty)$.
(a) The equation of the line tangent to $y = \Gamma(x + 1)$ at the point (0, 1) is

$$y = 1 + \Gamma'(1)x = 1 - \gamma x.$$

Since the gamma function is convex, this implies that for $x > -1$,

$$\Gamma(x + 1) \geq 1 - \gamma x.$$

Applying this with $x = 1/n$ yields

$$\Gamma(1/n) = n\Gamma(1/n + 1) \geq n(1 - \gamma/n) = n - \gamma.$$

Also, since $\Gamma(1) = \Gamma(2) = 1$, by convexity $\Gamma(x + 1) \leq 1$ for $0 \leq x \leq 1$. Hence

$$\Gamma(1/n) = n\Gamma(1/n + 1) \leq n.$$

Since $n - \gamma \leq \Gamma(1/n) \leq n$ and $\gamma < 1$, we conclude that $\lceil \Gamma(1/n) \rceil = n$.
(b) The solution to part **(a)** shows that γ satisfies the required condition. Now let c be any constant such that $\Gamma(1/n) \geq n - c$ for all n. We have

$$c \geq n - \Gamma(1/n) = n - n\Gamma(1/n + 1) = -\frac{\Gamma(1 + 1/n) - 1}{1/n}.$$

Letting n approach ∞ yields

$$c \geq \lim_{n\to\infty} -\frac{\Gamma(1 + 1/n) - 1}{1/n} = -\Gamma'(1) = \gamma.$$

Thus, γ is the smallest such c.

Also solved by R. A. Agnew, K. F. Andersen (Canada), P. Bracken, H. Chen, G. Fera (Italy), D. Fleischman, J.-P. Grivaux (France), J. A. Grzesik (Canada), L. Han, N. Hodges (UK), O. Kouba (Syria), O. P. Lossers (Netherlands), I. Manzur (UK) & M. Graczyk (France), R. Molinari, M. Omarjee (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), T. Wiandt, J. Yan (China), L. Zhou, and the proposer.

A Recursive Sequence That Is Convergent or Eventually Periodic

12226 [2021, 88]. *Proposed by Jovan Vukmirovic, Belgrade, Serbia.* Let x_1, x_2, and x_3 be real numbers, and define x_n for $n \geq 4$ recursively by $x_n = \max\{x_{n-3}, x_{n-1}\} - x_{n-2}$. Show that the sequence $x_1, x_2, \ldots$ is either convergent or eventually periodic, and find all triples (x_1, x_2, x_3) for which it is convergent.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. Let λ_1 be the unique real root of $\lambda^3 + \lambda - 1$, so

$$\lambda_1 = \left(\frac{9 + \sqrt{93}}{18}\right)^{1/3} + \left(\frac{9 - \sqrt{93}}{18}\right)^{1/3} = 0.682327803828\ldots.$$

The sequence converges if and only if $(x_1, x_2, x_3) = (x_1, x_1\lambda_1, x_1\lambda_1^2)$ with $x_1 > 0$ or $(x_1, x_2, x_3) = (x_1, 0, 0)$ with $x_1 \leq 0$. Otherwise, it is eventually periodic with period 4.

Given such a sequence $x_1, x_2, \ldots$, let $i \in \mathbb{N}$ be of *type A* if $x_i \leq x_{i+2}$ and *type B* if $x_i > x_{i+2}$. We claim that if i is of type A and $i + 1$ is of type B, then $x_j = x_{j+4}$ for $j \geq i + 3$. To see this, let $(a, b, c) = (x_i, x_{i+1}, x_{i+2})$. We have $a \leq c$ and $x_{i+3} = c - b$, so $b > c - b$ and $x_{i+4} = b - c$.

If $c \leq b - c$, which with $b > c - b$ implies $b > c$, then the sequence continues

$$x_{i+5} = 2b - 2c, \quad x_{i+6} = b - c, \quad x_{i+7} = c - b, \quad x_{i+8} = b - c, \quad x_{i+9} = 2b - 2c.$$

With $(x_{i+7}, x_{i+8}, x_{i+9}) = (x_{i+3}, x_{i+4}, x_{i+5})$, the claim follows. If $c > b - c$, then the sequence continues

$$x_{i+5} = b, \quad x_{i+6} = c, \quad x_{i+7} = c - b,$$

yielding $(x_{i+5}, x_{i+6}, x_{i+7}) = (x_{i+1}, x_{i+2}, x_{i+3})$. In both cases, the sequence has period 4 beginning no later than x_{i+3} and hence does not converge.

If i of type A is never followed by $i+1$ of type B, then either all i are of type B or there exists some integer $k \geq 1$ such that i is of type A if and only if $i \geq k$. If all i are of type B, then $x_n = -x_{n-2} + x_{n-3}$ for $n \geq 4$. The characteristic polynomial $\lambda^3 + \lambda - 1$ is strictly increasing with unique real root λ_1 between 0 and 1. The complex conjugate roots λ_2 and λ_3 have magnitude greater than 1.

It follows that $x_n = c_1\lambda_1^n + \Re(c_2\lambda_2^n)$ for some real c_1 and complex c_2, where $\Re(z)$ denotes the real part of z. Since $|\lambda_2| > 1$ and $x_{n-3} > x_{n-1}$ for $n \geq 4$, we conclude $c_2 = 0$ and therefore $x_n = c_1\lambda_1^n$, where $c_1 > 0$ to satisfy $x_n > x_{n+2}$. This is a strictly decreasing convergent solution, not eventually periodic.

Finally, if i is of type A if and only if $i \geq k$, then $x_{k+1}, x_{k+2}, \ldots$ satisfies $x_n = x_{n-1} - x_{n-2}$ for $n \geq k+3$. Therefore,

$$\begin{aligned}
x_{k+3} &= x_{k+2} - x_{k+1} \geq x_{k+1}, \\
x_{k+4} &= -x_{k+1} \geq x_{k+2}, \\
x_{k+5} &= -x_{k+2}, \\
x_{k+6} &= x_{k+1} - x_{k+2} \geq -x_{k+1}, \\
x_{k+7} &= x_{k+1} \geq -x_{k+2}.
\end{aligned}$$

From $-x_{k+1} \geq x_{k+2}$ and $x_{k+1} \geq -x_{k+2}$ we conclude $x_i = 0$ for $i \geq k+1$. Since k is of Type A, also $x_k \leq 0$. If $k > 1$, then $x_{k+2} = x_{k-1} - x_k > x_{k+1} - x_k = -x_k \geq 0$, which contradicts $x_{k+2} = 0$. Therefore, k must equal 1, and the convergent sequences that are also eventually periodic are given by $(x_1, x_2, x_3) = (x_1, 0, 0)$ with $x_1 \leq 0$.

Also solved by C. Curtis & J. Boswell, G. Fera (Italy), N. Hodges (UK), Y. J. Ionin, P. Lalonde (Canada), M. Reid, R. Stong, L. Zhou, and the proposer.

Sum of Reciprocals of Consecutive Integers

12227 [2021, 88]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Prove that for any integer n with $n \geq 3$ there exist infinitely many pairs (A, B) such that A is a set of n consecutive positive integers, B is a set of fewer than n positive integers, A and B are disjoint, and $\sum_{k\in A} 1/k = \sum_{k\in B} 1/k$.

Solution by Rory Molinari, Beverly Hills, MI. For positive integers t and n, let

$$A_n(t) = \begin{cases} \{t-m, t-m+1, \ldots, t+m\} & \text{if } n = 2m+1, \\ \{t-m, t-m+1, \ldots, t+m-1\} & \text{if } n = 2m, \end{cases}$$

where m is an integer. For a set X of nonzero numbers, let $S(X) = \sum_{i\in X} 1/i$.

First consider the odd case: $n = 2m+1 \geq 3$. Fix a positive integer p. Using $1/(np) = 1/p - (n-1)/(np)$, we compute

$$\begin{aligned}
S(A_n(np)) &= \frac{1}{p} - \frac{n-1}{np} + \sum_{i=1}^{m}\left(\frac{1}{np-i} + \frac{1}{np+i}\right) \\
&= \frac{1}{p} + \sum_{i=1}^{m}\left(\frac{1}{np-i} + \frac{1}{np+i} - \frac{2}{np}\right) \\
&= \frac{1}{p} + \sum_{i=1}^{m}\frac{2i^2}{np(n^2p^2-i^2)} = \frac{1}{p} + \sum_{i=1}^{m}\frac{1}{b(np,i)},
\end{aligned}$$

where $b(x, y) = x(x^2 - y^2)/(2y^2)$. If we choose p to be a multiple of $2m!$, then $b(np, i)$ is an integer for $1 \leq i \leq m$. By taking $A = A_n(np)$ and $B = \{p, b(np, 1), \ldots, b(np, m)\}$, we see that B is a set of fewer than n distinct positive integers and $S(A) = S(B)$. Since $b(np, i) = \Theta(p^3)$, the sets A and B are disjoint for sufficiently large p.

The case $n = 2m$ is similar. We compute

$$\begin{aligned} S(A_n(np)) &= \frac{1}{p} + \frac{1}{np-m} - \frac{n-1}{np} + \sum_{i=1}^{m-1}\left(\frac{1}{np-i} + \frac{1}{np-i}\right) \\ &= \frac{1}{p} + \left(\frac{1}{np-m} - \frac{1}{np}\right) + \sum_{i=1}^{m-1}\left(\frac{1}{np-i} + \frac{1}{np-i} - \frac{2}{np}\right) \\ &= \frac{1}{p} + \frac{1}{np(2p-1)} + \sum_{i=1}^{m-1}\frac{1}{b(np,i)}. \end{aligned}$$

Setting $A = A_p(np)$ and $B = \{p, np(2p-1), b(np,1), \ldots, b(np, m-1)\}$ suffices when we take p to be a sufficiently large multiple of $2(m-1)!$.

Also solved by C. Curtis & J. Boswell, K. Gatesman, J.-P. Grivaux (France), N. Hodges (UK), P. Lalonde (Canada), O. P. Lossers (Netherlands), I. Manzur (UK) & M. Graczyk (France), A. Pathak (India), C. R. Pranesachar (India), M. Reid, E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), L. Zhou, Missouri State University Problem Solving Group, and the proposers.

An Integral for Catalan Squared

12228 [2021, 89]. *Proposed by Hervé Grandmontagne, Paris, France.* Prove

$$\int_0^1 \frac{(\ln x)^2 \ln\left(2\sqrt{x}/(x^2+1)\right)}{x^2-1}\,dx = 2G^2,$$

where G is Catalan's constant $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$.

Solution by Li Zhou, Polk State College, Winter Haven, FL. It is well known that $2G = \int_0^\infty (x/\cosh x)\,dx$. (See, e.g., I. S. Gradshteyn and I. M. Ryzhik (2015), *Table of Integrals, Series, and Products*, 8th ed., Waltham, MA: Academic Press, equation 3.521(2).) Therefore, using the change of variables $u = x + y$, $v = x - y$, we have

$$\begin{aligned} 2G^2 &= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{xy}{\cosh x\cosh y}\,dx\,dy \\ &= \frac{1}{4}\int_0^\infty\int_0^\infty \frac{(x+y)^2-(x-y)^2}{\cosh(x+y)+\cosh(x-y)}\,dx\,dy \\ &= \frac{1}{8}\int_0^\infty\int_{-u}^{u} \frac{u^2-v^2}{\cosh u+\cosh v}\,dv\,du = \frac{1}{4}\int_0^\infty\int_0^{u} \frac{u^2-v^2}{\cosh u+\cosh v}\,dv\,du \\ &= \frac{1}{4}\left[\int_0^\infty u^2\int_0^u \frac{1}{\cosh u+\cosh v}\,dv\,du - \int_0^\infty v^2\int_v^\infty \frac{1}{\cosh u+\cosh v}\,du\,dv\right] \\ &= \frac{1}{4}\int_0^\infty u^2\left[\int_0^u \frac{dv}{\cosh u+\cosh v} - \int_u^\infty \frac{dv}{\cosh u+\cosh v}\right]du. \end{aligned}$$

To evaluate the inner integrals, we use

$$\begin{aligned} \int \frac{dv}{\cosh u+\cosh v} &= \int \frac{\tanh((u+v)/2)+\tanh((u-v)/2)}{2\sinh u}\,dv \\ &= \frac{1}{\sinh u}\ln\left(\frac{\cosh((u+v)/2)}{\cosh((u-v)/2)}\right) + C, \end{aligned}$$

which yields

$$\int_0^u \frac{dv}{\cosh u + \cosh v} = \frac{\ln \cosh u}{\sinh u} \quad \text{and} \quad \int_u^\infty \frac{dv}{\cosh u + \cosh v} = \frac{u - \ln \cosh u}{\sinh u}.$$

Hence

$$2G^2 = \frac{1}{4}\int_0^\infty \frac{u^2(2\ln\cosh u - u)}{\sinh u}\, du = \int_0^1 \frac{(\ln x)^2 \ln\left(2\sqrt{x}/(x^2+1)\right)}{x^2-1}\, dx,$$

where the last equality follows from the substitution $u = -\ln x$.

Also solved by F. R. Ataev (Uzbekistan), A. Berkane (Algeria), N. Bhandari (Nepal), H. Chen, G. Fera (Italy), M. L. Glasser, D. Henderson, N. Hodges (UK), O. Kouba (Syria), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), M. Wildon (UK), and the proposer.

A Sum of Secants from a Triangle

12231 [2021, 178]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* For an acute triangle ABC with circumradius R and inradius r, prove

$$\sec\left(\frac{A-B}{2}\right) + \sec\left(\frac{B-C}{2}\right) + \sec\left(\frac{C-A}{2}\right) \le \frac{R}{r} + 1.$$

Solution by UM6P Math Club, Mohammed VI Polytechnic University, Ben Guerir, Morocco. Since $\left(\cos((B-C)/2) - 2\sin(A/2)\right)^2 \ge 0$, we have

$$\cos^2\left(\frac{B-C}{2}\right) \ge 4\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{A}{2}\right) - 4\sin^2\left(\frac{A}{2}\right).$$

Using the well-known formula $4\sin(A/2)\sin(B/2)\sin(C/2) = r/R$, we obtain

$$\begin{aligned}
4\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{A}{2}\right) - 4\sin^2\left(\frac{A}{2}\right) &= 4\sin\left(\frac{A}{2}\right)\left(\cos\left(\frac{B-C}{2}\right) - \sin\left(\frac{A}{2}\right)\right)\\
&= 4\sin\left(\frac{A}{2}\right)\left(\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right)\right)\\
&= 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \frac{2r}{R}.
\end{aligned}$$

Thus $\sec\big((B-C)/2\big) \le \sqrt{R/(2r)}$. Similarly

$$\sec\big((A-B)/2\big) \le \sqrt{R/(2r)} \quad \text{and} \quad \sec\big((C-A)/2\big) \le \sqrt{R/(2r)},$$

and summing these inequalities yields

$$\sec\left(\frac{A-B}{2}\right) + \sec\left(\frac{B-C}{2}\right) + \sec\left(\frac{C-A}{2}\right) \le 3\sqrt{\frac{R}{2r}}.$$

To complete the proof, it suffices to show

$$3\sqrt{\frac{R}{2r}} \le \frac{R}{r} + 1.$$

Setting $t = \sqrt{R/(2r)}$, the required inequality becomes $3t \le 2t^2 + 1$, or $(2t-1)(t-1) \ge 0$. This holds because $t \ge 1$, by Euler's inequality $R \ge 2r$.

Editorial comment. The assumption that the triangle is acute is not needed.

Also solved by M. Bataille (France), H. Chen (China), C. Chiser (Romania), C. Curtis, N. S. Dasireddy (India), P. De (India), H. Y. Far, G. Fera (Italy), O. Geupel (Germany), N. Hodges (UK), W. Janous (Austria), K.-W. Lau (China), M. Lukarevski (North Macedonia), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), M. Vowe (Switzerland), T. Wiandt, L. Wimmer (Germany), and the proposer.

CLASSICS

C8. *(Due to Leonard Räde, suggested by the editors)*. Start with n fair coins. Flip all of them. After this first flip, take all coins that show tails and flip them again. After the second flip, take all coins that still show tails and flip them again. Repeat until all coins show heads. Let q_n be the probability that the last flip involved only a single coin. What is $\lim_{n\to\infty} q_n$?

Are $\mathbb{R}$ and $\mathbb{C}$ Isomorphic Under Addition?

C7. *Contributed by Alan D. Taylor, Union College, Schenectady, NY*. Are the additive group of real numbers and the additive group of complex numbers isomorphic?

Solution. Each of the given groups is a vector space over the set $\mathbb{Q}$ of rational numbers. Because every vector space has a basis, we can let B_1 be a basis for $\mathbb{R}$ and B_2 a basis for $\mathbb{C}$. Because $\mathbb{Q}$ is countable while $\mathbb{R}$ is not, in order for B_1 to span $\mathbb{R}$, the cardinality of B_1 must equal the cardinality of $\mathbb{R}$. The same holds for $\mathbb{C}$. Because $\mathbb{R}$ and $\mathbb{C}$ have the same cardinality, there is a bijection $f : B_1 \to B_2$. The bijection can be extended to an isomorphism of the groups as follows: for each $x \in \mathbb{R}$ write x (uniquely) as a finite sum $\sum_{i=1}^{n} q_i b_i$, where $q_i \in \mathbb{Q} \setminus \{0\}$ and $b_i \in B_1$ and define $f(x)$ to be $\sum_{i=1}^{n} q_i f(b_i)$. It is easy to verify that $f(x+y) = f(x) + f(y)$, so f is a group isomorphism.

Editorial comment. The result of the problem is folklore. The theorem that every vector space has a basis relies on the axiom of choice (denoted AC). A simple proof uses Zorn's lemma to show that there is a maximal linearly independent set of vectors; such a set must be a basis. It is well known that Zorn's lemma is equivalent to AC. It turns out that the statement that every vector space has a basis is also equivalent to AC (A. Blass (1984), Existence of bases implies the axiom of choice, *Contemp. Math.* 31, 31–33). The question therefore arises whether the existence of an isomorphism from $\mathbb{R}$ to $\mathbb{C}$ can be proved without using AC. We sketch a proof that it cannot.

A set of reals has the *property of Baire* if it differs from an open set by a meager set (i.e., a countable union of nowhere dense sets), and a function has the property of Baire if the inverse image of any open set has the property of Baire (so it is "almost continuous"). Let ZF be the axiomatic theory whose axioms are the Zermelo–Fraenkel axioms (AC not included). Let PB be the assertion that "all sets of reals have the property of Baire" and let ZF + PB be the theory in which PB is added to ZF as an additional axiom. The theory ZF + PB is known to be consistent, assuming ZF is consistent (S. Shelah (1984), Can you take Solovay's inaccessible away?, *Isr. J. Math.* 48, 1–47).

We now show that, in ZF + PB, the additive groups $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are not isomorphic. An *involution* of a group is an automorphism of order 2. The complex numbers admit at least two involutions: $z \mapsto -z$ and $z \mapsto \bar{z}$. Any automorphism f of $\mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$, and it is a classic result (W. Sierpiński (1924), Sur un propriété des fonctions de M. Hamel, *Fund. Math.* 5, 334–336) that any function with the property of Baire that satisfies this functional equation has the form $x \mapsto cx$ for some real c. Therefore, by PB, the only involution of $\mathbb{R}$ is $x \mapsto -x$. Because $\mathbb{C}$ has more than one involution, $\mathbb{C}$ cannot be isomorphic to $\mathbb{R}$.

SOLUTIONS

A Common Coefficient

12209 [2020, 852]. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.* Prove

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+2n-2k+1}{m}=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k+1}{m-n}$$

for all integers m and n with $m \geq n \geq 0$.

Solution by Michel Bataille, France. We show that both sides equal the coefficient of x^m in the polynomial P defined by

$$P(x)=(1+x)^{m+1}(2x+x^2)^n=(1+x)^{m+1}((1+x)^2-1)^n.$$

Using the binomial theorem twice yields

$$P(x)=(1+x)^{m+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+x)^{2(n-k)}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+x)^{2n-2k+m+1}$$

$$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{2n-2k+m+1}\binom{2n-2k+m+1}{j}x^j.$$

This expresses the left side of the identity as the coefficient of x^m in the expansion of $P(x)$.

Also,

$$P(x)=(1+x)^{m+1}(x(2+x))^n=x^n(1+x)^{m+1}(1+(1+x))^n,$$

so another two uses of the binomial theorem yield

$$P(x)=x^n(1+x)^{m+1}\sum_{k=0}^{n}\binom{n}{k}(1+x)^k=\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{m+k+1}\binom{m+k+1}{j}x^{n+j}.$$

This shows that the coefficient of x^m in the expansion of $P(x)$ is also the right side of the identity, completing the proof.

Also solved by R. Boukharfane (Saudi Arabia), Ó. Ciaurri (Spain), J. Boswell & C. Curtis, G. Fera (Italy), N. Hodges (UK), M. Kaplan & M. Goldenberg, O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), M. Maltenfort, E. Schmeichel, A. Stadler (Switzerland), R. Stong, F. A. Velandia (Colombia), M. Vowe (Switzerland), J. Vukmirović (Serbia), J. Wangshinghin, M. Wildon (UK), X. Ye (China), and the proposer.

A Median Inequality

12214 [2020, 853]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let x, y, and z be the lengths of the medians of a triangle with area F. Prove

$$\frac{xyz(x+y+z)}{xy+zx+yz} \geq \sqrt{3}F.$$

Solution by Oliver Geupel, Brühl, Germany. The Cauchy–Schwarz inequality implies that $x^2+y^2+z^2 \geq xy+yz+zx$, and therefore

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq 3(xy+yz+zx). \tag{1}$$

It is well known that the medians of a triangle with area F are the sides of a triangle with area $K = 3F/4$ (see, for example, sections 91–93 in N. Altschiller-Court (1952), *College Geometry*, New York: Barnes and Noble). Moreover, it is known that a triangle with sides x, y, and z and area K satisfies the inequality

$$\frac{9xyz}{x+y+z} \geq 4\sqrt{3}K \tag{2}$$

(see item 4.13 on p. 45 of O. Bottema et al. (1969), *Geometric Inequalities*, Groningen: Wolters-Noordhoff). Combining (1) and (2), we obtain

$$\frac{xyz(x+y+z)}{xy+yz+zx} \geq \frac{3xyz(x+y+z)}{(x+y+z)^2} = \frac{3xyz}{x+y+z} \geq \frac{4\sqrt{3}K}{3} = \sqrt{3}F.$$

Editorial comment. Inequality (2) appeared as part of elementary problem E1861 [1966, 199; 1967, 724] from this MONTHLY, proposed by T. R. Curry and solved by Leon Bankoff. The equation $K = 3F/4$ is also featured as Theorem 10.4 on p. 165 of C. Alsina and R. B. Nelsen (2010), *Charming Proofs: A Journey Into Elegant Mathematics*, Washington, DC: Mathematical Association of America.

Also solved by A. Alt, H. Bai (Canada), M. Bataille (France), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), I. Borosh, R. Boukharfane (Saudi Arabia), P. Bracken, S. H. Brown, C. Curtis, N. S. Dasireddy (India), A. Dixit (India) & S. Pathak (UK), H. Y. Far, G. Fera (Italy), N. Hodges (UK), W. Janous (Austria), M. Kaplan & M. Goldenberg, P. Khalili, O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), M. Lukarevski (Macedonia), A. Pathak (India), C. R. Pranesachar (India), C. Schacht, V. Schindler (Germany), A. Stadler (Switzerland), N. Stanciu & M. Drăgan (Romania), R. Stong, B. Suceavă, M. Vowe (Switzerland), J. Vukmiroviıc (Serbia), T. Wiandt, X. Ye (China), M. R. Yegan (Iran), Davis Problem Solving Group, and the proposer.

Another Incenter-Centroid Inequality

12217 [2020, 944]. *Proposed by Giuseppe Fera, Vicenza, Italy.* Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$\frac{3}{2} < \frac{AI}{AG} + \frac{BI}{BG} + \frac{CI}{CG} \leq 3.$$

Solution by Haoran Chen, Suzhou, China. Let $a = BC$, $b = CA$, and $c = AB$. Also let $s = (a + b + c)/2$. Let m_a be the length of the median from A, r the radius of the incircle, and K the point of tangency of the incircle with AB. By the triangle inequality,

$$2m_a < \left(\frac{a}{2} + b\right) + \left(\frac{a}{2} + c\right) = 2s.$$

Also, $AG = 2m_a/3$ and $AI > AK = s - a$. Therefore

$$\frac{AI}{AG} = \frac{3AI}{2m_a} > \frac{3(s-a)}{2s}.$$

Summing this with the other two analogous inequalities establishes the strict lower bound of 3/2.

For the upper bound, note that

$$rs = \text{area of } \triangle ABC = \frac{bc \sin A}{2},$$

and therefore

$$AI^2 = \frac{AK}{\cos(A/2)} \cdot \frac{r}{\sin(A/2)} = \frac{(s-a)r}{(1/2)\sin A} = \frac{bc(s-a)}{s}.$$

Also, by Apollonius's theorem,

$$4m_a^2 = 2b^2 + 2c^2 - a^2 = (b + c + a)(b + c - a) + (b - c)^2 \geq 4s(s - a).$$

Therefore

$$\frac{AI}{AG} = \frac{3AI}{2m_a} \leq \frac{3\sqrt{bc}}{2s} \leq \frac{3(b+c)}{4s}.$$

Summing this with the other two analogous inequalities establishes the upper bound of 3.

Editorial comment. Problem 12175 [2020, 372; 2021, 952] establishes

$$\frac{AI^2}{AG^2} + \frac{BI^2}{BG^2} + \frac{CI^2}{CG^2} \leq 3.$$

This can be used to give an alternative proof of the upper bound: By the Cauchy–Schwarz inequality,

$$\frac{AI}{AG} + \frac{BI}{BG} + \frac{CI}{CG} \leq \sqrt{3\left(\frac{AI^2}{AG^2} + \frac{BI^2}{BG^2} + \frac{CI^2}{CG^2}\right)} \leq 3.$$

Also solved by A. Alt, S. Gayen (India), P. Khalili, S. Lee (Korea), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, and the proposer.

Composing All Permutations of $[n]$ to Do Nothing

12218 [2020, 944]. *Proposed by Richard Stong, Center for Communications Research, La Jolla, CA, and Stan Wagon, Macalester College, St. Paul, MN.* For which positive integers n does there exist an ordering of all permutations of $\{1, \ldots, n\}$ so that their composition in that order is the identity?

Solution by S. M. Gagola Jr., Kent State University, Kent, OH. Such an ordering of permutations is possible for $n = 1$ (trivially) and for all n at least 4.

When n is 2 or 3, the number of permutations with odd parity is odd, so no composition in these cases can have even parity like the identity. Note, however, that when $n = 3$ the product of the three distinct transpositions always equals the middle factor ($t_1 t_2 t_3 = t_2$).

Before considering $n \geq 4$, it is useful to note that any group of even order has an odd number of elements of order 2. To see this, pair the elements of the group with their inverses. The identity element and the elements of order two (involutions) are self-paired, while the remaining elements form sets of size 2. Since the group has even order, the number of involutions is therefore odd.

If in a group of even order a product of the involutions (in some order) can be shown to equal the identity, then the remaining elements can be paired with their inverses to yield a product of all the elements equaling the identity. Hence it suffices to show that for $n \geq 4$, the involutions of the symmetric group S_n can be ordered so that their product is the identity.

The nine involutions in S_4 can be partitioned into three triples as follows:

$$\{(12), (34), (12)(34)\}, \quad \{(13), (24), (13)(24)\}, \quad \{(14), (23), (14)(23)\}.$$

The product of the three involutions in any one subset (in any order) equals the identity; this completes the $n = 4$ case.

For $n = 5$, we partition the involutions in S_5 into sets $I_1, \ldots, I_5$ and order each set to obtain a product yielding the identity. For I_1 we take the nine involutions on $\{2, 3, 4, 5\}$. By the $n = 4$ case, there is a product of these yielding the identity. For $j \geq 2$, let I_j consist of all involutions that exchange 1 and j. One element is $(1j)$, and each of the other three elements is the product of $(1j)$ and a transposition of two of the three elements of $\{2, 3, 4, 5\} - \{j\}$. Each of the four elements of I_j transposes 1 and j, and we have noted that the product of the three transpositions on a set of size 3 can be ordered to yield any one of the three transpositions. We can therefore choose orderings of each of I_2, I_3, I_4, and I_5 so that their products are (45), (45), (23), and (23), respectively. Combining these orderings completes the $n = 5$ case.

The solutions for $n = 4$ and $n = 5$ provide a basis for a proof by induction. We write $[n]$ for $\{1, \ldots, n\}$. For $n \geq 6$, partition the involutions of S_n into the n sets $I_1, \ldots, I_n$, where I_1 consists of all the involutions on $[n] - \{1\}$, and I_j for $j \geq 2$ consists of all involutions exchanging 1 and j. The $n - 1$ case yields an ordering of I_1 that produces the identity. For $j \geq 2$, each element of I_j consists of the transposition $(1j)$ times an element of the symmetric group on $[n] - \{1, j\}$ that is the identity or an involution. As noted earlier, I_j thus has even size, and hence any product of the elements of I_j leaves 1 and j in place. Furthermore, the $n - 2$ case guarantees that the elements of I_j other than $(1j)$ can be ordered so that their effect on $[n] - \{1, j\}$ is the identity. Doing this independently for all I_j completes the proof.

Editorial comment. The problem is a special case of a result from J. Dénes and P. Hermann (1982), On the product of all elements in a finite group, in E. Mendelsohn, ed., *Algebraic and geometric combinatorics*, North-Holland Math. Stud. 65, Amsterdam: North-Holland, pp. 105–109. A special case of their theorem that still includes the problem here is proved more simply in M. Vaughan-Lee and I. M. Wanless (2003), Latin squares and the Hall–Paige conjecture. *Bull. London Math. Soc.* 35, no. 2, 191–195.

The solver Gagola noted that if a group G of even order has a cyclic Sylow 2-subgroup, then there is a normal 2-complement N, and the product of the elements of G taken in any order always represents a coset of order 2 in the factor group G/N. Therefore, this product can never equal the identity element. He then asked whether a group of even order that does not have a cyclic Sylow 2-subgroup always has an ordering of the elements so that the resulting product produces the identity. As Vaughan-Lee and Wanless wrote, “The Hall–Paige conjecture deals with conditions under which a finite group G will possess a complete mapping, or equivalently a Latin square based on the Cayley table of G will possess a transversal. Two necessary conditions are known to be: (i) that the Sylow 2-subgroups of G are trivial or noncyclic, and (ii) that there is some ordering of the elements of G which yields a trivial product. These two conditions are known to be equivalent, but the first direct, elementary proof that (i) implies (ii) is given here." Thus the answer to Gagola's question is yes.

Also solved by F. Chamizo & Y. Fuertes (Spain), D. Dima (Romania), O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin (USA) & B. M. Bekker (Russia), O. P. Lossers (Netherlands), M. Reid, A. Stadler (Switzerland), R. Tauraso (Italy), T. Wilde (UK), and the proposers.

A Vanishing Sum of Stirling Numbers

12219 [2020, 944]. *Proposed by Brad Isaacson, New York City College of Technology, New York, NY.* Let k and m be positive integers with $k < m$. Let $c(m,k)$ be the number of permutations of $\{1,\dots,m\}$ consisting of k cycles. (The numbers $c(m,k)$ are known as unsigned Stirling numbers of the first kind.) Prove

$$\sum_{j=k}^{m} \frac{(-2)^j\binom{m}{j}c(j,k)}{(j-1)!} = 0$$

whenever m and k have opposite parity.

Solution by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy. Let

$$F_m(x) = \sum_{k=1}^{m}(-x)^k \sum_{j=k}^{m} \frac{(-2)^j\binom{m}{j}c(j,k)}{(j-1)!}.$$

Here $F_m(x)$ is a generating function for the desired sum, evaluated at the negative of the formal variable. We aim to show that the coefficients of odd powers of x are 0 when m is even, and the coefficients of even powers of x are 0 when m is odd. For this it suffices to show

$$F_m(-x) = (-1)^m F_m(x).$$

The well-known generating function for the unsigned Stirling numbers of the first kind is given by $\sum_{k=1}^{j} c(j,k)y^k = \prod_{i=0}^{j-1}(y+i)$ (easily proved combinatorially). Setting $y=-x$ yields $\sum_{k=1}^{j}(-1)^{j-k}c(j,k)x^k = \prod_{i=0}^{j-1}(x-i)$.

We interchange the order of summation to take advantage of this identity. Let x be an integer with $x \geq m$. We compute

$$\begin{aligned}
F_m(x) &= \sum_{j=1}^{m} \frac{2^j\binom{m}{j}}{(j-1)!}\sum_{k=1}^{j}(-1)^{j-k}c(j,k)x^k = \sum_{j=1}^{m} \frac{2^j\binom{m}{j}}{(j-1)!}\prod_{i=0}^{j-1}(x-i) \\
&= m\sum_{j=1}^{m} 2^j\binom{m-1}{j-1}\binom{x}{j} = m\sum_{j=1}^{m}\binom{m-1}{m-j}\binom{x}{j}2^j \\
&= m[z^m](1+z)^{m-1}(1+2z)^x = m[z^m](1+z)^{x+m-1}\left(1+\frac{z}{1+z}\right)^x,
\end{aligned}$$

where $[z^m]$ is the "coefficient operator" extracting the coefficient of z^m in the expression that follows it.

To extract the coefficient of z^m in a different way, we apply the binomial theorem twice to obtain

$$\begin{aligned}(1+z)^{x+m-1}\left(1+\frac{z}{1+z}\right)^x &= \sum_{j=0}^{x}(1+z)^{x+m-j-1}\binom{x}{j}z^j\\ &= \sum_{j=0}^{x}\binom{x}{j}z^j\sum_{k=0}^{x+m-j-1}\binom{x+m-j-1}{k}z^k.\end{aligned}$$

To extract all the contributions to the coefficient of z^m, restrict j to run from 0 to m, and set $k = m - j$ in the inner sum. This leads to the formula

$$F_m(x) = m[z^m](1+z)^{x+m-1}\left(1+\frac{z}{1+z}\right)^x = m\sum_{j=0}^{m}\binom{x+m-j-1}{m-j}\binom{x}{j}.$$

Viewing $\binom{x}{j}$ as a polynomial in x, this is a polynomial equation that holds for every integer x with $x \geq m$. It therefore holds for all real numbers x. Thus, by reversing the index of summation and using

$$\binom{-y}{r} = (-1)^r\binom{y+r-1}{r},$$

we obtain

$$\begin{aligned}F_m(-x) &= m\sum_{j=0}^{m}\binom{-x+m-j-1}{m-j}\binom{-x}{j} = m\sum_{j=0}^{m}\binom{-(x-j+1)}{j}\binom{-x}{m-j}\\ &= m\sum_{j=0}^{m}(-1)^j\binom{x}{j}\cdot(-1)^{m-j}\binom{x+m-j-1}{m-j} = (-1)^m F_m(x),\end{aligned}$$

as desired.

Editorial comment. In addition to the polynomials studied above, solvers used induction, contour integration, generating function manipulations, or primitive Dirichlet characters.

There is a direct combinatorial proof of the needed identity

$$\sum_{j=1}^{m}2^j\binom{m-1}{j-1}\binom{x}{j} = \sum_{j=0}^{m}\binom{x+m-j-1}{m-j}\binom{x}{j}$$

in the proof given above. Both sides count the distinguishable ways to place m balls in x boxes, where balls may be black or white, with each box having at most one white ball but any number of black balls. On the left side, j is the number of boxes that have balls: Pick the boxes, distribute the balls with a positive number in each chosen box, and decide for each chosen box whether to make one of the balls white. On the right side, j is the number of white balls: Pick boxes for them, and independently distribute $m - j$ black balls into the x boxes with repetition allowed.

Also solved by N. Hodges (UK), O. Kouba (Syria), P. Lalonde (Canada), A. Stadler (Switzerland), J. Wangshinghin (Canada), and the proposer.

A Limit Related to the Basel Problem

12220 [2020, 944]. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $a_n = \sum_{k=1}^{n} 1/k^2$ and $b_n = \sum_{k=1}^{n} 1/(2k-1)^2$. Prove

$$\lim_{n\to\infty} n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{3}{\pi^2}.$$

Solution by Charles Curtis, Missouri Southern State University, Joplin, MO. Note that

$$b_n = \sum_{k=1}^{2n} \frac{1}{k^2} - \frac{1}{4}\sum_{k=1}^{n} \frac{1}{k^2} = \frac{3}{4}\sum_{k=1}^{n} \frac{1}{k^2} + \sum_{k=n+1}^{2n} \frac{1}{k^2} = \frac{3}{4}a_n + \sum_{k=n+1}^{2n} \frac{1}{k^2}.$$

Therefore

$$n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{n}{a_n}\sum_{k=n+1}^{2n} \frac{1}{k^2} = \frac{n}{a_n}\sum_{k=1}^{n} \frac{1}{(n+k)^2} = \frac{1}{a_n}\left[\frac{1}{n}\sum_{k=1}^{n} \frac{1}{(1+k/n)^2}\right].$$

It is well known that a_n converges to $\pi^2/6$ (this is often called the *Basel problem*). The expression in square brackets can be interpreted as a Riemann sum, yielding

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{(1+k/n)^2} = \int_1^2 \frac{1}{x^2}\,dx = \frac{1}{2}.$$

Hence we get the desired result.

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), F. R. Ataev (Uzbekistan), M. Bataille (France), N. Batir (Turkey), A. Berkane (Algeria), N. Bhandari (Nepal), R. Boukharfane (Morocco), P. Bracken, B. Bradie, V. Brunetti & J. Garofali & A. Aurigemma (Italy), F. Chamizo (Spain), H. Chen, C. Chiser (Romania), G. Fera (Italy), D. Fleischman, O. Geupel (Germany), D. Goyal (India), N. Grivaux (France), J. A. Grzesik, L. Han, J.-L. Henry (France), E. A. Herman, N. Hodges (UK), F. Holland (Ireland), R. Howard, W. Janous (Austria), O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), G. Lavau (France), S. Lee, P. W. Lindstrom, O. P. Lossers (Netherlands), C. J. Lungstrom, J. Magliano, R. Molinari, A. Natian, S. Omar (Morocco), M. Omarjee (France), M. Reid, S. Sharma (India), J. Singh (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, M. Tang, R. Tauraso (Italy), D. Terr, D. B. Tyler, D. Văcaru (Romania), J. Vinuesa (Spain), M. Vowe (Switzerland), J. Wangshinghin (Canada), T. Wiandt, Q. Zhang (China), Missouri State University Problem Solving Group, and the proposer.

A Logarithmic Integral Evaluated by Residues

12221 [2020, 945]. *Proposed by Necdet Batır, Nevşehir Hacı Bektaş Veli University, Nevşehir, Turkey.* Prove

$$\int_0^1 \frac{\log(x^6+1)}{x^2+1}\,dx = \frac{\pi}{2}\log 6 - 3G,$$

where G is Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. Let I denote the requested integral. Writing I as a sum of two integrals and then making the change of variable $t = 1/x$ in the first integral, we obtain

$$I = \int_0^1 \frac{\log(1+1/x^6)}{1+x^2}\,dx + 6\int_0^1 \frac{\log x}{1+x^2}\,dx = \int_1^\infty \frac{\log(1+t^6)}{1+t^2}\,dt + 6\int_0^1 \frac{\log x}{1+x^2}\,dx,$$

and therefore

$$2I = \int_0^\infty \frac{\log(1+x^6)}{1+x^2}\,dx + 6\int_0^1 \frac{\log x}{1+x^2}\,dx.$$

To evaluate the last integral, we express $1/(1+x^2)$ as an infinite series:

$$\int_0^1 \frac{\log x}{1+x^2}\,dx = \int_0^1 \left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right)\log x\,dx.$$

Since the partial sums of the series are bounded in absolute value by 1, the dominated convergence theorem justifies interchanging the order of summation and integration, and then an integration by parts yields

$$\int_0^1 \frac{\log x}{1+x^2}\,dx = \sum_{k=0}^{\infty}(-1)^k \int_0^1 x^{2k}\log x\,dx = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)^2} = -G.$$

Thus,

$$2I = \int_0^{\infty} \frac{\log(1+x^6)}{1+x^2}\,dx - 6G,$$

so the required result follows from

$$\int_{-\infty}^{\infty} \frac{\log(1+x^6)}{1+x^2}\,dx = 2\pi\log 6, \tag{1}$$

which we now prove using the method of residues.

For $z = |z|e^{i\theta}$ with $|z| > 0$ and $-\pi < \theta \le \pi$, define $\operatorname{Log} z = \log|z| + i\theta$. The function $\operatorname{Log} z$ is analytic on the open upper half-plane. For $R > 1$ let C_R denote the contour $z = Re^{i\theta}$, $0 \le \theta \le \pi$. Let

$$P_1(z) = z + i, \quad P_2(z) = z - \sqrt{3}/2 + i/2, \quad \text{and} \quad P_3(z) = z + \sqrt{3}/2 + i/2.$$

For $j \in \{1, 2, 3\}$, the function $\operatorname{Log} P_j(z)$ is analytic on the closed upper half-plane, and therefore the residue theorem yields

$$\begin{aligned}\int_{-R}^{R} \frac{\operatorname{Log} P_j(x)}{1+x^2}\,dx + \int_{C_R} \frac{\operatorname{Log} P_j(z)}{1+z^2}\,dz &= 2\pi i \operatorname{Res}\left(\frac{\operatorname{Log} P_j(z)}{1+z^2}, i\right)\\ &= \pi \operatorname{Log} P_j(i).\end{aligned} \tag{2}$$

Since

$$\left|\int_{C_R} \frac{\operatorname{Log} P_j(z)}{1+z^2}\,dz\right| \le \pi R\frac{(\log(R+1)+\pi)}{R^2-1},$$

letting $R \to \infty$ in (2) and then taking the real part of the resulting identity yields

$$\int_{-\infty}^{\infty} \frac{\log|P_j(x)|}{1+x^2}\,dx = \pi\log|P_j(i)|.$$

Finally, since

$$\begin{aligned} x^6 + 1 &= \big(x^2+1\big)\big(x^2 - \sqrt{3}x + 1\big)\big(x^2 + \sqrt{3}x + 1\big)\\ &= \big(x^2+1\big)\big((x - \sqrt{3}/2)^2 + 1/4\big)\big((x + \sqrt{3}/2)^2 + 1/4\big)\\ &= |P_1(x)|^2|P_2(x)|^2|P_3(x)|^2,\end{aligned}$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\log(1+x^6)}{1+x^2}\,dx &= \sum_{j=1}^{3}\int_{-\infty}^{\infty}\frac{2\log|P_j(x)|}{1+x^2}\,dx\\ &= \sum_{j=1}^{3} 2\pi\log|P_j(i)| = 2\pi\big(\log 2 + \log\sqrt{3} + \log\sqrt{3}\big)\\ &= 2\pi\log 6,\end{aligned}$$

which completes the proof of (1).

Editorial comment. Several solvers noted that a similar problem appeared as problem 2107 in *Math. Mag.* 93 (2020), p. 389.

Also solved by U. Abel & V. Kushnirevych (Germany), F. R. Ataev (Uzbekistan), M. Bataille (France), A. Berkane (Algeria), N. Bhandari (Nepal), K. N. Boyadzhiev, P. Bracken, B. Bradie, V. Brunetti & J. Garofali & J. D'Aurizio (Italy), H. Chen, B. E. Davis, G. Fera (Italy), M. L. Glasser, R. Gordon, H. Grandmontagne (France), J. A. Grzesik, L. Han, D. Henderson, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), P. Khalili, O. Kouba (Syria), Z. Lin (China), O. P. Lossers (Netherlands), T. M. Mazzoli (Austria), M. Omarjee (France), V. Schindler (Germany), J. Singh (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), D. Văcaru (Romania), T. Wiandt, M. R. Yegan (Iran), and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

C7. *Contributed by Alan D. Taylor, Union College, Schenectady, NY.* Are the additive group of real numbers and the additive group of complex numbers isomorphic?

Random Tetrahedra Inscribed in a Sphere

C6. *Contributed by David Aldous, University of California, Berkeley, CA*. Consider four random points on the surface of a sphere, chosen uniformly and independently. Prove that the probability that the tetrahedron determined by the points contains the center of the sphere is $1/8$.

Solution. Assume the sphere is in $\mathbb{R}^3$ centered at the origin O. Fix the point P_4 and then choose P_1, P_2, P_3 by randomly choosing three diameters, D_1, D_2, and D_3, and then choosing, randomly, an end of each. There are eight ways to choose the endpoints. The probability conclusion follows from the observation that, for almost all choices of diameters, exactly one of the eight choices of endpoints yields a tetrahedron containing O.

To see this, assume that P_1, P_2, and P_3 are chosen so that no three of the points P_1, P_2, P_3, P_4 are linearly dependent as vectors in $\mathbb{R}^3$. (The opposite case has probability 0.) The equation $-P_4 = xP_1 + yP_2 + zP_3$ has a unique solution in nonzero real numbers x, y, and z. Write this as $O = xP_1 + yP_2 + zP_3 + P_4$. The eight choices of endpoints now correspond to the eight choices of signs in the expression $O = \pm xP_1 \pm yP_2 \pm zP_3 + P_4$. The tetrahedron contains O if and only if there is a representation $O = a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4$ where $a_i > 0$ for all i. This happens if and only if the coefficients $\pm x$, $\pm y$, $\pm z$ are all positive, and that occurs for exactly one of the eight equally likely choices.

Editorial comment. This was problem A6 on the 1992 Putnam Competition. For a geometric explanation of what is happening, see the 3blue1brown video "The hardest problem on the hardest test" at youtube.com/watch?v=OkmNXy7er84. In J. G. Wendel (1962), A problem in geometric probability, *Math. Scand.* 11: 109–111, it is proved that for k points on the sphere in $\mathbb{R}^n$, the probability $p_{n,k}$ that the convex hull of the points contains the origin is $\sum_{j=n}^{k-1}\binom{k-1}{j}/2^{k-1}$. A corollary is the surprising duality formula $p_{m,m+n} + p_{n,m+n} = 1$. According to Wendel, the problem goes back to R. E. Machol and was first solved by L. J. Savage.

Some further generalizations can be found in R. Howard and P. Sisson (1996), Capturing the origin with random points: Generalizations of a Putnam problem, *College Math. J.*, 27(3): 186–192.

SOLUTIONS

Dominated Convergence of an Integral

12207 [2020, 753]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $\int_0^1 f(x)\,dx = 1$. Evaluate

$$\lim_{n\to\infty} \frac{n}{\ln n} \int_0^1 x^n f(x^n) \ln(1-x)\,dx.$$

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. Substituting $t = x^n$, we get

$$\frac{n}{\ln n} \int_0^1 x^n f(x^n) \ln(1-x)\,dx = -\int_0^1 f(t) u_n(t)\,dt,$$

where

$$u_n(t) = -\frac{t^{1/n} \ln(1-t^{1/n})}{\ln n}.$$

For fixed $t \in (0, 1)$, letting $y = 1/n$ and applying L'Hôpital's rule twice yields

$$\lim_{n\to\infty} u_n(t) = \lim_{y\to 0^+} \frac{\ln(1-t^y)}{\ln y} = \lim_{y\to 0^+} \frac{t^y \ln t/(t^y-1)}{1/y} = \lim_{y\to 0^+} \frac{y \ln t}{t^y - 1} = \lim_{y\to 0^+} \frac{\ln t}{t^y \ln t} = 1.$$

Moreover, by Bernoulli's inequality, for $n \geq 3$ we have

$$0 \leq u_n(t) \leq -\frac{\ln(1-t^{1/n})}{\ln n} \leq -\frac{\ln((1-t)/n)}{\ln n} = 1 - \frac{\ln(1-t)}{\ln n} \leq 1 - \ln(1-t).$$

Since f is bounded and $\int_0^1 (1 - \ln(1-t))\,dt = 2 < \infty$, the dominated convergence theorem applies, and we conclude that

$$\lim_{n\to\infty} \frac{n}{\ln n} \int_0^1 x^n f(x^n) \ln(1-x)\,dx = -\lim_{n\to\infty} \int_0^1 f(t) u_n(t)\,dt = -\int_0^1 f(t)\,dt = -1.$$

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), C. Antoni (Italy), R. Boukharfane (Saudi Arabia), N. Caro (Brazil), R. Gordon, N. Grivaux (France), L. Han (USA) & X. Tang (China), E. A. Herman, N. Hodges (UK), F. Holland (Ireland), E. J. Ionaşcu, Y. Jinhai, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), A. Stadler (Switzerland), R. Stong, T. Wilde (UK), Y. Xiang (China), and the proposer.

Three Wise Women

12208 [2020, 753]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* (In memory of John Horton Conway, 1937–2020.) Three wise women, Alice, Beth, and Cecily, sit around a table. A card with a positive integer on it is attached to each woman's forehead, so she can see the other two numbers but not her own. The women know that one of the three integers is equal to the sum of the other two. The same question, "Can you determine the number on your forehead?", is addressed to the wise women in the following order: Alice, Beth, Cecily, Alice, Beth, Cecily, ... The answer is either "No" or "Yes, the number is ___," and the other wise women hear the answer. The questioning ends as soon as the positive answer is obtained. (Assume that the women are logical and honest, they all know this, they all know that they all know this, and so on.)
(a) Prove that whichever numbers are assigned to the wise women, an affirmative answer is obtained eventually.
(b) Suppose that Alice's second answer is "Yes, the number is 50." Determine the numbers assigned to Beth and Cecily.
(c) Suppose the numbers assigned to Alice, Beth, and Cecily are 1492, 1776, and 284, respectively. Determine who will give the affirmative answer and how many negative answers she will give before that.
Solution by Mark D. Meyerson, US Naval Academy, Annapolis, MD. We describe each assignment of numbers with a triple (a, b, c) giving Alice's, Beth's, and Cicely's positive numbers in that order. Note that one of the entries must be the sum of the other two.

We claim that for all triples, if a woman says "Yes" on some turn, then her number must be the largest. Suppose not, and choose a counterexample (a, b, c) for which the "Yes" answer occurs as early as possible. Suppose, for example, Alice says "Yes" on turn n, but Beth has the largest number, so $b = a + c$. (Other cases are similar.) Alice, seeing the numbers $a + c$ and c, knows from the beginning that her number must be either a or $a + 2c$. To say "Yes" on turn n, she must be able to rule out the triple $(a + 2c, a + c, c)$ for the first time on that turn, and this will happen only if either Cicely or Beth would have said "Yes" on turn $n - 1$ or $n - 2$ on that triple. But this is ruled out by the minimality of n, since neither Beth nor Cicely has the largest number in that triple.

Let f be the function that assigns to a triple the number of the turn on which the answer "Yes" occurs. Part (**a**) asks us to show that f is defined for every triple. If the triple has the form $(2x, x, x)$, for some positive integer x, then Alice will say "Yes" on her first turn, so $f(2x, x, x) = 1$. If it has the form $(x, 2x, x)$, then Alice will think she could have either x or $3x$, so she will say "No," and then Beth will say "Yes." Therefore $f(x, 2x, x) = 2$. Similarly, for triples of the form $(x, x, 2x)$, Cicely will say "Yes" on her first turn, and $f(x, x, 2x) = 3$.

Now consider triples in which the numbers are distinct. If some triple never yields an affirmative answer, then let (a, b, c) be such a triple whose largest element is as small as possible. If $c = a + b$, then $(a, b, |a - b|)$ has a smaller largest element, so $f(a, b, |a - b|)$ is defined. If $f(a, b, |a - b|) = n$, then on turn $n + 1$ or $n + 2$, depending on which of a or b is larger, Cecily can eliminate the triple $(a, b, |a - b|)$, since Alice or Beth would previously have said "Yes." Cecily then answers "Yes" with $a + b$ on her turn. The argument is similar when a or b is the largest entry in (a, b, c). This completes the solution to (**a**).

(**b**) Using the reasoning from part (**a**), we can now determine, for every n, the triples (a, b, c) for which $f(a, b, c) = n$. If $f(a, b, c) = 1$, then (a, b, c) must have the form $(2x, x, x)$, for some positive integer x. For $f(a, b, c) = 2$, we must have $b = a + c$. If $a = c$ then (a, b, c) has the form $(x, 2x, x)$. If not, then $f(a, |a - c|, c)$ must be 1, so $(a, |a - c|, c)$ has the form $(2x, x, x)$, and therefore $(a, b, c) = (2x, 3x, x)$. Thus, the triples (a, b, c) such that $f(a, b, c) = 2$ are those of the form $(x, 2x, x)$ or $(2x, 3x, x)$. If $f(a, b, c) = 3$, then $c = a + b$, and either (a, b, c) has the form $(x, x, 2x)$ or $f(a, b, |a - b|)$ is either 1 or 2, in which case (a, b, c) has the form $(2x, x, 3x)$, $(x, 2x, 3x)$, or $(2x, 3x, 5x)$. A similar argument shows that the triples (a, b, c) with $f(a, b, c) = 4$ are those of the form $(3x, 2x, x)$, $(4x, 3x, x)$, $(3x, x, 2x)$, $(4x, x, 3x)$, $(5x, 2x, 3x)$, or $(8x, 3x, 5x)$. Since 50 is not divisible by any number in $\{3, 4, 8\}$, the only way Alice will say "Yes, my number is 50" on her second turn ($n = 4$) is for x to be 10 in the fifth triple, so Beth has 20 and Cecily has 30.

(c) Working from $(1492, 1776, 284)$ to determine the turn on which that triple will be resolved, we iteratively replace the biggest number by the difference of the other two to undo the decision process. The successive triples after $(1492, 1776, 284)$ are these: $(1492, 1208, 284)$, $(924, 1208, 284)$, $(924, 640, 284)$, $(356, 640, 284)$, $(356, 72, 284)$, $(212, 72, 284)$, $(212, 72, 140)$, $(68, 72, 140)$, $(68, 72, 4)$, $(68, 64, 4)$, $(60, 64, 4)$, $(60, 56, 4)$, $(52, 56, 4)$, $(52, 48, 4)$, $(44, 48, 4)$, $(44, 40, 4)$, $(36, 40, 4)$, $(36, 32, 4)$, $(28, 32, 4)$, $(28, 24, 4)$, $(20, 24, 4)$, $(20, 16, 4)$, $(12, 16, 4)$, $(12, 8, 4)$, $(4, 8, 4)$. The last triple would be resolved by Beth on turn 2, the one before it by Alice on turn 4. Working backward, Yes comes on the following turns for these triples:

$$2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 27, 28, 30, 31, 32, 34, 35, 37, 38.$$

Since $38 = 3 \cdot 12 + 2$, the affirmative answer is by Beth after giving 12 negative answers. (Tracking only the two smaller entries in each triple, the decision process parallels the Euclidean algorithm.)

Also solved by E. Curtin, J. Boswell & C. Curtis, N. Hodges (UK), E. J. Ionaşcu, G. Lavau (France), O. P. Lossers (Netherlands), K. Schilling, E. Schmeichel, R. Stong, F. A. Velandia & J. F. Gonzalez (Colombia), T. Wilde (UK), Eagle Problem Solvers, The Zurich Logic Coffee (Switzerland), and the proposer.

Asymptotics of a Recursively Defined Sequence

12210 [2020, 852]. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.* Let $x_1 = 1$, and let

$$x_{n+1} = \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right)^2$$

when $n \geq 1$. For $n \in \mathbb{N}$, let $a_n = 2n + (1/2)\log n - x_n$. Show that the sequence $a_1, a_2, \ldots$ converges.

Solution by Peter W. Lindstrom, Saint Anselm College, Manchester, NH. By the recurrence for x_n, we have $x_{n+1} = x_n + 2 + 1/x_n > x_n + 2$, and therefore by induction $x_n \geq 2n$ when $n > 1$.

Let $z_k = x_k - 2k$. Since $z_{k+1} - z_k = x_{k+1} - x_k - 2 = 1/x_k$, we have

$$z_n = z_1 + \sum_{k=1}^{n-1}(z_{k+1} - z_k) = -1 + \sum_{k=1}^{n-1} \frac{1}{x_k}$$

for $n > 1$. Thus

$$0 \le \frac{1}{2k} - \frac{1}{x_k} = \frac{z_k}{2kx_k} = \frac{\sum_{j=1}^{k-1} 1/x_j - 1}{2kx_k} \le \frac{\sum_{j=2}^{k-1} 1/x_j}{(2k)^2} \le \frac{(1/2)\sum_{j=2}^{k-1} 1/j}{4k^2} < \frac{\log k}{8k^2}$$

for $k > 2$. Since $\sum_{k=1}^{\infty} \log k/(8k^2)$ is convergent, so is $\sum_{k=1}^{\infty}(1/(2k) - 1/x_k)$. Let

$$\zeta = \sum_{k=1}^{\infty}\left(\frac{1}{2k} - \frac{1}{x_k}\right).$$

For $n > 1$,

$$\begin{aligned} a_n = 2n + \frac{\log n}{2} - x_n = -z_n + \frac{\log n}{2} &= 1 - \sum_{k=1}^{n-1} \frac{1}{x_k} + \frac{\log n}{2} \\ &= 1 - \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) + \sum_{k=1}^{n-1}\left(\frac{1}{2k} - \frac{1}{x_k}\right) + \frac{1}{2n}. \end{aligned}$$

Thus $\lim_{n\to\infty} a_n = 1 - \gamma/2 + \zeta$, where γ is the Euler–Mascheroni constant.

Also solved by G. Aggarwal (India), K. F. Andersen (Canada), M. Bataille (France), R. Boukharfane (Saudi Arabia), H. Chen, C. Chiser (Romania), Ó. Ciaurri (Spain), C. Degenkolb, A. Dixit (India) & S. Pathak (USA), G. Fera (Italy), J. Freeman (Netherlands), R. Gordon, J.-P. Grivaux (France), L. Han, R. Hang, D. Henderson, E. A. Herman, N. Hodges (UK), Y. Jinhai (China), O. Kouba (Syria), Z. Lin (China), J. H. Lindsey II, O. P. Lossers (Netherlands), S. Omar (Morocco), M. Omarjee (France), P. Palmieri & C. Antoni (Italy), A. Pathak (India), R. K. Schwartz, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. Terr, D. B. Tyler, E. I. Verriest, J. Vukmirović (Serbia), T. Wiandt, L. Wimmer (Germany), L. Zhou, and the proposer.

A Truncated Tetrahedron

12211 [2020, 852]. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania.* On each of the six edges of a tetrahedron, identify the point that is coplanar with the incenter of the tetrahedron and with the two vertices incident to the opposite edge. Prove that the volume of the octahedron formed by these six points is no more than half the volume of the tetrahedron, and determine the conditions for equality.

Solution by Elton Bojaxhiu, Tirana, Albania, and Enkel Hysnelaj, Sydney, Australia. Let A, B, C, and D be the vertices of the tetrahedron, and let w, x, y, and z denote the areas of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$, respectively.

Let p_{AB} be the plane passing through C, D, and the incenter of the tetrahedron, and let P_{AB} denote the intersection of p_{AB} with AB. Let h_A and h_B be the altitudes from A and B, respectively, to the line CD, and let d_A and d_B be the distances from A and B, respectively, to the plane p_{AB}. Since p_{AB} bisects the angle between the planes containing $\triangle ACD$ and $\triangle BCD$, we have

$$\frac{AP_{AB}}{BP_{AB}} = \frac{d_A}{d_B} = \frac{h_A}{h_B} = \frac{y}{z}.$$

Similarly, if P_{AC}, P_{AD}, P_{BC}, P_{BD}, and P_{CD} are the vertices of the octahedron that lie on the other edges of the tetrahedron, then we have

$$\frac{AP_{AC}}{CP_{AC}} = \frac{x}{z}, \quad \frac{AP_{AD}}{DP_{AD}} = \frac{w}{z}, \quad \frac{BP_{BC}}{CP_{BC}} = \frac{x}{y}, \quad \frac{BP_{BD}}{DP_{BD}} = \frac{w}{y}, \quad \text{and} \quad \frac{CP_{CD}}{DP_{CD}} = \frac{w}{x}.$$

The octahedron is constructed from the tetrahedron $ABCD$ by removing the four smaller tetrahedra $AP_{AB}P_{AC}P_{AD}$, $BP_{AB}P_{BC}P_{BD}$, $CP_{AC}P_{BC}P_{CD}$, and $DP_{AD}P_{BD}P_{CD}$. If t is the volume of the tetrahedron $ABCD$ and t_A is the volume of $AP_{AB}P_{AC}P_{AD}$, then

$$\frac{t_A}{t} = \frac{AP_{AD}}{AD} \cdot \frac{AP_{AC}}{AC} \cdot \frac{AP_{AB}}{AB} = \frac{w}{w+z} \cdot \frac{x}{x+z} \cdot \frac{y}{y+z}.$$

Combining this with similar formulas for the other small tetrahedra, we see that it suffices to show

$$\frac{wxy}{(w+z)(x+z)(y+z)}+\frac{wxz}{(w+y)(x+y)(z+y)} \\ +\frac{wyz}{(w+x)(y+x)(z+x)}+\frac{xyz}{(x+w)(y+w)(z+w)}\geq\frac{1}{2}. \qquad (*)$$

Let a, b, c, and d denote the elementary symmetric polynomials in w, x, y, and z:

$$\begin{aligned} a &= w+x+y+z, \\ b &= wx+wy+wz+xy+xz+yz, \\ c &= wxy+wxz+wyz+xyz, \\ d &= wxyz. \end{aligned}$$

By multiplying out and rearranging, we find that $(*)$ is equivalent to

$$abc-5a^2d\geq c^2.$$

From Newton's inequalities for the elementary symmetric polynomials, we have $(a/4)(c/4)\leq(b/6)^2$ and $(b/6)d\leq(c/4)^2$. Consequently,

$$b\geq\frac{3\sqrt{ac}}{2} \quad \text{and} \quad d\leq\frac{3c^2}{8b}\leq\frac{3c^2}{12\sqrt{ac}}=\frac{c^{3/2}}{4\sqrt{a}}.$$

Also, by Maclaurin's inequality, $a/4\geq\sqrt[3]{c/4}$, so $a^{3/2}\geq 4\sqrt{c}$. Therefore

$$abc-5a^2d\geq ac\cdot\frac{3\sqrt{ac}}{2}-5a^2\cdot\frac{c^{3/2}}{4\sqrt{a}}=\frac{a^{3/2}c^{3/2}}{4}\geq\frac{4\sqrt{c}\cdot c^{3/2}}{4}=c^2,$$

as required.

Equality holds if and only if $w=x=y=z$; that is, all faces of the tetrahedron have the same area. It is well known that this is true precisely when the tetrahedron is *isosceles*, which means that each pair of opposite edges have the same length.

Editorial comment. There are several other ways to establish $(*)$, as indicated by multiple solvers. For instance, one could cite Muirhead's inequality; alternatively, assume without loss of generality that $w\leq x\leq y\leq z$, write $x=w+s$, $y=w+s+t$, and $z=w+s+t+u$ for $s,t,u\geq 0$, and note that expanding and rearranging $(*)$ yields $f(w,s,t,u)\geq 0$, where f is a polynomial with all nonnegative coefficients.

Also solved by C. Curtis, G. Fera (Italy), O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Stong, J. Vukmirović, and the proposer.

An Application of Farkas's Lemma

12212 [2020, 852]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $x_1,\ldots,x_m$ and $y_1,\ldots,y_m$ be two lists of m vectors in $\mathbb{R}^n$, and suppose

$$\langle x_i-x_j, y_i-y_j\rangle\geq 0$$

for all i and j in $\{1,\ldots,m\}$. Prove that there exists a vector y in $\mathbb{R}^n$ such that

$$\langle x_i, y_i\rangle\geq\langle x_i, y\rangle$$

for all i in $\{1,\ldots,m\}$.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. The following is a variant of Farkas's lemma (see for example Corollary 7.1(e) in A. J. Schrijver, *Theory of Linear and Integer Programming*, John Wiley and Sons, Chichester, UK, 1986).

> If A is a p-by-q matrix, and $b \in \mathbb{R}^p$, then exactly one of the following two assertions is true:
>
> (1) The system $Au \le b$ has a solution $u \in \mathbb{R}^q$.
>
> (2) The system $v^T A = 0$ has a solution $v \in \mathbb{R}^p$ with $v \ge 0$ and $v^T b < 0$.

Let X and Y be the n-by-m matrices that have the vectors x_i and y_i, respectively, for their columns. Let $A = X^T Y$; in particular, the (i, j)-entry of A is $\langle x_i, y_j \rangle$. Let b be the vector consisting of the main diagonal entries of A. If some vector u satisfies $Au \le b$, then the vector y defined by

$$y = Yu = \sum_{j=1}^{m} u_j y_j$$

has the desired property, because

$$\langle x_i, y \rangle = \sum_j u_j \langle x_i, y_j \rangle = \sum_j u_j a_{i,j} = (Au)_i \le b_i = \langle x_i, y_i \rangle.$$

If there is no such vector u, then by the variant of Farkas's lemma there exists $v \in \mathbb{R}^m$ such that $v^T A = 0$ with $v \ge 0$ and $v^T b < 0$. The condition $\langle x_i - x_j, y_i - y_j \rangle \ge 0$ expands to the condition $a_{ii} - a_{ij} - a_{ji} + a_{jj} \ge 0$ on the entries of A. Hence,

$$\begin{aligned}
0 &\le \sum_{i,j} v_i v_j (a_{ii} - a_{ij} - a_{ji} + a_{jj}) \\
&= \sum_j v_j \sum_i v_i a_{ii} - \sum_j v_j \sum_i v_i a_{ij} - \sum_i v_i \sum_j v_j a_{ji} + \sum_i v_i \sum_j v_j a_{jj} \\
&= \sum_j v_j v^T b - \sum_j v_j 0 - \sum_i v_i 0 + \sum_i v_i v^T b = 2v^T b \sum_i v_i < 0,
\end{aligned}$$

which is a contradiction.

Also solved by R. Stong and the proposer.

A Sum of Tails of the Zeta Function

12215 [2020, 853]. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+4)^2} + \cdots \right) - \frac{1}{2n} \right).$$

Solution by Gaurav Aggarwal, student, Guru Nanak Dev University, Amritsar, India. The sum equals $\pi^2/16 + 1/2$. Let

$$S_N = \sum_{n=1}^{N} \left(\left(\frac{1}{n^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+4)^2} + \cdots \right) - \frac{1}{2n} \right).$$

The term

$$\left(\frac{1}{n^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+4)^2}+\cdots\right)-\frac{1}{2n}$$

clearly approaches 0 as n approaches infinity, since the part in parentheses is bounded by $\sum_{k=n}^{\infty} 1/k^2$, which itself goes to 0. Therefore, it suffices to prove

$$\lim_{N\to\infty} S_{2N} = \pi^2/16 + 1/2.$$

We compute

$$\begin{aligned} S_{2N} &= \sum_{i=1}^{N} i\left(\frac{1}{(2i-1)^2}+\frac{1}{(2i)^2}\right) + N\sum_{i=2N+1}^{\infty}\frac{1}{i^2} - \sum_{i=1}^{2N}\frac{1}{2i} \\ &= \sum_{i=1}^{N}\left(\frac{i}{(2i-1)^2}+\frac{i}{(2i)^2}-\frac{1}{2(2i-1)}-\frac{1}{2(2i)}\right) + N\sum_{i=2N+1}^{\infty}\frac{1}{i^2} \\ &= \sum_{i=1}^{N}\frac{1}{2(2i-1)^2} + N\sum_{i=2N+1}^{\infty}\frac{1}{i^2}. \end{aligned}$$

Noting that $\zeta(2) = \pi^2/6$, where ζ is the Riemann zeta function, we have

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{1}{2(2i-1)^2} = \frac{1}{2}\left(1-\frac{1}{2^2}\right)\zeta(2) = \frac{\pi^2}{16}.$$

We use telescoping series again and the squeeze theorem to show that the remaining term tends to $1/2$:

$$\begin{aligned} \frac{N}{2N+1} &= N\sum_{i=2N+1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+1}\right) = N\sum_{i=2N+1}^{\infty}\frac{1}{i(i+1)} < N\sum_{i=2N+1}^{\infty}\frac{1}{i^2} \\ &< N\sum_{i=2N+1}^{\infty}\frac{1}{(i-1)i} = N\sum_{i=2N+1}^{\infty}\left(\frac{1}{i-1}-\frac{1}{i}\right) = \frac{N}{2N} = \frac{1}{2}. \end{aligned}$$

Hence $\lim_{N\to\infty} S_N = \lim_{N\to\infty} S_{2N} = \pi^2/16 + 1/2$.

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), R. Boukharfane (Saudi Arabia), K. N. Boyadzhiev, P. Bracken, B. Bradie, V. Brunetti & A. Aurigemma & G. Bramanti & J. D'Aurizio & D. B. Malesani (Italy), B. S. Burdick, H. Chen, C. Curtis, T. Dickens, G. Fera (Italy), M. L. Glasser, H. Grandmontagne (France), J.-P. Grivaux (France), J. A. Grzesik, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), Y. Jinhai (China), O. Kouba (Syria), K.-W. Lau (China), G. Lavau (France), O. P. Lossers (Netherlands), R. Molinari, A. Natian, M. Omarjee (France), P. Palmieri (Italy), K. Schilling, A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), D. Terr, D. B. Tyler, J. Vukmirović (Serbia), T. Wiandt, Y. Xiang (China), FAU Problem Solving Group, Missouri State Problem Solving Group, and the proposer.

Rotating an Icosahedron

12216 [2020, 944]. *Proposed by Zachary Franco, Houston, TX.* A regular icosahedron with volume 1 is rotated about an axis connecting opposite vertices. What is the volume of the resulting solid?

Solution by Albert Stadler, Herrliberg, Switzerland. It is known (see for example en.wikipedia.org/wiki/Regular_icosahedron) that if the edge length of a regular icosahedron is a, then the radius of the circumscribed sphere is

$$R = \frac{a}{4}\sqrt{10 + 2\sqrt{5}},$$

while the volume is

$$V = \frac{5}{12}(3 + \sqrt{5})a^3.$$

We place the icosahedron in $\mathbb{R}^3$ in such a way that its 12 vertices have the following coordinates:

$$\begin{aligned}
P_1 &: (0, 0, R),\\
P_2\text{–}P_6 &: \frac{R}{\sqrt{5}}\left(2\cos\left(\frac{2k\pi}{5}\right), 2\sin\left(\frac{2k\pi}{5}\right), 1\right), \quad k \in \{0, \dots, 4\},\\
P_7\text{–}P_{11} &: \frac{R}{\sqrt{5}}\left(2\cos\left(\frac{(2k+1)\pi}{5}\right), 2\sin\left(\frac{(2k+1)\pi}{5}\right), -1\right), \quad k \in \{0, \dots, 4\},\\
P_{12} &: (0, 0, -R).
\end{aligned}$$

The segment connecting the two points P_2 and P_7 is given by

$$s(t) = \frac{R}{\sqrt{5}}\left[t\left(2, 0, 1\right) + (1-t)\left(2\cos\left(\frac{\pi}{5}\right), 2\sin\left(\frac{\pi}{5}\right), -1\right)\right], \quad 0 \le t \le 1.$$

This segment generates the boundary of the middle part of the solid formed when the icosahedron is rotated about the z-axis. The other two parts are cones whose boundaries are generated by rotating the segment connecting P_1 and P_2 and the segment connecting P_7 and P_{12}.

The distance of $s(t)$ from the z-axis equals

$$\frac{R}{\sqrt{5}}\left\|t\left(2, 0, 0\right) + (1-t)\left(2\cos\left(\frac{\pi}{5}\right), 2\sin\left(\frac{\pi}{5}\right), 0\right)\right\| = R\sqrt{\frac{4 - 2(3-\sqrt{5})t(1-t)}{5}}.$$

Therefore, the volume of the rotated icosahedron equals

$$V_{\text{rot}} = \frac{2}{3}\pi\left(R - \frac{R}{\sqrt{5}}\right)\left(\frac{2R}{\sqrt{5}}\right)^2 + \pi R^2\frac{2R}{\sqrt{5}}\int_0^1\left(\frac{4 - 2(3-\sqrt{5})t(1-t)}{5}\right)dt.$$

The first term in this formula is the volume of the two cones, and the second is the volume of the middle part. Evaluating the integral and simplifying we obtain

$$V_{\text{rot}} = \frac{2}{15}(5 + \sqrt{5})\pi R^3 = \frac{\sqrt{2}}{240}\left(5 + \sqrt{5}\right)^{5/2}\pi a^3.$$

If the volume of the icosahedron is 1, then a is determined by

$$a^3 = \frac{12}{5(3 + \sqrt{5})}.$$

Substituting this into our formula for V_{rot} gives a volume of

$$V_{\text{rot}} = \frac{\pi}{5}\sqrt{\frac{5 + \sqrt{5}}{2}} \approx 1.19513.$$

Also solved by F. Chamizo (Spain), C. Curtis & J. Boswell, G. Fera (Italy), O. Geupel (Germany), J.-P. Grivaux (France), N. Hodges (UK), M. J. Knight, G. Lavau (France), O. P. Lossers (Netherlands), M. D. Meyerson, R. Stong, D. Terr, T. Wiandt, L. Zhou, Davis Problem Solving Group, Eagle Problem Solvers, and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

C6. *Due to R. E. Machol and L. J. Savage, contributed by David Aldous, University of California, Berkeley, CA*. Consider four random points on the surface of a sphere, chosen uniformly and independently. Prove that the probability that the tetrahedron determined by the points contains the center of the sphere is $1/8$.

The Affine Hull of Four Points in Space

C5. *Contributed by the editors*. Given a set S in $\mathbb{R}^n$, let $L(S)$ be the set of all points lying on some line determined by two points in S. For example, if S is the set of vertices of an equilateral triangle in $\mathbb{R}^2$, then $L(S)$ is the union of the three lines that extend the sides of the triangle, and $L(L(S))$ is all of $\mathbb{R}^2$. If S is the set of vertices of a regular tetrahedron, then what is $L(L(S))$?

Solution. There are precisely four points that are not in $L(L(S))$. Inscribe the tetrahedron in a cube with the vertices of the tetrahedron at four of the corners of the cube. The four other corners of the cube are the missing points.

To see that these points are missed, observe that $L(S)$ consists of all the points on the extended edges of the tetrahedron. A line through points on adjacent extended edges lies in the plane of a tetrahedral face and so misses the unused corners. Also, a line connecting one such corner to a nearby extended edge of the tetrahedron lies in the plane of a face of the cube and so misses any of the skew edges.

We now show that all other points in $\mathbb{R}^3$ are included. Let P_1 be the plane containing the top face of the cube and let P_2 be the plane containing the bottom face. Let l_1 and l_2 be the tetrahedral edges lying in P_1 and P_2, respectively. Notice that P_1 is the unique plane containing l_1 that is parallel to l_2, and similarly for P_2. Suppose that Q is a point that does not lie on either P_1 or P_2. Let P be the plane containing Q and l_1. Since Q does not lie on P_1, P is not equal to P_1, so it is not parallel to l_2. Therefore it intersects l_2, say at R. The line QR lies in the plane P, which contains l_1. Since Q does not lie on P_2, QR is not parallel to l_1. Therefore QR must intersect l_1, say at T. But now Q, R, and T are collinear, so Q is in $L(L(S))$.

This argument shows that $L(L(S))$ contains all points that do not lie in either the plane of the top of the cube or the plane of the bottom. Similarly, it contains all points that do not lie on either the plane of the left side or the right side, and all points that do not lie on either the plane of the front or back. This means that the only points that can be missed are the corners of the cube.

Editorial comment. The problem was proposed by Victor Klee as Problem 1413 in *Math. Mag.* 66 (1993) 56, with solution in *Math. Mag.* 67 (1993) 68–69. See also V. Klee (1963), The generation of affine hulls, *Acta Scient. Math. (Szeged)* 24, 60–81.

SOLUTIONS

Non-divisors of Translated Sums of Squares

12200 [2020, 660]. *Proposed by Ibrahim Suat Evren, Denizli, Turkey.* Prove that for every positive integer m, there is a positive integer k such that k does not divide $m + x^2 + y^2$ for any positive integers x and y.

Solution by Peter W. Lindstrom, Saint Anselm College, Manchester, NH. We prove that $4m^2$ has the desired property. Let $k = 4m^2$, and let c be a positive integer, so $ck - m = m(4cm - 1)$. Since $4cm - 1 \equiv -1 \pmod 4$, the prime factorization of $4cm - 1$ must have an odd power of a prime p with $p \equiv -1 \pmod 4$. Also, since m and $4cm - 1$ are relatively prime, p cannot divide m, so the prime factorization of $ck - m$ has p to an odd power.

The "sum of two squares" theorem in number theory states that the prime factorization of a number of the form $x^2 + y^2$ has even exponent for each prime congruent to $-1 \pmod 4$. Hence no integers c, x, and y satisfy $x^2 + y^2 + m = ck$. This makes it impossible for k to divide $x^2 + y^2 + m$ for any integers x and y.

Also solved by R. Boukharfane (Saudi Arabia), R. Chapman (UK), C. Curtis & J. Boswell, S. M. Gagola Jr., N. Hodges (UK), E. J. Ionaşcu, Y. J. Ionin, J. S. Liu, O. P. Lossers (Netherlands), S. Miao (China), C. R. Pranesachar (India), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), K. Williams (Canada), L. Zhou, FAU Problem Solving Group, and the proposer.

A Large Vector Sum from Probability or Polygons

12202 [2020,752]. *Proposed by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China.* Let V be a finite set of vectors in $\mathbb{R}^2$ such that $\sum_{v\in V} |v| = \pi$. Prove that there exists a subset U of V such that $|\sum_{v\in U} v| \geq 1$.

Solution I by Oliver Geupel, Brühl, Germany. Choose at random a ray h starting from the origin. For $v \in V$, let X_v be the length of the projection of v onto h if the angle between them is acute, and 0 otherwise. The expected value of X_v is

$$E[X_v] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} |v|\cos\phi\, d\phi = \frac{|v|}{\pi}.$$

Therefore $E[\sum_{v\in V} X_v] = \sum_{v\in V} E[X_v] = 1$, so there is some ray h such that $\sum_{v\in V} X_v \geq 1$. We can now let $U = \{v \in V : \text{the angle between } h \text{ and } v \text{ is acute}\}$.

Solution II by Elton Bojaxhiu, Tirana, Albania, and Enkel Hysnelaj, Sydney, Australia. Let $V = \{v_1, \dots, v_n\}$, and define v_{n+1} so that $v_1 + \cdots + v_{n+1} = 0$. For any vector v, let $\theta(v)$ be the angle from the positive x-axis to v, with $0 \leq \theta(v) < 2\pi$, and let $v_1', \dots, v_{n+1}'$ be a permutation of $v_1, \dots, v_{n+1}$ such that $\theta(v_1') \leq \cdots \leq \theta(v_{n+1}')$. The endpoints of the partial sums $\sum_{i=1}^r v_i'$ form the vertices of a (possibly degenerate) convex polygon. Let p and d be the perimeter and diameter of this polygon; it is known that $p < \pi d$. Thus

$$\pi = \sum_{v\in V} |v| \leq \sum_{k=1}^{n+1} |v_k| = p < \pi d,$$

so $d > 1$. The set U can be chosen to be a collection of vectors (not including v_{n+1}) whose sum gives a diameter of the polygon.

Editorial comment. Kevin Byrnes and Nicolás Caro pointed out that this problem appears as exercise 14.9 in J. Michael Steele (2004), *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, Cambridge: Cambridge Univ. Press, and also in W. W. Bledsoe (1970), An inequality about complex numbers, this MONTHLY 77, pp. 180–182. If p and d are the perimeter and diameter of a convex m-gon, then the inequality $p < \pi d$ follows from $p \leq 2m \sin(\pi/(2m))d$, proved in H. Sedrakyan and N. Sedrakyan (2017), *Geometric Inequalities: Methods of Proving*, Cham, Switzerland: Springer, p. 379. Radouan Boukharfane and Tom Wilde extended the problem to $\mathbb{R}^n$, where the constant π generalizes to $2\sqrt{\pi}\,\Gamma((n+1)/2)/\Gamma(n/2)$.

Also solved by R. Boukharfane (Saudi Arabia), K. M. Byrnes, N. Caro (Brazil), R. Chapman (UK), R. Frank (Germany), Y. J. Ionin, Y. Jeong (Korea), J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, K. Schilling, E. Schmeichel, R. Stong, R. Tauraso (Italy), T. Wilde (UK), and the proposer.

A Family of Sums with Logarithmic Powers

12203 [2020, 752]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Let m be a nonnegative integer, and let μ be the Möbius function on $\mathbb{Z}^+$, defined by setting $\mu(k)$ equal to $(-1)^r$ if k is the product of r distinct primes and equal to 0 if k has a square prime factor. Evaluate

$$\lim_{n\to\infty} \frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right).$$

Solution by Albert Stadler, Herrliberg, Switzerland. The limit is $m+1$.

For a fixed $j \geq 1$, we show that there is a positive constant c such that

$$\sum_{k=1}^{n} \frac{\mu(k)}{k}(-1)^j \ln^j k = \frac{d^j}{ds^j}\frac{1}{\zeta(s)}\bigg|_{s=1} + O\left(e^{-c\sqrt{\ln n}}\right), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta function. We start with

$$\frac{d^j}{ds^j}\frac{1}{\zeta(s)} - \sum_{k=1}^{n} \frac{\mu(k)}{k^s}(-1)^j \ln^j(k) = \sum_{k=n+1}^{\infty} \frac{\mu(k)}{k^s}(-1)^j \ln^j(k) \tag{2}$$

for $s > 1$, which follows from Dirichlet's expansion of $1/\zeta(s)$. We now show that (2) holds also in the case $s = 1$.

Let $M(k) = \sum_{i=1}^{k} \mu(i)$. The function M is known as the Mertens function. Partial summation yields

$$\sum_{k=n+1}^{\infty} \frac{\mu(k)}{k^s}(-1)^j \ln^j(k) = \sum_{k=n+1}^{\infty} \frac{M(k)}{k^s}(-1)^j \ln^j(k) - \sum_{k=n}^{\infty} \frac{M(k)}{(k+1)^s}(-1)^j \ln^j(k+1)$$

$$= \frac{M(n)}{n^s}(-1)^{j+1} \ln^j(n) + \sum_{k=n}^{\infty} M(k)(-1)^j \left(\frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s} \right).$$

For $s \geq 1$ and $x > e^j$,

$$\frac{d}{dx}\frac{\ln^j(x)}{x^s} = \frac{\ln^j(x)}{x^{s+1}}\left(\frac{j}{\ln x} - s\right) < 0.$$

Moreover,

$$\frac{d}{dx}\frac{\ln^j(x)}{x^s} > -s\,\frac{\ln^j(x)}{x^{s+1}},$$

with the latter increasing in x. Thus, by the mean value theorem,

$$\left| \frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s} \right| < s\,\frac{\ln^j(k)}{k^{s+1}} \leq 2\,\frac{\ln^j(k)}{k^{s+1}}$$

for $1 \leq s \leq 2$ and $k > e^j$. Since $M(k) = O\left(ke^{-2c\sqrt{\ln k}}\right)$ for a suitable positive constant c (see, for instance, E. Landau (1974), *Handbuch der Lehre von der Verteilung der Primzahlen*, v. 2, AMS Chelsea Publishing: Providence, p. 570) and since $\ln^{j+2}(k) = O\left(e^{c\sqrt{\ln k}}\right)$, we have

$$\left| M(k)(-1)^j \left(\frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s} \right) \right| = O\left(e^{-c\sqrt{\ln k}}\,\frac{1}{k\ln^2(k)} \right).$$

From this we deduce

$$\left| \frac{M(n)}{n^s}(-1)^{j+1} \ln^j(n) + \sum_{k=n}^{\infty} M(k)(-1)^j \left(\frac{\ln^j(k)}{k^s} - \frac{\ln^j(k+1)}{(k+1)^s} \right) \right|$$

$$= O\left(e^{-c\sqrt{\ln n}}\right) + \sum_{k=n}^{\infty} O\left(e^{-c\sqrt{\ln k}}\,\frac{1}{k\ln^2(k)} \right)$$

$$= O\left(e^{-c\sqrt{\ln n}}\right) + O\left(e^{-c\sqrt{\ln n}}\frac{1}{\ln n} \right) = O\left(e^{-c\sqrt{\ln n}}\right).$$

The convergence of the series is uniform for $s \in [1, 2]$, so both sides of (2) are continuous on $[1, 2]$. Therefore, (2) is valid at $s = 1$, proving (1).

We conclude

$$\frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right) = \frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} (\ln n - \ln k)^{m+1}$$

$$= \frac{1}{\ln^m(n)} \sum_{j=0}^{m+1} \binom{m+1}{j} \ln^{m+1-j}(n) \sum_{k=1}^{n} \frac{\mu(k)}{k}(-1)^j \ln^j(k)$$

$$= \frac{1}{\ln^m(n)} \sum_{j=0}^{m+1} \binom{m+1}{j} \ln^{m+1-j}(n) \left(\left.\frac{d^j}{ds^j}\frac{1}{\zeta(s)}\right|_{s=1} + O\left(e^{-c\sqrt{\ln n}}\right) \right).$$

As $n \to \infty$, all error terms have limit 0. Since $\zeta(s)$ is meromorphic with a simple pole of residue 1 at $s = 1$, the function $1/\zeta(s)$ is holomorphic at $s = 1$, and its Taylor series expansion begins $(s-1) + \cdots$. The main term vanishes for $j = 0$ and has limit 0 for $j > 1$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{1}{\ln^m(n)} \sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right) = \binom{m+1}{1} \frac{d}{ds} \frac{1}{\zeta(s)}\bigg|_{s=1} = m + 1.$$

Editorial comment. The proof of the bound on the Mertens function is similar to one for the prime number theorem. Some solvers used other bounds, shortening their solutions. Bounds on sums of the form $\sum_{k=1}^{n} \mu(k) \ln^q(k)/k$ (Landau, pp. 568–570, 594–595) allow one to begin with the binomial expansion of $\ln n - \ln k$. For $m > 0$, the solution follows immediately from

$$\sum_{k=1}^{n} \frac{\mu(k)}{k} \ln^{m+1}\left(\frac{n}{k}\right) = (m+1)\ln^m(n) + \sum_{k=1}^{m-1} c_k(m) \ln^k(n) + O(1),$$

which appears on p. 489 of H. N. Shapiro (1950), On a theorem of Selberg and generalizations, *Ann. Math.*, 485–497.

Also solved by W. Janous (Austria), A. Stenger, R. Stong, and the proposer.

The Sum of Cosines in a Convex Quadrilateral

12204 [2020, 752]. *Proposed by Florentin Visescu, Bucharest, Romania.* Prove that the absolute value of the sum of the cosines of the four angles in a convex quadrilateral is less than 1/2.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. Denote the angles by α_i for $i \in \{1, 2, 3, 4\}$, with $0 < \alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4 < \pi$. We have $\sum \alpha_i = 2\pi$. Let $a = \alpha_1 + \alpha_2$, and note that $a \le \pi$ and $\alpha_3 + \alpha_4 = 2\pi - a$. If $a = \pi$, then all four angles are $\pi/2$, so $\sum \cos(\alpha_i) = 0$, so $\sum \alpha_i = 0$ and the required inequality holds. We may therefore assume $a < \pi$.

For the sum of the first two cosines,

$$\cos\alpha_1 + \cos\alpha_2 = 2\cos\left(\frac{a}{2}\right)\cos\left(\frac{\alpha_2 - \alpha_1}{2}\right). \tag{1}$$

Since $0 < \alpha_1 \le \alpha_2$, we have

$$0 \le \frac{\alpha_2 - \alpha_1}{2} < \frac{\alpha_1 + \alpha_2}{2} = \frac{a}{2} < \frac{\pi}{2},$$

and therefore

$$\cos\left(\frac{a}{2}\right) < \cos\left(\frac{\alpha_2 - \alpha_1}{2}\right) \le 1.$$

Multiplying by $2\cos(a/2)$, which is positive, we conclude

$$2\cos^2\left(\frac{a}{2}\right) < 2\cos\left(\frac{a}{2}\right)\cos\left(\frac{\alpha_2 - \alpha_1}{2}\right) \le 2\cos\left(\frac{a}{2}\right),$$

which by (1) implies

$$2\cos^2\left(\frac{a}{2}\right) < \cos\alpha_1 + \cos\alpha_2 \le 2\cos\left(\frac{a}{2}\right). \tag{2}$$

Since $0 < \pi - \alpha_4 \le \pi - \alpha_3 < \pi$ and

$$(\pi - \alpha_4) + (\pi - \alpha_3) = 2\pi - (\alpha_3 + \alpha_4) = a,$$

we can apply the same reasoning to $\pi - \alpha_4$ and $\pi - \alpha_3$ to obtain

$$2\cos^2\left(\frac{a}{2}\right) < \cos(\pi - \alpha_4) + \cos(\pi - \alpha_3) \le 2\cos\left(\frac{a}{2}\right),$$

or equivalently

$$-2\cos\left(\frac{a}{2}\right) \le \cos\alpha_3 + \cos\alpha_4 < -2\cos^2\left(\frac{a}{2}\right). \tag{3}$$

Adding (2) and (3), and putting $x = \cos(a/2)$, we get

$$2x^2 - 2x < \sum \alpha_i < 2x - 2x^2.$$

Since the quadratic $2x - 2x^2$ has maximum value 1/2 at $x = 1/2$, this proves the inequality.

Editorial comment. The problem statement assumes that all angles are strictly less than π. If one allows an angle to equal π, then one can achieve a cosine sum of 1/2 by beginning with an equilateral triangle and adding a fourth vertex along one side, obtaining a four-sided figure with angles $\pi/3, \pi/3, \pi/3$, and π. One can obtain quadrilaterals with all angles less than π and cosine sum arbitrarily close to $1/2$ by using angles $\pi/3 + \epsilon$, $\pi/3 + \epsilon$, $\pi/3 + \epsilon$, and $\pi - 3\epsilon$.

Nicolás Caro solved the more general problem of bounding $\sum_{i=1}^n \cos x_i$, given that $0 < x_i < \pi$ and $\sum_{i=1}^n x_i = j\pi$; the stated problem is the case $n = 4$, $j = 2$.

Also solved by E. Bojazhiu (Albania) & E. Hysnelaj (Australia), R. Boukharfane (Saudi Arabia), N. Caro (Brazil), R. Chapman (UK), C. Chiser (Romania), G. Fera & G. Tescaro (Italy), L. Giugiuc (Romania), J.-P. Grivaux (France), N. Hodges (UK), E. J. Ionaşcu, Y. J. Ionin, W. Janous (Austria), A. B. Kasturiarachi, O. Kouba (Syria), K.-W. Lau (China), Z. Lin (China), J. H. Lindsey II, K. Park (Korea), C. Schacht, E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, L. Zhou, and the proposer.

Minimizing a Ratio of Integrals

12205 [2020, 752]. *Proposed by Christian Chiser, Elena Cuza College, Craiova, Romania.* Find the minimum value of

$$\frac{\int_0^1 x^2 (f'(x))^2\,dx}{\int_0^1 x^2 (f(x))^2\,dx}$$

over all nonzero continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ with $f(1) = 0$.

Solution by Jinhai Yan, Fudan University, Shanghai, China. We show that the minimum value is π^2.

Let

$$g(x) = \begin{cases} \sin(\pi x)/x, & \text{if } x \ne 0, \\ \pi, & \text{if } x = 0. \end{cases}$$

Note that $g \in C^\infty[0, 1]$, $g(1) = 0$, and g satisfies the Euler–Lagrange equation

$$\frac{d}{dx}\left(x^2 g'(x)\right) = -\pi^2 x^2 g(x).$$

Therefore, for any f as in the problem statement,

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2 g'(x)}{g(x)} f(x)^2\right) &= x^2\left(\frac{2g'(x)}{g(x)} f(x) f'(x) - \pi^2 f(x)^2 - \frac{g'(x)^2}{g(x)^2} f(x)^2\right) \\ &= x^2\left(f'(x)^2 - \pi^2 f(x)^2\right) - x^2\left(f'(x) - \frac{g'(x)}{g(x)} f(x)\right)^2.\end{aligned}$$

Note that the singularity at $x = 1$ on both sides of this equation is removable, because

$$\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} \frac{f'(x)}{g'(x)} = -\frac{f'(1)}{\pi} \in \mathbb{R}.$$

It follows that

$$\int_0^1 \left(x^2 \left(f'(x)^2 - \pi^2 f(x)^2 \right) - x^2 \left(f'(x) - \frac{g'(x)}{g(x)} f(x) \right)^2 \right) dx = \left. \frac{x^2 g'(x)}{g(x)} f(x)^2 \right|_0^1 = 0.$$

Thus

$$\int_0^1 x^2 f'(x)^2 \, dx - \pi^2 \int_0^1 x^2 f(x)^2 \, dx = \int_0^1 x^2 \left(f'(x) - \frac{g'(x) f(x)}{g(x)} \right)^2 dx \geq 0,$$

with equality if $f = g$, and the desired conclusion follows.

Also solved by K. F. Andersen (Canada), R. Boukharfane (Saudi Arabia), P. Bracken, H. Chen, T. Dickens, L. Han, O. Kouba (Syria), P. W. Lindstrom, A. Natian, M. Omarjee (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, and the proposer.

A Skew-Harmonic Formula for Apéry's Constant

12206 [2020, 752]. *Proposed by Seán Stewart, Bomaderry, Australia.* Prove

$$\sum_{n=1}^{\infty} \frac{\overline{H}_{2n}}{n^2} = \frac{3}{4}\zeta(3),$$

where $\overline{H}_n$ is the nth *skew-harmonic number* $\sum_{k=1}^{n}(-1)^{k+1}/k$ and $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$.

Solution by Michel Bataille, Rouen, France. With $H_0 = 0$ and $H_n = \sum_{k=1}^{n} 1/k$,

$$\overline{H}_{2m} = H_{2m} - 2\sum_{k=1}^{m} \frac{1}{2k} = H_{2m} - H_m = \sum_{k=1}^{m} \frac{1}{m+k}. \tag{1}$$

Also note that

$$H_{2m-1} - H_{m-1} - \sum_{j=m}^{m+N} \left(\frac{1}{j} - \frac{1}{j+m} \right) = H_{2m+N} - H_{m+N} = \sum_{j=m+N+1}^{2m+N} \frac{1}{j}.$$

As N tends to ∞, the right side tends to 0, so

$$\sum_{j=m}^{\infty} \left(\frac{1}{j} - \frac{1}{j+m} \right) = H_{2m-1} - H_{m-1}. \tag{2}$$

Let $S = \sum_{n=1}^{\infty} \overline{H}_{2n}/n^2$. By (1),

$$\begin{aligned} S = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{n+k} &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} \left(\frac{1}{n} - \frac{1}{n+k} \right) \\ &= \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{nk(n+k)}. \end{aligned} \tag{3}$$

We consider the two terms in this expression separately. First

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \sum_{n=1}^{\infty} \left(\frac{H_{n-1}}{n^2} + \frac{1}{n^3} \right) = \sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2} + \zeta(3) = 2\zeta(3)$$

by Euler's formula $\sum_{n=1}^{\infty} H_{n-1}/n^2 = \zeta(3)$.

To evaluate the double sum in the second term of (3), interchange the order of summation, use (2), and then manipulate the harmonic terms and use the first part of (1) to obtain

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{nk(n+k)} = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{n=k}^{\infty} \left(\frac{1}{n} - \frac{1}{n+k} \right) = \sum_{k=1}^{\infty} \frac{H_{2k-1} - H_{k-1}}{k^2}$$

$$= \sum_{k=1}^{\infty} \frac{H_{2k} - H_k + 1/(2k)}{k^2} = \sum_{k=1}^{\infty} \frac{\overline{H}_{2k}}{k^2} + \frac{\zeta(3)}{2} = S + \frac{\zeta(3)}{2}.$$

Thus

$$S = 2\zeta(3) - \left(S + \frac{\zeta(3)}{2} \right),$$

and the result follows.

Editorial comment. A simple proof of Euler's formula for $\zeta(3)$ appears in this MONTHLY 127 (2020), 855. That issue contains the solutions to Problem 12091 and Problem 12102, both of which also link $\zeta(3)$ to infinite series involving harmonic sums.

Many solvers expressed harmonic numbers as integrals from 0 to 1 of the formula for the sum of a finite geometric series and then performed interchanges. This led to various integrals with logarithmic integrands and/or dilogarithms. Two known definite integrals that played a role in many solutions were

$$\int_1^1 \frac{\log^2(1-x)}{x}\,dx = 2\zeta(3)$$

and

$$\int_0^1 \frac{\log(1-x)\log(1+x)}{x}\,dx = -\frac{5}{8}\zeta(3).$$

Also solved by A. Berkane (Algeria), N. Bhandari (Nepal), R. Boukharfane (Saudi Arabia), K. N. Boyadzhiev, P. Bracken, B. Bradie, N. Caro (Brazil), A. C. Castrillón (Colombia), H. Chen, N. S. Dasireddy (India), G. Fera (Italy), M. L. Glasser, R. Gordon, H. Grandmontagne (France), L. Han, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), I. Mezö (China), R. Molinari, V. H. Moll & T. Amdeberhan, K. Nelson, M. Omarjee (France), S. Sharma (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Wangshinghin (Canada), T. Wiandt, Y. Xiang (China), and the proposer.

A Fibonacci Inequality

12213 [2020, 853]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove

$$\sum_{k=1}^{n} \sqrt{F_{k-1}F_{k+2}} \leq \sqrt{F_{n+1}F_{n+4}} - \sqrt{5}.$$

Solution by Rory Molinari, Beverly Hills, MI. More generally, consider a sequence $\langle a \rangle$ of nonnegative real numbers such that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. For $n \geq 2$ and d a nonnegative integer, we prove

$$\sum_{k=1}^{n-1} \sqrt{a_{k-1}a_{k+d-1}} \leq \sqrt{a_n a_{n+d}} - \sqrt{a_1 a_{d+1}}.$$

Setting $a_n = F_{n+1}$ and $d = 3$ proves the desired inequality.

The identity $\sum_{k=j}^{m} a_k = a_{m+2} - a_{j+1}$ is easily shown by induction on m. By the Cauchy–Schwarz inequality,

$$\sum_{k=1}^{n-1} \sqrt{a_{k-1}a_{k+d-1}} \leq \left(\sum_{k=1}^{n-1} a_{k-1}\right)^{1/2} \left(\sum_{k=1}^{n-1} a_{k+d-1}\right)^{1/2} = \sqrt{(a_n - a_1)(a_{n+d} - a_{d+1})}.$$

By the AM-GM inequality,

$$\begin{aligned}(a_n - a_1)(a_{n+d} - a_{d+1}) &= a_n a_{n+d} + a_1 a_{d+1} - a_1 a_{n+d} - a_{d+1} a_n \\ &\leq a_n a_{n+d} + a_1 a_{d+1} - 2\sqrt{a_1 a_{n+d} a_{d+1} a_n} \\ &= \left(\sqrt{a_n a_{n+d}} - \sqrt{a_1 a_{d+1}}\right)^2.\end{aligned}$$

Editorial comment. The majority of solvers proved the inequality by induction, showing

$$\sqrt{F_{n+1}F_{n+4}} + \sqrt{F_n F_{n+3}} \leq \sqrt{F_{n+2}F_{n+5}}$$

by squaring both sides and applying the AM-GM inequality. Doyle Henderson used this approach to generalize to a sequence of real numbers satisfying $a_n \geq a_{n-1} + a_{n-2}$ for $n \geq 2$ and $\sqrt{a_0 a_3} \leq \sqrt{a_2 a_5} - \sqrt{a_5}$, obtaining

$$\sum_{k=1}^{n} \sqrt{a_{k-1}a_{k+2}} \leq \sqrt{a_{n+1}a_{n+4}} - \sqrt{a_5}.$$

Also solved by K. F. Andersen (Canada), M. Bataille (France), B. D. Beasley, R. Boukharfane (Saudi Arabia), P. Bracken, B. Bradie, Ó. Ciaurri (Spain), C. Curtis, A. Dixit (India) & S. Pathak (USA), G. Fera (Italy), D. Fleischman, O. Geupel (Germany), R. Gordon, D. Henderson, N. Hodges (UK), Y. J. Ionin, W. Janous (Austria), M. Kaplan & M. Goldenberg, K. T. L. Koo (China), O. Kouba (Syria), W.-K. Lai, P. Lalonde (Canada), K.-W. Lau (China), O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), J. Pak (Canada), A. Pathak (India), Á. Plaza (Spain), E. Schmeichel, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. B. Tyler, J. Van hamme (Belgium), M. Vowe (Switzerland), J. Vukmirović (Serbia), T. Wiandt, L. Wimmer (Germany), X. Ye (China), A. Zaidan, L. Zhou, FAU Problem Solving Group, and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

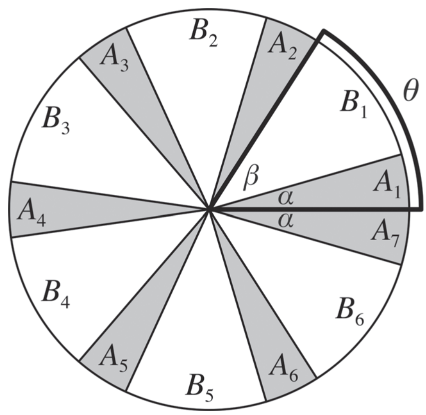
C5. *Due to Victor Klee, contributed by the editors*. Given a set S in $\mathbb{R}^n$, let $L(S)$ be the set of all points lying on some line determined by two points in S. For example, if S is the set of vertices of an equilateral triangle in $\mathbb{R}^2$, then $L(S)$ is the union of the three lines that extend the sides of the triangle, and $L(L(S))$ is all of $\mathbb{R}^2$. If S is the set of vertices of a regular tetrahedron, then what is $L(L(S))$?

Returning the Icing to the Top

C4. *From the 1968 Moscow Mathematical Olympiad, contributed by the editors*. A round cake has icing on the top but not the bottom. Cut a piece of the cake in the usual shape of a sector with vertex angle one radian and with vertex at the center of the cake. Remove the piece, turn it upside down, and replace it in the cake to restore roundness. Next, move one radian around the cake, cut another piece with the same vertex angle adjacent to the first, remove it, turn it over, and replace it. Keep doing this, moving around the cake one radian at a time, inverting each piece. Show that, after a finite number of steps, all the icing will again be on the top.

Solution. We solve the general problem in which the central angle of every slice is θ radians. If $2\pi/\theta$ is an integer n, then clearly n flips put all the icing on the bottom, and n more flips return it all to the top. Otherwise, let $n = \lfloor 2\pi/\theta \rfloor$. We show that the icing returns to the top for the first time after $2n(n+1)$ steps. In the case $\theta = 1$, we have $n = 6$, and therefore it takes 84 steps for the icing to return to the top.

Let $\alpha = 2\pi - n\theta$. Clearly $0 < \alpha < \theta$. Let $\beta = \theta - \alpha$, so that $\alpha + \beta = \theta$. Cut n consecutive pieces with angle θ (these are the first n pieces to be flipped), leaving a piece with angle α. Cut each of the n pieces into two pieces of angle α and β, as in the figure. Reading counterclockwise, you now have pieces of width $\alpha, \beta, \alpha, \beta, \ldots, \alpha$, with the last α adjacent to the first. Let $A_1, \ldots, A_{n+1}$ be the pieces with angle α, and let $B_1, \ldots, B_n$ be the pieces with angle β, with B_i between A_i and A_{i+1}, as shown here. You may now discard the knife; no further cutting is necessary.



Imagine that the cake is on a rotating cake plate and we rotate the cake plate clockwise through an angle of θ after each piece is flipped. In the first step, we flip the piece consisting of A_1 and B_1 and then rotate the plate clockwise. Piece A_1 is now upside down in the original location of piece A_{n+1}, and B_1 is now upside down in the original location of piece B_n. All other pieces simply rotate clockwise without being flipped, so for $2 \le i \le n+1$, A_i moves to the original location of A_{i-1}, and for $2 \le i \le n$, B_i moves to the original location of B_{i-1}. At the end of this operation the cuts are in the same positions as they were in originally; the net effect of one step is simply to permute the A and B pieces cyclically, with one of each being flipped.

It is now clear that after n steps the B pieces have completed a full rotation, with each piece being flipped once, so they are back in their original positions upside down, and after another n steps they are in their original positions right side up again. Similarly, it takes $2(n+1)$ steps for all the A pieces to return to right side up, in their original positions. It follows that the number of steps needed to return all icing to the top is the least common multiple of $2n$ and $2(n+1)$, which is $2n(n+1)$. Indeed, after this many steps, not only is the icing on top, but the cake is fully restored to its original configuration.

Editorial comment. This problem appeared, in a somewhat different form, as problem 31.2.8.3 in the 1968 Moscow Mathematical Olympiad. The version given here appears in P. Winkler (2007), *Mathematical Mind-Benders*, A K Peters/CRC Press, Wellesley, MA.

SOLUTIONS

An Euler–Mascheroni Sum

12194 [2020, 564]. *Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bîrlad, Romania.* Let $\gamma_n = -\ln n + \sum_{k=1}^{n} 1/k$, and let γ be the Euler–Mascheroni constant $\lim_{n\to\infty} \gamma_n$. Evaluate

$$\sum_{n=1}^{\infty} \left(\gamma_n - \gamma - \frac{1}{2n} \right).$$

Solution by Abdelhak Berkane, Université Frères Mentouri, Constantine, Algeria. We show that the answer is $\bigl(1 + \gamma - \ln(2\pi)\bigr)/2$. Let H_n denote the nth harmonic number, so that $\gamma_n = -\ln n + H_n$, and let S_n denote the nth partial sum of the series in the problem. Applying the formula $\sum_{k=1}^{n} H_k = (n+1)H_n - n$ (which is easily verified by induction), we find that

$$S_n = \sum_{k=1}^{n} \left(H_k - \ln k - \gamma - \frac{1}{2k} \right) = \left(n + \frac{1}{2} \right) H_n - n - \ln(n!) - n\gamma.$$

Using the known asymptotic formulas

$$H_n = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \qquad \text{and}$$

$$\ln(n!) = n \ln n - n + \frac{\ln(2\pi n)}{2} + O\left(\frac{1}{n}\right),$$

we obtain

$$\begin{aligned} S_n &= \left(n + \frac{1}{2}\right)\left(\ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right) - n - \left(n \ln n - n + \frac{\ln(2\pi n)}{2} + O\left(\frac{1}{n}\right)\right) - n\gamma \\ &= \frac{1 + \gamma - \ln(2\pi)}{2} + O\left(\frac{1}{n}\right). \end{aligned}$$

Let $n \to \infty$ to get the desired sum.

Also solved by U. Abel & V. Kushnirevych (Germany), T. Akhmetov (Russia), K. F. Andersen (Canada), M. Bataille (France), N. Bhandari (Nepal), R. Boukharfane (Saudi Arabia), P. Bracken, B. Bradie, N. Caro (Brazil), R. Chapman (UK), H. Chen, C. Chiser (Romania), B. E. Davis, A. Dixit (India) & S. Pathak (US), S. P. I. Evangelou (Greece), G. Fera (Italy), D. Fleischman, S. Gayen (India), O. Geupel (Germany), J. A. Grzesik, E. A. Herman, N. Hodges (UK), W. Janous (Austria), M. Kaplan & M. Goldenberg, K. T. L. Koo (China), O. Kouba (Syria), S. S. Kumar, K.-W. Lau (China), G. Lavau (France), O. P. Lossers (Netherlands), I. Mezo (Canada), R. Molinari, A. Natian, K. Nelson, M. Omarjee (France), N. Osipov (Russia), A. Pathak, Á. Plaza (Spain), K. Sarma (India), K. Schilling, S. Sharma (India), S. Singhania (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), Y. Xiang (China), L. Zhou, and the proposer.

A Mean Inequality

12196 [2020, 659]. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.* Determine which positive integers n have the following property: If $a_1, \dots, a_n$ are n real numbers greater than or equal to 1, and A, G, and H are their arithmetic mean, geometric mean, and harmonic mean, respectively, then

$$G - H \geq \frac{1}{G} - \frac{1}{A}.$$

Composite solution by Radouan Boukharfane, Extreme Computing Research Center, Thuwal, Saudi Arabia, Nigel Hodges, Cheltenham, UK, the proposer, and the editors. The property holds for $n \leq 5$ but fails for $n \geq 6$.

If $a_1 = a_2 = \cdots = a_{n-1} = 1$ and $a_n = n+1$, then the inequality becomes

$$\sqrt[n]{n+1} - \frac{n+1}{n} \geq \frac{1}{\sqrt[n]{n+1}} - \frac{1}{2}. \tag{1}$$

We claim that this inequality is false for $n \geq 6$. To see why, we first note that $(5/4)^{12} > 13$, and therefore $\sqrt[12]{13} < 5/4$. It is easily verified that the sequence $\{\sqrt[n]{n+1}\}$ is decreasing, so $\sqrt[n]{n+1} < 5/4$ for $n \geq 12$, and therefore

$$\sqrt[n]{n+1} - \frac{n+1}{n} < \frac{5}{4} - 1 = \frac{1}{4} \qquad \text{and} \qquad \frac{1}{\sqrt[n]{n+1}} - \frac{1}{2} > \frac{4}{5} - \frac{1}{2} = \frac{3}{10} > \frac{1}{4}.$$

Thus, (1) is false for $n \geq 12$. One can check numerically that it is also false for $n = 6, \dots, 11$, so the property in the problem does not hold for $n \geq 6$.

To prove that it holds for $n \leq 5$, let

$$F(a_1, \dots, a_n) = G - \frac{1}{G} - H + \frac{1}{A}.$$

Suppose $C > 1$. We show that if $n \leq 5$ and $1 \leq a_1 \leq \cdots \leq a_n \leq C$, then $F(a_1, \dots, a_n) \geq 0$. Since C is arbitrary, this will establish that the property holds for $n \leq 5$.

Since we have restricted our attention to a compact domain, F achieves a minimum value on that domain. We need the following fact about where the minimum occurs.

Lemma. *If the minimum value of $F(a_1, \dots, a_n)$ for $1 \leq a_1 \leq \cdots \leq a_n \leq C$ is negative, and F achieves that minimum value at a sequence $(a_1, \dots, a_n)$, then $a_j = 1$ whenever $1 \leq j \leq n/2 + 1$.*

Proof. Suppose that the minimum value is negative. Note that if $a_1 = \cdots = a_n$, then $F(a_1, \dots, a_n) = 0$, so the minimum must occur at a nonconstant sequence. We proceed now by induction on j.

For the base case, suppose that F achieves its minimum at a sequence $(a_1, \dots, a_n)$ with $1 < a_1 \le \cdots \le a_n \le C$. Since the sequence is not constant, $H < G < A$. With $b_i = a_i/a_1$, we have

$$F(b_1, \dots, b_n) = \frac{1}{a_1}(G - H) - a_1\left(\frac{1}{G} - \frac{1}{A}\right) < G - H - \left(\frac{1}{G} - \frac{1}{A}\right) = F(a_1, \dots, a_n),$$

contradicting the assumption that F achieves its minimum at $(a_1, \dots, a_n)$. This establishes the base case.

For the induction step, assume that $j \ge 1$, the claim holds for $1, \dots, j$, and $j + 1 \le n/2 + 1$; that is, $j \le n/2$. Suppose F achieves its minimum at a sequence $(a_1, \dots, a_n)$ with $a_{j+1} > 1$. By the induction hypothesis, $a_1 = \cdots = a_j = 1$. We have $A = S/n$ and $H = n/T$, where

$$S = a_1 + \cdots + a_n = j + a_{j+1} + \cdots + a_n, \quad T = \frac{1}{a_1} + \cdots + \frac{1}{a_n} = j + \frac{1}{a_{j+1}} + \cdots + \frac{1}{a_n}.$$

Let $b_i = a_i$ for $i \notin \{j, j+1\}$, and let $b_j = b_{j+1} = \sqrt{a_{j+1}}$. The sequence $(b_1, \dots, b_n)$ has the same geometric mean as $(a_1, \dots, a_n)$, and its arithmetic and harmonic means are S'/n and n/T', respectively, where

$$\begin{aligned} S' &= S - 1 - a_{j+1} + 2\sqrt{a_{j+1}} = S - (\sqrt{a_{j+1}} - 1)^2, \\ T' &= T - 1 - \frac{1}{a_{j+1}} + \frac{2}{\sqrt{a_{j+1}}} = T - \frac{(\sqrt{a_{j+1}} - 1)^2}{a_{j+1}}. \end{aligned}$$

Therefore

$$\begin{aligned} F(a_1, \dots, a_n) - F(b_1, \dots, b_n) &= \left(G - \frac{1}{G} - \frac{n}{T} + \frac{n}{S}\right) - \left(G - \frac{1}{G} - \frac{n}{T'} + \frac{n}{S'}\right) \\ &= \left(\frac{n}{T - (\sqrt{a_{j+1}} - 1)^2/a_{j+1}} - \frac{n}{T}\right) - \left(\frac{n}{S - (\sqrt{a_{j+1}} - 1)^2} - \frac{n}{S}\right) \\ &= n(\sqrt{a_{j+1}} - 1)^2\left(\frac{1}{T(Ta_{j+1} - (\sqrt{a_{j+1}} - 1)^2)} - \frac{1}{S(S - (\sqrt{a_{j+1}} - 1)^2)}\right). \end{aligned} \tag{2}$$

Clearly, $T \le S$, and using the fact that $j \le n/2$ we obtain

$$\begin{aligned} Ta_{j+1} &= \left(j + \frac{1}{a_{j+1}} + \cdots + \frac{1}{a_n}\right)a_{j+1} \le \left(j + \frac{n-j}{a_{j+1}}\right)a_{j+1} = ja_{j+1} + n - j \\ &= n + j(a_{j+1} - 1) \le n + (n - j)(a_{j+1} - 1) = j + (n - j)a_{j+1} \\ &\le j + a_{j+1} + \cdots + a_n = S. \end{aligned}$$

Combining this with (2), we conclude $F(a_1, \dots, a_n) - F(b_1, \dots, b_n) \ge 0$, which implies $F(b_1, \dots, b_n) \le F(a_1, \dots, a_n)$ and hence F achieves its minimum at $(b_1, \dots, b_n)$. But $b_j = \sqrt{a_{j+1}} > 1$, so this contradicts the induction hypothesis. ■

We are now ready to complete the solution. The case $n = 1$ is trivial. If $n = 2$ and the minimum of F is negative, then by the lemma this minimum must occur at the sequence $(1, 1)$. But $F(1, 1) = 0$, so this is impossible.

If $n = 3$ and the minimum of F is negative, then by the lemma the minimum occurs at some sequence $(1, 1, a_3)$. Writing $a_3 = (x + 1)^3$ for some $x \ge 0$, we have

$$F(1, 1, (x + 1)^3) < 0.$$

On the other hand,

$$F(1, 1, (x+1)^3) = (x+1) - \frac{1}{x+1} - \frac{3}{2+1/(x+1)^3} + \frac{3}{2+(x+1)^3}$$
$$= \frac{x^3(2+x)(6+12x+15x^2+9x^3+2x^4)}{(1+x)(3+3x+3x^2+x^3)(3+6x+6x^2+2x^3)} \geq 0,$$

so this is a contradiction.

Similarly, if $n = 4$ and the minimum of F is negative, then by the lemma we have $F(1, 1, 1, (x+1)^4) < 0$ for some $x \geq 0$, and we get a contradiction from the calculation

$$F(1, 1, 1, (x+1)^4)$$
$$= \frac{x^3(2+x)(8+24x+60x^2+80x^3+56x^4+20x^5+3x^6)}{(1+x)(4+4x+6x^2+4x^3+x^4)(4+12x+18x^2+12x^3+3x^4)} \geq 0.$$

Finally, if $n = 5$ and the minimum of F is negative, then by the lemma we have $F(1, 1, 1, (x+1)^5, (x+y+1)^5) < 0$ for some $x, y \geq 0$. A calculation similar to those in the previous cases shows that $F(1, 1, 1, (x+1)^5, (x+y+1)^5)$ is a rational function with all coefficients positive, which is a contradiction.

Editorial comment. When $n = 6$, $F(1, 1, 1, 1, 1, (x+1)^6)$ is a rational function whose numerator is

$$x^3(2+x)(-30-150x-111x^2+456x^3+1328x^4$$
$$+1758x^5+1431x^6+764x^7+264x^8+54x^9+5x^{10}),$$

which is negative for x positive and close to 0.

No other complete solutions were received.

A Pell-type Equation in Disguise

12197 [2020, 659]. *Proposed by Nicolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Prove that the equation

$$(a^2+1)(b^2-1) = c^2 + 3333$$

has no solutions in integers a, b, and c.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We may clearly assume $a, b, c \geq 0$. If $a = 0$, then $b^2 - c^2 = 3334$, which has no solutions since $3334 \equiv 2 \pmod 4$. If $b \in \{0, 1\}$, then the left side is nonpositive and there are no solutions. Thus we may assume $a > 0$ and $b > 1$. Hence neither $a^2 + 1$ nor $b^2 - 1$ is a perfect square. We rewrite the equation as $c^2 - da^2 = b^2 - 3334$, where $d = b^2 - 1$, in order to apply known results about Pell-type equations.

In the Pell-type equation $x^2 - dy^2 = n$, where $d > 0$ and d is not a perfect square, with any solution (x, y) we can associate an algebraic number α by setting $\alpha = x + y\sqrt{d}$. Since $\alpha = x + \sqrt{x^2 - n}$, and $x + \sqrt{x^2 - n}$ increases with x for $x^2 > n$, minimizing x is equivalent to minimizing α.

With a solution (u, v) in positive integers to $u^2 - dv^2 = 1$ we associate another algebraic number β by setting $\beta = u + v\sqrt{d}$. Note that $\beta^{-1} = u - v\sqrt{d}$. We compute

$$\alpha\beta^{-1} = (x + y\sqrt{d})(u - v\sqrt{d}) = (xu - dyv) + (yu - xv)\sqrt{d}.$$

Setting $x' = xu - dyv$ and $y' = yu - xv$ gives another solution to $x^2 - dy^2 = n$. Suppose that (x, y) is the solution in nonnegative integers that minimizes x and hence also minimizes α. Since $\beta > 1$, we have $\alpha\beta^{-1} < \alpha$, so x' or y' must be negative. They cannot both be negative, because $\alpha\beta^{-1} > 0$. Since $(x')^2 - d(y')^2 = n$, we have

$$n\beta\alpha^{-1} = (x' + y'\sqrt{d})(x' - y'\sqrt{d})\beta\alpha^{-1} = \alpha\beta^{-1}(x' - y'\sqrt{d})\beta\alpha^{-1} = x' - y'\sqrt{d}.$$

Since exactly one of x' and y' is negative, $|x' - y'\sqrt{d}| = |x'| + |y'|\sqrt{d}$, and hence $|n|\beta\alpha^{-1} = |x'| + |y'|\sqrt{d}$. Since $(|x'|, |y'|)$ is a solution to $x^2 - dy^2 = n$, the minimality of α implies $\alpha \le |n|\beta\alpha^{-1}$, and hence $\alpha \le \sqrt{|n|\beta}$.

Now consider a solution (a, b, c) to the original equation that minimizes c. Write the equation as

$$c^2 - (b^2 - 1)a^2 = b^2 - 3334 = n,$$

and note that $(u, v) = (b, 1)$ satisfies $u^2 - (b^2 - 1)v^2 = 1$. Letting $\alpha = c + a\sqrt{b^2 - 1}$ and $\beta = b + \sqrt{b^2 - 1}$, we obtain

$$c + a\sqrt{b^2 - 1} = \alpha \le \sqrt{|n|\beta} = \sqrt{|b^2 - 3334|(b + \sqrt{b^2 - 1})} < \sqrt{|b^2 - 3334|(2b)}.$$

We next prove that $b < 117$. If $b \ge 117$ (in fact, whenever $b \ge 58$), then

$$c^2 - (b^2 - 1)a^2 = b^2 - 3334 > 0,$$

so $c > a\sqrt{b^2 - 1}$. Hence,

$$2a\sqrt{b^2 - 1} < \sqrt{2b|b^2 - 3334|}. \tag{$*$}$$

Now rewrite the original equation as

$$c^2 - (a^2 + 1)b^2 = -a^2 - 3334.$$

Note that $(u, v) = (2a^2 + 1, 2a)$ satisfies $u^2 - (a^2 + 1)v^2 = 1$. Take $\alpha = c + b\sqrt{a^2 + 1}$ and $\beta = (2a^2 + 1) + 2a\sqrt{a^2 + 1}$ in the preceding, and note that $\beta = (a + \sqrt{a^2 + 1})^2 < 4(a^2 + 1)$. We obtain

$$c + b\sqrt{a^2 + 1} = \alpha \le \sqrt{|n|\beta} = \sqrt{(a^2 + 3334)\beta} < 2\sqrt{(a^2 + 3334)(a^2 + 1)}.$$

Since $c > 0$, we conclude $b < 2\sqrt{a^2 + 3334}$. Combining this with $(*)$, we obtain

$$b^2 < 4(a^2 + 3334) < \frac{2b(b^2 - 3334)}{b^2 - 1} + 13336.$$

The largest real root of $t^2(t^2 - 1) - 2t(t^2 - 3334) - 13336(t^2 - 1)$ is less than 117, so $b < 117$.

Thus the problem is reduced to checking values of b up to 116 and values of a up to $\sqrt{b|b^2 - 3334|/(2(b^2 - 1))}$ and then evaluating c. This is easily done on a computer, yielding no solutions with integral c.

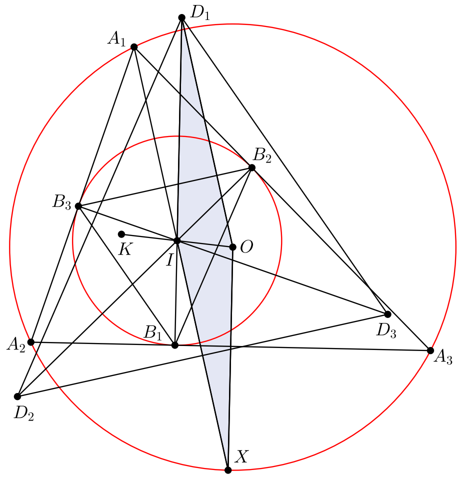
Also solved by R. Chapman (UK), A. Stenger, and the proposer.

Dilating Kimberling's Center X_{65} from the Incenter

12198 [2020, 659]. *Proposed by Michel Bataille, Rouen, France.* Let $A_1A_2A_3$ be a nonequilateral triangle with incenter I, circumcenter O, and circumradius R. For $i \in \{1, 2, 3\}$, let B_i be the point of tangency of the incircle of $A_1A_2A_3$ with the side of the triangle opposite A_i, and let C_i be the point of intersection between the circle centered at

I of radius R and the ray IB_i. Let K be the orthocenter of $C_1C_2C_3$. Prove that I is the midpoint of OK.

Solution by Lienhard Wimmer, Isny im Allgäu, Germany. For $i \in \{1, 2, 3\}$, let D_i be the reflection of C_i through I. It suffices to show that O is the orthocenter of $D_1D_2D_3$, because this orthocenter is the reflection of K through I. Extend A_1I to intersect the circumcircle of $A_1A_2A_3$ at X.



Since A_1X bisects $\angle A_2A_1A_3$, arcs A_2X and A_3X are equal. Therefore OX is the perpendicular bisector of A_2A_3, so $OX \parallel D_1I$. By construction, $D_1I = R = OX$. Thus D_1IXO is a parallelogram, which implies $D_1O \parallel A_1X$. The isosceles triangles $\triangle IB_2B_3$ and $\triangle ID_2D_3$ are similar, and therefore $B_2B_3 \parallel D_2D_3$. Since $A_1X \perp B_2B_3$, we conclude that $D_1O \perp D_2D_3$. Likewise, $D_2O \perp D_3D_1$, completing the proof.

Editorial comment. Oliver Geupel and Nigel Hodges point out that the orthocenter of $B_1B_2B_3$ is center X_{65} in Clark Kimberling's Encyclopedia of Triangle Centers (faculty.evansville.edu/ck6/encyclopedia/etc.html), and I divides OX_{65} in the ratio of $R : r$. The result in the problem follows immediately, because $\triangle C_1C_2C_3$ is the image of $\triangle B_1B_2B_3$ under a dilation with center I and ratio R/r.

Also solved by R. Boukharfane (Saudi Arabia), H. Chen (China), G. Fera (Italy), O. Geupel (Germany), N. Hodges (UK), E. J. Ionaşcu, W. Janous (Austria), M. Kaplan & M. Goldenberg, L. Kiernan, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, T. Wiandt, L. Zhou, and the proposer.

The Basel Problem in Disguise

12199 [2020, 660]. *Proposed by Shivam Sharma, Delhi University, New Delhi, India.* Prove

$$\int_0^\infty \frac{x \sinh(x)}{3 + 4\sinh^2(x)}\, dx = \frac{\pi^2}{24}.$$

Solution by Robin Chapman, University of Exeter, Exeter, UK. Observe that for $x > 0$,

$$\begin{aligned}
\frac{2\sinh x}{3 + 4\sinh^2 x} &= \frac{e^x - e^{-x}}{3 + (e^x - e^{-x})^2} = \frac{1}{e^x - e^{-x}} \cdot \frac{(e^x - e^{-x})^2}{3 + (e^x - e^{-x})^2} \\
&= \frac{1}{e^x - e^{-x}} \left(1 - \frac{3}{3 + (e^x - e^{-x})^2}\right) \\
&= \frac{1}{e^x - e^{-x}} - \frac{3}{e^{3x} - e^{-3x}} = \frac{1}{2\sinh x} - \frac{3}{2\sinh(3x)}.
\end{aligned}$$

Therefore

$$\int_0^\infty \frac{x \sinh x}{3 + 4\sinh^2 x}\, dx = \frac{1}{4}\int_0^\infty \frac{x\, dx}{\sinh x} - \frac{3}{4}\int_0^\infty \frac{x\, dx}{\sinh(3x)}.$$

A simple substitution gives

$$\int_0^\infty \frac{x\,dx}{\sinh(3x)} = \frac{1}{9}\int_0^\infty \frac{x\,dx}{\sinh x},$$

so

$$\begin{aligned}
\int_0^\infty \frac{x\sinh x}{3+4\sinh^2 x}\,dx &= \frac{1}{6}\int_0^\infty \frac{x\,dx}{\sinh x} = \frac{1}{3}\int_0^\infty \frac{x\,dx}{e^x - e^{-x}} \\
&= \frac{1}{3}\int_0^\infty \sum_{k=0}^\infty x e^{-(2k+1)x}\,dx = \frac{1}{3}\sum_{k=0}^\infty \int_0^\infty x e^{-(2k+1)x}\,dx \\
&= \frac{1}{3}\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \frac{1}{3}\left[\sum_{k=1}^\infty \frac{1}{k^2} - \sum_{k=1}^\infty \frac{1}{(2k)^2}\right] \\
&= \frac{1}{3}\left(1-\frac{1}{4}\right)\sum_{k=1}^\infty \frac{1}{k^2} = \frac{1}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{24}.
\end{aligned}$$

Also solved by Z. Ahmed (India), T. Akhmetov (Russia), K. F. Andersen (Canada), F. R. Ataev (Uzbekistan), S. Attaoui & M. Slimane (Algeria), M. Bataille (France), N. Batir (Turkey), A. Berkane (Algeria), N. Bhandari (Nepal), R. Boukharfane (Saudi Arabia), P. Bracken, B. Bradie, V. Brunetti (India), C. Burnette, H. Chen, B. E. Davis, T. Dickens, G. A. Edgar, G. Fera (Italy), P. Fulop (Hungary), M. L. Glasser, H. Grandmontagne (France), N. Grivaux (France), J. A. Grzesik, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), E. J. Ionaşcu, W. Janous (Austria), J. E. Kampmeyer III, O. Kouba (Syria), K.-W. Lau (China), G. Lavau (France), J. Magliano, S. Miao (China), A. Natian, K. Nelson, Q. M. Nguyen (Canada), C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), D. Terr, D. B. Tyler, A. Tzarellas, E. I. Verriest, T. Wiandt, H. Widmer (Switzerland), Y. Xiang (China), M. R. Yegan (Iran), L. Zhou, FAU Problem Solving Group, and the proposer.

Group Algebras With Invariant Subsets

12201 [2020, 660]. *Proposed by Stephen M. Gagola, Jr., Kent State University, Kent, Ohio.* Let F be a field, and let G be a finite group. The *group algebra* $F[G]$ is the vector space of all formal sums $\sum_{g\in G} a_g g$, where $a_g \in F$, with multiplication defined by extending the multiplication in G via the distributive laws. A subset S of $F[G]$ is G*-invariant* if $s \in S$ and $g \in G$ imply $sg \in S$. In particular, the subset G is G-invariant, as is the singleton set $\{\sum_{g\in G} g\}$. Find all fields F and groups G such that there exists an F-linear transformation $\phi\colon F[G] \to F[G]$ that is not right multiplication by an element of G but that nevertheless sends every G-invariant subset to itself.

Solution by Kenneth Schilling, University of Michigan, Flint, MI. The field F must be the field of order 2, and the group G must be a cyclic group of order 3, 4, or 5.

Let $F[G]$ be a group algebra, and let $\phi\colon F[G] \to F[G]$ be an F-linear transformation that preserves G-invariant sets but is not right-multiplication by an element of G. Let e be the identity element of G. It follows that the map $\psi\colon F[G] \to F[G]$ given by $\psi(x) = \phi(x)(\phi(e))^{-1}$ is also an F-linear transformation of $F[G]$ that preserves G-invariant sets but is not right-multiplication by an element of G and has the additional property that $\psi(e) = e$. We may therefore assume henceforth without loss of generality that $\phi(e) = e$.

Claim 1: *For every finite subset $\{g_1, \ldots, g_k\}$ of G, there exists $h \in G$ such that*

$$\{\phi(g_1), \ldots, \phi(g_k)\} = \{g_1 h, \ldots, g_k h\}.$$

In particular, ϕ maps G injectively into itself, and hence ϕ is injective on $F[G]$.

Proof. Since G is G-invariant, $\phi(G) \subset G$. Since the set $\{g_1 h + \cdots + g_k h : h \in G\}$ is also G-invariant and contains $g_1 + \cdots + g_k$, there exists $h' \in G$ such that

$$\phi(g_1 + \cdots + g_k) = \phi(g_1) + \cdots + \phi(g_k) = g_1 h' + \cdots + g_k h'.$$

The claim now follows from the fact that G is a linearly independent set in the vector space $F[G]$ and $\phi(g_i)$ and $g_i h'$ belong to G for all i. ■

Claim 2: *For all $g \in G$, $\phi(g) \in \{g, g^{-1}\}$.*

Proof. For $g \in G - \{e\}$, Claim 1 implies that $\{e, \phi(g)\} = \{h, gh\}$ for some $h \in G$. Thus either $e = h$ and $\phi(g) = gh$, in which case $\phi(g) = g$, or $e = gh$ and $\phi(g) = h$, in which case $\phi(g) = g^{-1}$. ■

Claim 3: *If $\phi(g_1) \neq g_1$ and $\phi(g_2) \neq g_2$ for distinct elements $g_1, g_2 \in G$, then $g_1 = g_2^{-1}$ or $g_1 = g_2^2$ or $g_2 = g_1^2$.*

Proof. By Claims 1 and 2, $\{\phi(e), \phi(g_1), \phi(g_2)\} = \{e, g_1^{-1}, g_2^{-1}\} = \{h, g_1 h, g_2 h\}$ for some $h \in G$. If $e = h$, then $\{e, g_1^{-1}, g_2^{-1}\} = \{e, g_1, g_2\}$, and $g_1 = g_2^{-1}$ follows from $\phi(g_2) = g_2^{-1} \neq g_2$. If $e = g_1 h$, then $\{e, g_1^{-1}, g_2^{-1}\} = \{g_1^{-1}, e, g_2 g_1^{-1}\}$, so $g_2^{-1} = g_2 g_1^{-1}$, which yields $g_1 = g_2^2$. By symmetry, $g_2 = g_1^2$ when $e = g_2 h$. ■

Claim 4: *If $\phi(g_1) = g_1$ and $\phi(g_2) \neq g_2$ for $g_1, g_2 \in G - \{e\}$, then g_1 and e are the only elements of G fixed by ϕ. Also, $g_1^2 = e$, and $g^2 = g_1$ for all $g \in G - \{e, g_1\}$.*

Proof. By Claims 1 and 2, $\{e, \phi(g_1), \phi(g_2)\} = \{e, g_1, g_2^{-1}\} = \{h, g_1 h, g_2 h\}$ for some $h \in G$. If $e = h$, then $g_2^{-1} = g_2$, which contradicts $\phi(g_2) \neq g_2$. If $e = g_1 h$, then $\{e, g_1, g_2^{-1}\} = \{g_1^{-1}, e, g_2 g_1^{-1}\}$. Since $g_2^{-1} \neq g_1^{-1}$, we have $g_1 = g_1^{-1}$ and $g_2^{-1} = g_2 g_1^{-1}$, so $g_1^2 = e$ and $g_2^2 = g_1$. If $e = g_2 h$, then $\{e, g_1, g_2^{-1}\} = \{g_2^{-1}, g_1 g_2^{-1}, e\}$, so $g_1 = g_1 g_2^{-1}$, which contradicts $g_2 \neq e$.

We conclude $g_1^2 = e$ and $g_1 = g_2^2$. This implies that g_1 is the only element of $G - \{e\}$ that is fixed by ϕ. Furthermore, $g^2 = g_1$ for all $g \in G - \{e, g_1\}$. ■

Claim 5: *F is the field of order 2.*

Proof. If F has an element a that is neither 0 nor 1, then let g be any element of $G - \{e\}$. The set $\{h + agh : h \in G\}$ is G-invariant, and $e + ag$ is one of its elements, so there exists $h \in G$ such that $\phi(e + ag) = e + a\phi(g) = h + agh$. It follows that $e = h$ and $\phi(g) = gh$, so $\phi(g) = g$. In other words, ϕ is the identity transformation on G, and so also on $F[G]$, contrary to hypothesis. ■

We now find all possible groups G.

First, suppose that $G - \{e\}$ has elements g_1 and g_2 such that $\phi(g_1) = g_1$ and $\phi(g_2) = g_2^{-1} \neq g_2$. By Claim 4, $g_2^4 = g_1^2 = e$, so the group $\langle g_2 \rangle$ generated by g_2 is a cyclic group of order 4 and contains g_1, which equals g_2^2. Furthermore, we claim $G = \langle g_2 \rangle$. If there exists $h \in G - \langle g_2 \rangle$, then $\phi(h) \neq h$ by Claim 4. Applying Claim 3 to g_2 and h now yields either $g_2^2 = h$ (forbidden by $h \notin \langle g_2 \rangle$) or $h^2 = g_2$ (forbidden by Claim 4 implying $h^2 = g_1$). With G being a cylic group of order 4, it is easy to check that $\phi(g) = g^{-1}$ satisfies the required conditions.

A second case is $G = \{e, g_1, g_1^{-1}\}$, where $\phi(g) = g^{-1}$ for $g \in G$. Here, G is a cylic group of order 3, and it is easy to check that $\phi(g) = g^{-1}$ satisfies the required conditions.

The only remaining case is that no element of $G - \{e\}$ is fixed by ϕ, and G contains at least two distinct pairs of inverse elements. Let $g_1, g_1^{-1}, g_2, g_2^{-1}$ be distinct elements of G. Assume without loss of generality that $g_2 = g_1^2$. We know that $g_1^{-1} = g_2^2$ or $g_2 = g_1^{-2}$. The second option is impossible (if true, then $g_2 = g_2^{-1}$, which would imply $\phi(g_2) = g_2$), so $g_1^{-1} = g_2^2$. Therefore, $g_1^{-1} = g_2^2 = g_1^4$, and the order of g_1 in G is 5. Furthermore, since $g_2 = g_1^2$ and $g_1 = g_2^{-2}$, each of g_1, g_2 belongs to the group generated by the other. Since g_1, g_2 were chosen arbitrarily, the entire group G is the group generated by g_1, a cyclic group of order 5. Once again it is easy to check that $\phi(g) = g^{-1}$ satisfies the required conditions.

Editorial comment. Kenneth Schilling observed that the hypothesis that G is finite is not needed, although the reference to the singleton set $\{\sum_{g \in G} g\}$ in the problem statement does not make sense without that hypothesis.

Also solved by N. Caro (Brazil), R. Chapman (UK), J. H. Lindsey II, and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

C4. *From the 1968 Moscow Mathematical Olympiad, contributed by the editors.* A round cake has icing on the top but not the bottom. Cut a piece of the cake in the usual shape of a sector with vertex angle one radian and with vertex at the center of the cake. Remove the piece, turn it upside down, and replace it in the cake to restore roundness. Next, move one radian around the cake, cut another piece with the same vertex angle adjacent to the first, remove it, turn it over, and replace it. Keep doing this, moving around the cake one radian at a time, inverting each piece. Show that, after a finite number of steps, all the icing will again be on the top.

The Game of Chomp

C3. *Attributed to Frederik Schuh, contributed by the editors.* Alice and Bob play a game in which they take turns removing squares from an m-by-n grid of squares. We label the square in row i and column j with the pair (i, j). A legal move in this game consists of selecting one of the remaining squares (i, j) and removing all the squares (a, b) with $i \le a \le m$ and $j \le b \le n$ that were not were not already removed by a previous move. The players alternate moves, with Alice going first, and the player who removes the square $(1, 1)$ loses. Show that Alice has a winning strategy.

Solution. Since the game is finite, either Alice or Bob has a winning strategy. Suppose it is Bob who has a winning strategy. If Alice removes just the single square (m, n) on her first move, then Bob has a winning response (i, j), leading to a position P from which Alice has no winning response. But Alice could have selected square (i, j) on her first move, and this would have been a winning move for Alice, since it leaves Bob to play from position P. This contradicts the assumption that Bob has a winning strategy, so it must be Alice who has a winning strategy.

Editorial comment. The solution illustrates the concept of *strategy stealing* from combinatorial game theory. It demonstrates that Alice has a winning move to open the game, although it does not tell her what that move is. Indeed, little is known about how Alice should play. It is easy to see that Alice's only winning opening move in the case $m = 1$ is $(1, 2)$ and in the case $m = 2$ is $(2, n)$. When $m = n$, Alice's only winning opening move is $(2, 2)$. Some progress on the $m = 3$ case is given in D. Zeilberger (2001), Three-rowed Chomp, *Adv. Appl. Math.* 26, 168–179.

The game goes back to Frederick Schuh, whose version of the game is played on the positive integers, with players alternately choosing divisors of a given integer, subject to the restriction that no choice can be a multiple of a previous choice. The version of the game that we have given here is due to David Gale. It is isomorphic to Schuh's game in the case that the integer is $2^m 3^n$.

SOLUTIONS

The Polytope of Parking Functions

12191 [2020, 563]. *Proposed by Richard Stanley, University of Miami, Coral Gables, FL.* A *parking function* of length n is a list $(a_1, a_2, \ldots, a_n)$ of positive integers whose increasing rearrangement $b_1 \le b_2 \le \cdots \le b_n$ satisfies $b_i \le i$. It is well known that the number of parking functions of length n is $(n+1)^{n-1}$. Let P_n denote the convex hull in $\mathbb{R}^n$ of all parking functions of length n.

(a) Find the number of vertices of the convex polytope P_n.

(b) Find the number of $(n-1)$-dimensional faces of P_n.

(c)* Find the number of integer points in P_n, i.e., the number of elements of $\mathbb{Z}^n \cap P_n$. For $n \le 8$ these numbers are 1, 3, 17, 144, 1623, 22804, 383415, 7501422.

(d)* Find the volume of P_n. For $n \le 5$ these volumes are 0, $1/2$, 4, $159/4$, 492.

Solution by Richard Stong, Center for Communications Research, San Diego, CA.
(a) Let a *tight* parking function be one whose increasing rearrangement consists of k copies of 1 followed by the numbers $k+1$ through n. Since parking functions remain parking functions when coordinates are reordered, there are $n!/k!$ parking functions with this increasing rearrangement and hence $\sum_{k=1}^{n} n!/k!$ tight parking functions of length n. The sum evaluates to $\lfloor n!(e-1) \rfloor$. We prove by induction on n that the tight parking functions are exactly the vertices of P_n.

Suppose first that n occurs in the parking function a. By the reordering criterion, n can only occur once. Every vertex of a face containing a in its interior must also have n in the same place. Deleting n from a parking function of length n always leaves a parking function of length $n-1$. This applies both to a and to the vertices of any face containing a. Thus if the parking function obtained by deleting n from a is a vertex in P_{n-1}, then a is a vertex of P_n. The converse holds as well. By the induction hypothesis, a is a vertex if and only if it is tight.

Suppose next that n does not occur in a. If every position in a is 1, then a minimizes the sum of entries over all parking functions. Hence it is a vertex; also it is tight. If some position in a is not 1, then a is not tight. Pick a largest entry of a, and let a^+ and a^- be the results of replacing this entry with n or 1, respectively. These are both parking functions: for a^- we have only lowered b, and for a^+ we have only changed b_n to n. Since the largest entry was not 1 or n, a is in the interior of the segment joining a^- and a^+ and hence is not a vertex.
(b) There are $2^n - 1$ such faces. The faces of a polytope are the sets of points where some linear function is maximized, and such a set is the convex hull of the vertices that achieve the maximum. By the reordering property of parking functions, when a linear function $x \mapsto \alpha \cdot x$ is maximized at a the coordinate values for α and a will be in the same order. That is, $\alpha \cdot a = \beta \cdot b$, where β is the increasing rearrangement of α and b is the increasing rearrangement of a.

Furthermore, if the first r entries of β are negative, then at a maximum the first r entries of b are all 1. Similarly, if the last s entries of β are positive, then at a maximum the last s entries of b are $(n+1-s, \dots, n)$ (after possibly re-sorting the places where β has a run of equal entries). That is, if β has m equal positive entries, then those m entries of b are m consecutive integers in some order; in particular, the sum of those m entries is fixed.

Putting this together, we see that if α has r negative entries and t distinct positive values, then the set of points a maximizing $\alpha \cdot x$ has codimension at least $r+t$ (we fix one entry for each negative entry in α and one sum of entries for each positive value).

Thus $(n-1)$-dimensional faces must correspond to α with $r+t=1$. Up to rescaling, faces must correspond either to α being 1 on some set S of coordinates and 0 elsewhere (which we denote by α_S), or to α being -1 in one coordinate and 0 elsewhere.

If $|S| \neq n-1$, then the codimension-1 hyperplane $\alpha_S \cdot x = n|S| - \binom{|S|}{2}$ passes through all the vertices whose coordinates in the S positions are a permutation of $\{n+1-|S|, \dots, n\}$ and in the other positions are any parking function of length $n-|S|$. Thus we obtain a codimension-1 face that is isometric to the product $P_{n-|S|} \times Q_{|S|}$, where Q_k is the convex hull of the points that are permutations of $(1, 2, \dots, k)$. Note that $P_{n-|S|}$ has dimension $n-|S|$, since $n-|S| \neq 1$, and $Q_{|S|}$ has dimension $|S|-1$. The product has dimension $n-1$. If $|S| = n-1$, then since P_1 is only a single point we obtain a face of codimension 2 and dimension $n-2$, contributing nothing to our count.

If α has a single -1 and zeroes elsewhere, then we get the face of codimension 1 (and dimension $n-1$) where that coordinate is fixed to 1.

The first case gives $2^n - n - 1$ faces (corresponding to nonempty subsets of coordinates with size other than $n-1$), and the second case gives n faces. Hence the number of faces of P_n is $2^n - 1$.

(c) No solution is available.

(d) Letting V_n denote the n-dimensional volume of P_n, we prove

$$V_n = \sum_{s=1}^{n} \binom{n-1}{s-1} \frac{n^{n-s}}{2^s} \sum_{m=0}^{s} (-1)^{s-m} \binom{s}{m} (2m-1)!!.$$

Let W_k denote the $(k-1)$-dimensional volume of the polytope Q_k in part (b). We first derive a closed formula for W_k. The polytope Q_k has $\binom{k}{r}$ faces isometric to $Q_r \times Q_{k-r}$, corresponding to fixing r coordinates with sum $r(r+1)/2$, leaving the remaining $k-r$ coordinates to sum to $k(k+1)/2 - r(r+1)/2$, which equals $(k-r)(k+r+1)/2$. (The proof of this is essentially the same as part **(b)** above.)

The distance from the center $((k+1)/2, (k+1)/2, \ldots, (k+1)/2)$ of Q_k to the plane of such a face is

$$\sqrt{r(k-r)^2/4 + (k-r)r^2/4} = \sqrt{kr(k-r)}/2.$$

Hence

$$W_k = \frac{1}{2(k-1)} \sum_{r=1}^{k-1} \binom{k}{r} W_r W_{k-r} \sqrt{kr(k-r)}.$$

This recurrence yields $W_k = k^{k-3/2}$ using induction and the identity

$$2(k-1)k^{k-2} = \sum_{r=1}^{k-1} \binom{k}{r} r^{r-1} (k-r)^{k-r-1}. \tag{$*$}$$

The identity $(*)$ appears in the book of Lovász (*Combinatorial Problems and Exercises*, North-Holland, 1979). It has both an analytic proof using generating functions and a bijective proof (due to L. Smiley) using Cayley's formula, which states that there are k^{k-2} trees with vertex set $[k]$, where $[k] = \{1, \ldots, k\}$. With n ways to distinguish one vertex as a root, there are k^{k-1} rooted trees with vertex set $[k]$. Both sides of the identity count the ordered pairs of rooted trees whose vertex sets have union $[k]$.

Splitting P_n into cones with vertex at the point $(1, , \ldots, 1)$, and invoking the solution of part (b), we see that P_n is the union, over values of k other than $n-1$, of $\binom{n}{k}$ cones with base $P_{n-k} \times Q_k$ and height $\sqrt{k}(2n-k-1)/2$. Thus

$$V_n = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} V_{n-k} W_k \frac{\sqrt{k}(2n-k-1)}{2} = \frac{1}{2n} \sum_{k=1}^{n} \binom{n}{k} V_{n-k} k^{k-1} (2n-k-1).$$

In this sum, we have included the term for $k = n-1$, but the computation remains correct since $V_1 = 0$. Let V be the exponential generating function of the sequence $\langle V_n \rangle$, so

$$V(z) = \sum_{n=0}^{\infty} \frac{V_n}{n!} z^n = \sum_{n=1}^{\infty} \frac{1}{2n} \sum_{k=1}^{n} \binom{n}{k} V_{n-k} k^{k-1} (2n-k-1) \frac{z^n}{n!}. \tag{$**$}$$

Let $F(z) = \sum_{n=1}^{\infty} n^{n-1} z^n / n!$. Differentiating $(**)$ and breaking the factor $2n-k-1$ into three pieces, we obtain

$$\begin{aligned}
2V'(z) &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{n}{k} V_{n-k} k^{k-1} (2n-2k+k-1) \frac{z^{n-1}}{n!} \\
&= 2F(z)V'(z) + F'(z)V(z) - \frac{F(z)V(z)}{z} \\
&= 2F(z)V'(z) + \left(F'(z) - \frac{F(z)}{z}\right) V(z).
\end{aligned} \tag{$***$}$$

We next study $F' - F/z$. Again using Cayley's formula, F is the exponential generating function (EGF) for rooted labeled trees: there are n^{n-1} with vertex set $[n]$. To form such a rooted tree, one chooses a root label and a rooted forest on the remaining labels, with any number of components. The EGF for choosing the root is just z, and the two choices are enumerated by the product of the EGFs, which yields the standard relation $F = ze^F$ (from which Cayley's formula can be obtained by Lagrange inversion).

Taking the logarithm of $F = ze^F$ yields $\log F = F + \log z$, and differentiating yields $F'/F = F' + 1/z$, or $F' - F/z = FF'$. Equation (***) then becomes

$$\frac{V'}{V} = \frac{FF'}{2(1-F)}.$$

Integrating yields $\log V = -\left(F + \log(1-F)\right)/2$, and hence

$$V(z) = \frac{e^{-F(z)/2}}{\sqrt{1-F(z)}}.$$

We now have both a recurrence and an EGF for V_n, and we have left the realm of geometry. A more explicit formula for V_n as a double sum can be derived from the generating function. The standard expansions of e^x and $(1-4x)^{-1/2}$ yield

$$\frac{e^{-F/2}}{\sqrt{1-F}} = \sum_{k=0}^{\infty} \frac{(-1)^k F^k}{k!2^k} \sum_{m=0}^{\infty} \binom{2m}{m} \frac{F^m}{4^m} = \sum_{s=0}^{\infty} \sum_{m=0}^{s} \frac{(-1)^{s-m}}{(s-m)!2^{s+m}} \binom{2m}{m} F^s.$$

Also, the series expansion for F^s is known to be

$$F^s(z) = \sum_{r=s}^{\infty} \frac{sr^{r-s-1}}{(r-s)!} z^r,$$

since the coefficient $[z^r]F^s(z)$ of z^r in $F^s(z)$ is given by

$$\frac{1}{2\pi i} \oint \frac{F^s(z)}{z^{r+1}}\, dz = \frac{1}{2\pi i} \oint \frac{(1-F)e^{rF}}{F^{r+1-s}}\, dF = [F^{r-s}](1-F)e^{rF}$$
$$= \frac{r^{r-s}}{(r-s)!} - \frac{r^{r-s-1}}{(r-s-1)!} = \frac{sr^{r-s-1}}{(r-s)!}.$$

Finally, set $r = n$ and plug this expression for the coefficient of z^n in F^s into the expansion of $e^{-F/2}/\sqrt{1-F}$ in terms of F. Since we defined V to be an EGF, we seek the coefficient of $z^n/n!$ and hence must introduce $n!$ also into the numerator. After a little algebra, we read off the formula

$$V_n = \sum_{s=1}^{n} \binom{n-1}{s-1} \frac{n^{n-s}}{2^s} \sum_{m=0}^{s} (-1)^{s-m} \binom{s}{m} (2m-1)!!.$$

Editorial comment. The inner sum in the formula for V_n is the well-known inclusion-exclusion formula for the number of ways to form s couples into pairs of people with no couple paired (sequence A053871 in the OEIS). Also, the generating function for V and standard techniques yield

$$V_n \sim \frac{2^{1/4}\pi^{1/2}}{\Gamma(1/4)e^{1/2}} \cdot n^{n-1/4} \left(1 + O(n^{-1/2})\right).$$

Parts **(a)** and **(b)** also solved by A. Amanbayeva & D. Wang and the proposer.

An Integral Bound

12193 [2020, 564]. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.* Suppose that $f:[0,1]\to\mathbb{R}$ has a continuous third derivative and $f(0)=f(1)$. Prove

$$\left|\int_0^1 f'(x)x^{k-1}(1-x)^{k-1}\,dx\right| \le \frac{(k-1)k!(k-1)!}{6\,(2k+1)!}\max_{0\le x\le 1}\left|f'''(x)\right|$$

where k is a positive integer.

Solution by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China. We proceed by induction on k. For the base case $k=1$, the left side is $|f(1)-f(0)|=0$ and the inequality is immediate.

For the inductive step, let $g_k(x)=x^k(1-x)^k$ and $I_k=\int_0^1 f'(x)g_{k-1}(x)\,dx$. One easily checks that

$$g_k''(x) = -2k(2k-1)g_{k-1}(x)+k(k-1)g_{k-2}(x). \tag{$*$}$$

For $k\ge 2$, $g_k(0)=g_k(1)=g_k'(0)=g_k'(1)=0$, so integrating by parts twice yields

$$\begin{aligned}\int_0^1 f'(x)g_k''(x)\,dx &= \Big[f'(x)g_k'(x)\Big]_0^1-\Big[f''(x)g_k(x)\Big]_0^1+\int_0^1 f'''(x)g_k(x)\,dx\\ &= \int_0^1 f'''(x)g_k(x)\,dx.\end{aligned}$$

Using ($*$) this yields

$$-2k(2k-1)I_k+k(k-1)I_{k-1}=\int_0^1 f'''(x)g_k(x)\,dx.$$

From the triangle inequality and $g_k(x)\ge 0$ for $x\in[0,1]$, we get

$$\begin{aligned}2k(2k-1)\,|I_k| &\le k(k-1)\,|I_{k-1}|+\left|\int_0^1 f'''(x)g_k(x)\,dx\right|\\ &\le k(k-1)\,|I_{k-1}|+\max_{0\le x\le 1}|f'''(x)|\left|\int_0^1 g_k(x)\,dx\right|.\end{aligned}$$

Recognizing $\int_0^1 g_k(x)\,dx$ as a beta integral, we have

$$\int_0^1 g_k(x)\,dx = B(k+1,k+1)=(k!)^2/(2k+1)!.$$

Using this together with the induction hypothesis gives

$$2k(2k-1)\,|I_k| \le \frac{(k-2)k!(k-1)!}{6(2k-1)!}\max_{0\le x\le 1}|f'''(x)|+\frac{(k!)^2}{(2k+1)!}\max_{0\le x\le 1}|f'''(x)|,$$

which after simplifying becomes

$$|I_k|\le \frac{(k-1)k!(k-1)!}{6(2k+1)!}\max_{0\le x\le 1}|f'''(x)|,$$

completing the induction.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), P. Bracken, R. Chapman (UK), C. Chiser (Romania), R. Guadalupe (Philippines), F. Holland (Ireland), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, M. Omarjee (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, L. Zhou, and the proposer.

Regular Polygons Inscribed in a Cube

12195 [2020, 659]. *Proposed by Joseph DeVincentis, Salem, MA, James Tilley, Bedford Corners, NY, and Stan Wagon, Macalester College, St. Paul, MN.* For which integers n with $n \geq 3$ can a regular n-gon be inscribed in a cube? The vertices of the n-gon must all lie on the cube but may not all lie on a single face.

Composite solution by Eugen J. Ionaşcu, Columbus State University, Columbus, GA, and Yury J. Ionin, Champaign, IL. An inscribed n-gon exists if and only if $3 \leq n \leq 9$ or $n = 12$. We work in the standard unit cube. We first show that regular n-gons embed in the cube for $n \in \{3, 4, 6, 8, 12\}$.

For $n = 3$, corners $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ yield an equilateral triangle.

For $n = 4$, points $(0, 0, 1/2)$, $(1, 0, 1/2)$, $(1, 1, 1/2)$, and $(0, 1, 1/2)$ determine a square embedded in the cube. Truncating it yields a regular octagon with all vertices on the faces of the cube, which takes care of $n = 8$.

For $n = 6$, points $(1/2, 0, 1)$, $(0, 1/2, 1)$, $(0, 1, 1/2)$, $(1/2, 1, 0)$, $(1, 1/2, 0)$, and $(1, 0, 1/2)$ determine a regular hexagon embedded in the cube. Truncating it yields a regular 12-gon with all vertices on the faces of the cube, which takes care of $n = 12$.

Next, we give constructions for $n \in \{5, 7, 9\}$, showing that a regular n-gon can be inscribed in a polygon embedded in the cube.

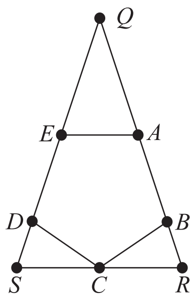


Figure 1a

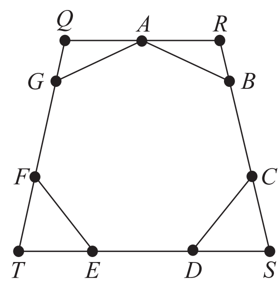


Figure 1b

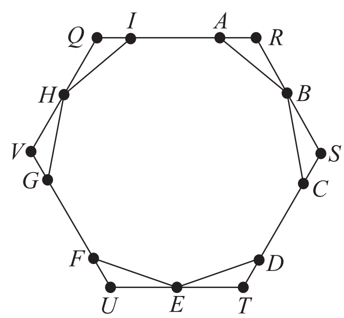


Figure 1c

For $n = 5$, we start with a regular pentagon $ABCDE$. Let lines AB and DE intersect at Q, and let the line through C perpendicular to CQ intersect lines AB and DE at R and S, respectively, as in Figure 1a. The isosceles triangle QRS has apex angle $\pi/5$. With $Q = (0, 0, a)$, $R = (0, b, 0)$, and $S = (b, 0, 0)$, the apex angle of QRS has cosine equal to $a^2/(a^2 + b^2)$. We may choose real numbers $a, b \in (0, 1)$ such that this equals $\cos(\pi/5)$.

For $n = 7$, we start with a regular heptagon $ABCDEFG$. Let the line through A parallel to DE intersect lines FG and BC at Q and R, respectively. Let line DE intersect lines BC and FG at S and T, respectively, as in Figure 1b. Now $QRST$ is an isosceles trapezoid with acute angles $3\pi/7$. An isosceles trapezoid is uniquely determined, up to similarity, by the measure of its acute angles and the ratio k of the shorter base to the longer base. By the law of sines,

$$k = \frac{QR}{ST} = \frac{2\sin(2\pi/7)/\sin(4\pi/7)}{1 + 2\sin(2\pi/7)/\sin(3\pi/7)} = \frac{2\sin(2\pi/7)}{\sin(3\pi/7) + 2\sin(2\pi/7)}.$$

Set $T = (a, 0, 0)$, $S = (0, a, 0)$, $Q = (ka, 0, 1)$, and $R = (0, ka, 1)$. It is required that $a \in (0, 1)$ satisfies

$$\cos(3\pi/7) = \frac{\overrightarrow{TQ} \cdot \overrightarrow{TS}}{TQ \cdot TS} = \frac{(1-k)a}{\sqrt{2} \cdot \sqrt{(1-k)^2a^2 + 1}}.$$

Solving for a yields

$$a = \frac{\sqrt{2}\cos(3\pi/7)}{(1-k)\sqrt{1-2\cos^2(3\pi/7)}} \approx 0.8633.$$

For $n = 9$, first observe that for $0 < a < 1$, the plane $x - y + z = a$ intersects the cube in a hexagon $QRSTUV$, where $Q = (0, 0, a)$, $R = (0, 1 - a, 1)$, $S = (a, 1, 1)$, $T = (1, 1, a)$, $U = (1, 1 - a, 0)$, and $V = (a, 0, 0)$. We compute that $QR = ST = UV = (1 - a)\sqrt{2}$, that $QV = RS = TU = a\sqrt{2}$, and that all six angles are equal. Hence they measure $2\pi/3$. Let B, E, and H be the midpoints of RS, TU, and QV, respectively. Let points A and I on QR, C and D on ST, and F and G on UV be such that angles ABR, CBS, DET, FEU, GHV, and IHQ, each measure $\pi/9$ (see Figure 1c). All the angles of nonagon $ABCDEFGHI$ measure $7\pi/9$. Finally, by the law of sines, six sides of the nonagon have length $a\sqrt{6}/(4\sin(2\pi/9))$ and three sides have length

$$(1-a)\sqrt{2} - 2 \cdot \frac{a\,\sin(\pi/9)}{\sqrt{2}\sin(2\pi/9)}.$$

Setting the two lengths equal and solving for a yields

$$a = \frac{4\sin(2\pi/9)}{\sqrt{3} + 4\sin(\pi/9) + 4\sin(2\pi/9)} \approx 0.4534.$$

We conclude by showing the impossibility of inscribing a regular n-gon for $n > 12$ and $n \in \{10, 11\}$.

The vertices of a regular n-gon inscribed in the unit cube lie in the intersection of the cube with the plane containing the n-gon. Thus a face of the cube contains at most two vertices of the n-gon, which yields $n \leq 12$.

Next, we exclude $n = 11$. Since no two sides of a regular 11-gon are parallel, opposite faces of the cube together contain at most three vertices of the 11-gon, but this limits the number of vertices to 9.

Finally, for $n = 10$, consider a regular inscribed 10-gon $ABCDEFGHIJ$. Since it lies in the intersection of a plane $\mathcal{P}$ with the cube, opposite faces of the cube cannot together contain exactly three vertices. Any four vertices on opposite faces must form opposite sides of the 10-gon. Also, the vertices of opposite sides of the 10-gon must form a rectangle. Thus the intersection of $\mathcal{P}$ with the cube must be a hexagon $QRSTUV$ with opposite sides parallel. We may assume that $\mathcal{P}$ intersects the plane $z = 1$ in QR and the xy-plane in TU with the 10-gon inscribed as in Figure 2.

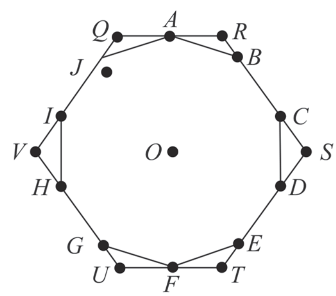


Figure 2.

The distance between sides QV and ST is the same as between sides RS and UV. This shows that the dihedral angle α between $\mathcal{P}$ and the yz-plane equals the dihedral angle β between $\mathcal{P}$ and the xz-plane. Consequently, $\mathcal{P}$ is symmetric with respect to the plane $x = y$. This symmetry implies $SV = \sqrt{2}$. Because triangles CDS and HIV are isosceles, the center O of the 10-gon is at the midpoint of SV.

We calculate the circumradius $r = OC$ using the law of sines as follows:

$$r = OS\,\frac{\sin(3\pi/10)}{\sin(3\pi/5)} = \frac{1}{2\sqrt{2}\cos(3\pi/10)}.$$

Let γ denote the dihedral angle between $\mathcal{P}$ and the xy-plane. It satisfies

$$\cos^2\gamma = 1 - \sin^2\gamma = 1 - \left(\frac{1}{2r}\right)^2 = 1 - 2\cos^2(3\pi/10) = \cos(2\pi/5) = \frac{\sqrt{5}-1}{4}.$$

The distance between sides RS and UV is $2r\cos(\pi/10)$, so

$$\cos^2\alpha = \cos^2\beta = 1 - \sin^2\alpha = 1 - \frac{2\cos^2(3\pi/10)}{\cos^2(\pi/10)} = 1 - 2\left(4\cos^2(\pi/10) - 3\right)^2 = \sqrt{5} - 2.$$

It is well known and easy to prove that if a plane has dihedral angles α, β, and γ with the yz-, xz-, and xy-planes, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

This yields a contradiction, because

$$2\cos^2\alpha + \cos^2\gamma = \frac{9\sqrt{5}-17}{4} \neq 1.$$

Editorial comment. A few solvers interpreted the problem as requiring that the entire n-gon be embedded in the cube, which is possible if and only if $n = 3, 4, 6$.

Also solved by R. Stong and the proposers.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

C3. *Attributed to Frederik Schuh, contributed by the editors.* Alice and Bob play a game in which they take turns removing squares from an m-by-n grid of squares. We label the square in row i and column j with the pair (i, j). A legal move in this game consists of selecting one of the remaining squares (i, j) and removing all the squares (a, b) with $i \le a \le m$ and $j \le b \le n$ that were not were not already removed by a previous move. The players alternate moves, with Alice going first, and the player who removes the square $(1, 1)$ loses. Show that Alice has a winning strategy.

A Curious Characterization of the Fibonacci Numbers

C2. *Ira Gessel [1972], contributed by the editors.* Prove that a positive integer n is a Fibonacci number if and only if either $5n^2 + 4$ or $5n^2 - 4$ is a perfect square.

Solution. The Fibonacci numbers are defined by: $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_k + F_{k+1}$ when $k \ge 0$. Using the well-known identity $F_{k-1}F_{k+1} = F_k^2 + (-1)^k$, we obtain

$$\begin{aligned} 5F_k^2 + (-1)^k 4 &= 5F_k^2 + 4(F_{k-1}F_{k+1} - F_k^2) \\ &= (F_{k+1} - F_{k-1})^2 + 4F_{k-1}F_{k+1} = (F_{k+1} + F_{k-1})^2. \end{aligned}$$

This shows that $5n^2 + 4$ or $5n^2 - 4$ is a perfect square when n is Fibonacci.

For the converse, we prove that if m and n are positive integers satisfying $5n^2 \pm 4 = m^2$, then there exists some positive integer k such that $n = F_k$ and $m = F_{k-1} + F_{k+1}$.

The proof is by induction on n. For $n = 1$, there are two cases: Either $m = 1$, in which case $n = F_1$ and $m = F_0 + F_2$, or $m = 3$, in which case $n = F_2$ and $m = F_1 + F_3$.

For the induction step, suppose $n \geq 2$, the result holds for smaller values of n, and for some positive integer m, $5n^2 \pm 4 = m^2$. Note that

$$m^2 \leq 5n^2 + 4 \leq 5n^2 + n^2 = 6n^2 < 9n^2,$$

so $m < 3n$. Also

$$m^2 \geq 5n^2 - 4 \geq 5n^2 - n^2 = 4n^2,$$

so $m \geq 2n$.

Let $n_1 = (m - n)/2$. Since the parities of n and m are the same, n_1 is an integer, and from $2n \leq m < 3n$ we get $n/2 \leq n_1 < n$. Let $m_1 = (5n - m)/2$. Again we see that m_1 is an integer and $m_1 > (5n - 3n)/2 = n$. So n_1 and m_1 are positive integers and $n_1 < n$. Also:

$$5n_1^2 = \frac{5(n^2 - 2nm + m^2)}{4} = \frac{5(6n^2 \pm 4 - 2nm)}{4} = \frac{15n^2 - 5nm}{2} \pm 5,$$

$$m_1^2 = \frac{25n^2 - 10nm + m^2}{4} = \frac{30n^2 \pm 4 - 10nm}{4} = \frac{15n^2 - 5nm}{2} \pm 1.$$

It follows that $5n_1^2 \mp 4 = m_1^2$. By the induction hypothesis, there is a positive integer k such that $n_1 = F_k$ and $m_1 = F_{k-1} + F_{k+1}$.

From the equations $n_1 = (m - n)/2$ and $m_1 = (5n - m)/2$, we get

$$n = \frac{n_1 + m_1}{2} = \frac{F_k + F_{k-1} + F_{k+1}}{2} = \frac{2F_{k+1}}{2} = F_{k+1} \quad \text{and}$$

$$m = \frac{5n_1 + m_1}{2} = \frac{5F_k + F_{k-1} + F_{k+1}}{2} = \frac{2F_k + 2F_{k+2}}{2} = F_k + F_{k+2}.$$

Editorial comment. The problem appeared as Problem H-187 in *Fibonacci Quarterly* 10 (1972) 417–419. The equation $5n^2 \pm 4 = m^2$ can be rearranged to read $m^2 - 5n^2 = \pm 4$, which is a variant of Pell's equation, and our proof that n in this equation must be a Fibonacci number is based on a standard method for solving Pell's equation. An alternative way to prove that n is a Fibonacci number is to let $j = (m + n)/2$ and then show that $\gcd(j, n) = 1$ and $|j/n - \phi| < 1/(2n^2)$, where ϕ is the golden mean $(1 + \sqrt{5})/2$. It follows that j/n is a convergent of the continued fraction for ϕ, and it is well known that these convergents are ratios of successive Fibonacci numbers (see G. H. Hardy and E. M. Wright (2008), *An Introduction to the Theory of Numbers*, 6th ed., Oxford: Oxford Univ. Press, pp. 190, 196). Yet another proof begins by rewriting $5n^2 \pm 4 = m^2$ in the form $(m + \sqrt{5}n)/2 \cdot (m - \sqrt{5})/2 = \pm 1$ and then using the fact that any unit in the ring $\mathbb{Z}[\phi]$ is of the form $\pm\phi^k$.

There is a connection to Hilbert's tenth problem about Diophantine equations. A set $X \subset \mathbb{N}^r$ is called *Diophantine* if there is a polynomial p with integer coefficients in $r + s$ variables such that $a \in X$ if and only if there exists $b \in \mathbb{N}^s$ such that $p(a, b) = 0$. This problem shows that the set of Fibonacci numbers is Diophantine, by setting $p(x, y) = (5x^2 + 4 - y^2)(5x^2 - 4 - y^2)$. In 1961, Martin Davis, Hilary Putnam, and Julia Robinson showed that a negative answer to Hilbert's tenth problem follows from the existence of a Diophantine set of the form $\{(n, f(n)) : n \in \mathbb{N}\}$, where f has exponential growth. In 1970, Y. V. Matiyasevich showed that the set $\{(n, F_{2n}) : n \in \mathbb{N}\}$ is Diophantine, settling Hilbert's problem. It is not hard to use this to prove that $\{(n, F_n) : n \in \mathbb{N}\}$ is Diophantine. The full story can be found in M. R. Davis (1973), Hilbert's tenth problem is unsolvable, *Amer. Math. Monthly* 80, 233–269.

SOLUTIONS

Evaluating an Integral with Leibniz's Help

12184 [2020, 461]. *Proposed by Paolo Perfetti, Universitá degli Studi di Roma "Tor Vergata," Rome, Italy.* Prove

$$\int_1^\infty \frac{\ln(x^4 - 2x^2 + 2)}{x\sqrt{x^2-1}}\,dx = \pi \ln(2+\sqrt{2}).$$

Solution by Warren P. Johnson, Connecticut College, New London, CT. For positive numbers a and b, we consider the integral

$$I(a,b) = \int_0^{\pi/2} \ln(a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta.$$

By substituting $\theta = \pi/2 - \phi$, we see that $I(a,b) = I(b,a)$. The Leibniz integral rule yields

$$\frac{\partial I}{\partial a} = \int_0^{\pi/2} \frac{2a\cos^2\theta\,d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} \quad \text{and} \quad \frac{\partial I}{\partial b} = \int_0^{\pi/2} \frac{2b\sin^2\theta\,d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}, \tag{1}$$

and it follows that

$$a\frac{\partial I}{\partial a} + b\frac{\partial I}{\partial b} = \int_0^{\pi/2} 2\,d\theta = \pi. \tag{2}$$

Also, using the substitution $b\tan\theta = a\tan\phi$ we see that

$$\begin{aligned} b\frac{\partial I}{\partial a} + a\frac{\partial I}{\partial b} &= \int_0^{\pi/2} \frac{2ab\,d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} \\ &= \int_0^{\pi/2} \frac{2ab\sec^2\theta\,d\theta}{a^2 + b^2\tan^2\theta} = \int_0^{\pi/2} 2\,d\phi = \pi. \end{aligned} \tag{3}$$

When $a \neq b$, the solution to (2) and (3) is

$$\frac{\partial I}{\partial a} = \frac{\partial I}{\partial b} = \frac{\pi}{a+b},$$

and it is easily checked from (1) that this is also correct when $a = b$.

Since I is symmetric in a and b, integrating with respect to either a or b gives $I(a,b) = \pi \ln(a+b) + K$ for some constant K. Setting $b = a$ we find

$$K = I(a,a) - \pi \ln(2a) = \pi \ln a - \pi \ln(2a) = -\pi \ln 2,$$

so

$$I(a,b) = \int_0^{\pi/2} \ln\left(a^2 \cos^2\theta + b^2 \sin^2\theta\right) d\theta = \pi \ln\left(\frac{a+b}{2}\right). \tag{4}$$

From this we can derive the well-known integral

$$\begin{aligned} \int_0^{\pi/2} \ln(\cos\theta)\, d\theta &= \lim_{b\to 0^+} \frac{1}{2} \int_0^{\pi/2} \ln(\cos^2\theta + b^2 \sin^2\theta)\, d\theta \\ &= \lim_{b\to 0^+} \frac{\pi}{2} \ln\left(\frac{1+b}{2}\right) = -\frac{\pi}{2} \ln 2. \end{aligned} \tag{5}$$

(We omit the justification of this limit calculation, since the result is well known.) Combining (4) and (5) we have

$$\begin{aligned} \int_0^{\pi/2} \ln(a^2 + b^2 \tan^2\theta)\, d\theta &= \int_0^{\pi/2} \ln(a^2 \cos^2\theta + b^2 \sin^2\theta) - 2\ln(\cos\theta)\, d\theta \\ &= \pi \ln\left(\frac{a+b}{2}\right) + \pi \ln 2 = \pi \ln(a+b). \end{aligned} \tag{6}$$

With this in hand, we turn to the integral in the problem, which we denote by P. Using the substitution $u = \sqrt{x^2-1}$, we obtain

$$P = \int_0^\infty \frac{\ln\left(1+u^4\right)}{u^2+1}\, du.$$

The further substitution $v = 1/u$ shows that we also have

$$P = \int_0^\infty \frac{\ln(1+1/u^4)}{u^2+1}\, du,$$

and averaging these two expressions yields

$$P = \int_0^\infty \frac{\ln(u^2+1/u^2)}{u^2+1}\, du.$$

Now substitute $v = u - 1/u$ to get

$$P = \int_{-\infty}^\infty \frac{\ln\left(v^2+2\right)}{v^2+4}\, dv = 2\int_0^\infty \frac{\ln\left(v^2+2\right)}{v^2+4}\, dv.$$

Finally, substituting $v = 2\tan\theta$ yields

$$P = \int_0^{\pi/2} \ln\left(2 + 4\tan^2\theta\right) d\theta,$$

which by (6) is $\pi \ln(2+\sqrt{2})$.

Also solved by Z. Ahmed (India), K. F. Andersen (Canada), F. R. Ataev (Uzbekistan), M. Bataille (France), N. Batir (Turkey), A. Berkane (Algeria), N. Bhandari (Nepal), K. N. Boyadzhiev, P. Bracken, B. Bradie, B. S. Burdick, W. Chang, R. Chapman (UK), H. Chen, Ó. Ciaurri (Spain), B. E. Davis, P. De & B. Sury (India), A. Eydelzon, G. Fera (Italy), P. Fulop (Hungary), M. L. Glasser, H. Grandmontagne (France), N. Grivaux (France), J. A. Grzesik, L. Han, E. A. Herman, N. Hodges (UK), E. J. Ionaşcu, W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), G. Lavau (France), K. Mahanta (India), L. Matejíčka (Slovakia), K. Nelson, Q. M. Nguyen (Canada), M. Omarjee (France), M. A. Prasad (India), K. Sarma (India), V. Schindler

(Germany), S. Sharma (India), F. Sinani (Kosovo), A. Stadler (Switzerland), A. Stenger, S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), E. I. Verriest, M. Vowe (Switzerland), T. Wiandt, H. Widmer (Switzerland), Y. Xiang (China), M. R. Yegan (Iran), FAU Problem Solving Group, and the proposer.

A Class of Matrices with Determinant 1

12185 [2020, 659]. *Proposed by George Stoica, Saint John, NB, Canada.* Let $n_1, \dots, n_k$ be pairwise relatively prime odd integers greater than 1. For $i \in \{1, \dots k\}$, let $f_i(x) = \sum_{m=1}^{n_i} x^{m-1}$. Let A be a $2k$-by-$2k$ matrix with real entries such that $\det f_j(A) = 0$ for all $j \in \{1, \dots, k\}$. Prove $\det A = 1$.

Solution by Nicolás Caro, Universidade Federal de Pernambuco, Recife, Brazil. For each i, the set U_i of complex roots of the polynomial f_i consists precisely of the n_ith roots of unity other than 1. When $i \neq j$, there exist integers r and s such that $rn_i + sn_j = 1$, and so if $\lambda \in \mathbb{C}$ satisfies $\lambda^{n_i} = \lambda^{n_j} = 1$, then $\lambda = (\lambda^{n_i})^r (\lambda^{n_j})^s = 1$. Thus the sets $U_1, \dots, U_k$ are pairwise disjoint. Moreover, $\lambda \in U_i$ implies $\overline{\lambda} \in U_i$ and $\overline{\lambda} \neq \lambda$ (because n_i is odd and greater than 1), and of course $\lambda\overline{\lambda} = 1$.

By the spectral mapping theorem, for each j there exists an eigenvalue λ_j of A such that $f_j(\lambda_j) = 0$, that is $\lambda_j \in U_j$. Since A is a real matrix, $\overline{\lambda_j}$ is also an eigenvalue of A, and therefore the $2k$ values $\lambda_1, \dots, \lambda_k, \overline{\lambda_1}, \dots, \overline{\lambda_k}$ are precisely the eigenvalues of A. Since $\det A$ is equal to the product of these eigenvalues, the determinant is 1.

Also solved by K. F. Andersen (Canada), R. Chapman (UK), J.-P. Grivaux (France), E. A. German, R. A. Horn, O. Kouba (Syria), G. Lavau (France), S. Miao (China), É. Pité, K. Sarma (India), A. Stadler (Switzerland), A. Stenger, R. Stong, B. Sury (India), E. I. Verriest, and the proposer.

A Median and Symmedian Produce Perpendicular Lines

12187 [2020, 462]. *Proposed by Khakimboy Egamberganov, Sorbonne University, Paris, France.* Given a scalene triangle ABC, let M be the midpoint of BC, and let m and s denote the median and symmedian lines, respectively, from A. (The *symmedian* line from A is the reflection of the median from A across the angle bisector from A.) Let K be the projection of C onto m, and let L be the projection of B onto s. Let P be the intersection of BL and CK, and let Q be the intersection of KL and BC. Prove that PM and AQ are perpendicular.

Solution by Haoran Chen, Jiangsu, China. We use a coordinate system in which A is the origin and the bisector of the angle at A is the positive x-axis. Thus the coordinates of B and C are (b, kb) and $(c, -kc)$, respectively, for some b, c, and k, where $b, c > 0$ and $k \neq 0$. Since the triangle is scalene, $b \neq c$. The coordinates of M are $((b+c)/2, k(b-c)/2)$, so the equations of m and s are $y = \lambda x$ and $y = -\lambda x$, respectively, where

$$\lambda = \frac{k(b-c)}{b+c}.$$

The line through C perpendicular to m has slope $-1/\lambda$, and therefore its equation is

$$y + kc = -\frac{x-c}{\lambda}. \tag{1}$$

Intersecting this line with m, we find that

$$K = \left(\frac{c(1-k\lambda)}{\lambda^2+1}, \frac{c\lambda(1-k\lambda)}{\lambda^2+1} \right).$$

Similarly, the equation of the line through B perpendicular to s is

$$y - kb = \frac{x-b}{\lambda}, \tag{2}$$

and therefore

$$L = \left(\frac{b(1-k\lambda)}{\lambda^2+1}, -\frac{b\lambda(1-k\lambda)}{\lambda^2+1} \right).$$

Equations (1) and (2) are the equations of the lines CK and BL, and intersecting them we find that

$$P = \left(\frac{(b+c)(1-k\lambda)}{2}, \frac{(c-b)(1-k\lambda)}{2\lambda} \right).$$

If $k\lambda = 1$, then $K = L = A = (0,0)$, but the statement of the problem presupposes that K and L determine a line. We therefore assume $k\lambda \neq 1$. Intersecting the lines KL and BC we obtain, after some calculation,

$$Q = \left(\frac{2bc}{(b+c)(\lambda^2+1)}, -\frac{2bck^2\lambda}{(b+c)(\lambda^2+1)} \right).$$

Finally, using the coordinates for P, M, A, and Q, we compute

$$\text{slope of } PM = \frac{b-c}{k\lambda^2(b+c)} = \frac{1}{k^2\lambda},$$
$$\text{slope of } AQ = -k^2\lambda,$$

and the conclusion follows.

Editorial comment. It is not necessary that $\triangle ABC$ be scalene; all that is required is the condition $AB \neq AC$.

There are some other interesting geometrical relationships in the configuration in this problem. Using the coordinates given above, we can compute

$$\text{slope of } KL = \frac{(b+c)\lambda}{c-b} = -k,$$
$$\text{slope of } AP = \frac{(c-b)}{\lambda(b+c)} = -\frac{1}{k}.$$

It follows that $KL \parallel AC$ and $AP \perp AB$.

The case $k\lambda = 1$, which was excluded in the solution above, occurs when $\angle CAM$ is a right angle. The configuration of the points and lines in this problem varies significantly depending on whether $\angle CAM$ is acute or obtuse and whether or not m and s are perpendicular. A few solvers gave synthetic solutions that were not completely general because they did not take into account the full range of possible configurations. Most solvers used analytic methods.

The proposer's solution shows that AQ is the radical axis of the circles with diameters AP and AM. This implies that AQ is perpendicular to the line through the centers of these two circles, which is parallel to PM.

Also solved by J. Chen (China), C. Curtis, G. Fera (Italy), J.-P. Grivaux (France), N. Hodges (UK), W. Janous (Austria), J. H. Lindsey II, C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, T. Wiandt, L. Zhou, and the proposer.

Perfect Paths through the Positive Integers

12188 [2020, 563]. *Proposed by H. A. ShahAli, Tehran, Iran.*

(a) Is there a permutation of the positive integers with the property that every pair of consecutive elements sums to a perfect square?

(b)* Is there a permutation of the positive integers with the property that every pair of consecutive elements sums to a perfect cube?

Solution by Texas State University Problem Solvers, San Marcos, TX. The answer to both questions is yes. We prove the more general claim that for every $k \in \mathbb{N}$ there is a permutation of $\mathbb{N}$ such that every pair of consecutive elements sums to a perfect kth power. This is trivial for $k = 1$, so consider $k \geq 2$.

Let G be the graph with vertex set $\mathbb{N}$ in which u and v are adjacent when $u + v$ is a kth power. It suffices to find an infinite path $n_1, n_2, \ldots$ through G that visits every vertex exactly once. For $u \in \mathbb{N}$, let G_u be the subgraph of G induced by $\{n \in \mathbb{N}\colon\ n \geq u\}$. For $x, y \in \mathbb{N}$, write $x \to y$ when G_x has a path from x to y.

We first prove the *xyz-property*: If $y, z \in V(G_x)$, then $x \to z$ and $y \to z$ imply $x \to y$. This holds because the actual edges of G in a path in G_u witnessing $u \to v$ are undirected. Following a path from x to z and then a path from z to y in G yields a walk from x to y in G, which contains a path from x to y. Furthermore, since the edges came from G_x and G_y, they all lie in G_x, so $x \to y$.

We prove $v \to v + k!$ for every positive integer v and then use this to show $v \to v + 1$ as well, establishing that G_v is connected for every $v \in \mathbb{N}$. We then inductively construct the desired path.

Define polynomials $g_1, \ldots, g_k$ by $g_1(m) = (m+1)^k - m^k$ and $g_j(m) = g_{j-1}(m+1) - g_{j-1}(m)$ for $2 \leq j \leq k$. Note inductively that g_j is a polynomial of degree $k - j$ with leading coefficient $\prod_{i=0}^{j-1}(k-i)$, and all of its coefficients are nonnegative. In particular, $g_k(m) = k!$. Also define polynomials $f_1, \ldots, f_k$ by $f_1(m) = 0$ and $f_j(m) = \sum_{i=2}^{j} g_i(m)$ for $2 \leq j \leq k$. Note that f_{j+1} is a polynomial of degree $k-2$ when $1 \leq j < k$. Since $g_i(n) \geq 0$ for all $n \in \mathbb{N}$, we have $0 \leq f_j(m) \leq f_{j+1}(m)$ for $m \in \mathbb{N}$ and $1 \leq j \leq k-1$. Choose $M \in \mathbb{N}$ so that $g_1(m) > 2f_k(m+1)$ when $m \geq M$, which we can do since g_1 has higher degree than f_k.

Given $1 \leq i \leq k$, we now prove by induction on i that $v \to v + g_i(m)$ when m and v are distinct positive integers such that $m \geq M$ and $m^k > 2v + 2f_i(m)$. For $i = 1$, the condition is $m^k > 2v$, and the list $(v, m^k - v, (m+1)^k - m^k + v)$ provides a path of length 2 from v to $v + g_1(v)$ in G_v, yielding $v \to v + g_1(m)$.

Now consider $i > 1$, with $m \geq M$ and $m^k > 2v + 2f_i(m)$. Since $f_i(m) \geq 0$ and $g_1(m) > f_k(m+1) \geq f_{i-1}(m+1)$, we have

$$(m+1)^k = m^k + g_1(m) > 2v + 2f_{i-1}(m+1),$$

so $v \to v + g_{i-1}(m+1)$ by applying the hypothesis for $i-1$ to $m+1$ and v. Also,

$$m^k > 2v + 2f_i(m)) = 2(v + g_i(m)) + 2f_{i-1}(m).$$

This allows us to apply the hypothesis for $i - 1$ to m and $v + g_i(m)$ to obtain $(v + g_i(m)) \to (v + g_i(m) + g_{i-1}(m))$. Since $g_i(m) + g_{i-1}(m) = g_{i-1}(m+1)$, this becomes $(v + g_i(m)) \to (v + g_{i-1}(m+1)$. Now the xyz-property yields $v \to v + g_i(m)$, establishing the claim.

Given $v \in \mathbb{N}$, we can choose m with $m \geq M$ and $m^k > v + f_k(m)$, because f_k is a polynomial of degree $k - 2$. We then have $v \to v + g_k(m) = v + k!$. It follows that $v \to v + n \cdot k!$ for all $n \in \mathbb{N}$. Let r be a multiple of $k!$ such that $r^k > 2v$. Since also $(r+1)^k > 2(2^k - v)$, the list $(v, r^k - v, (r+1)^k - (r^k - v))$ provides a path of length 2 in G_v showing $v \to (r+1)^k - (r^k - v)$. Since $(r+1)^k - (r^k - v) - (v+1)$ is a multiple of r, it is also a multiple of $k!$, so $v + 1 \to (r+1)^k - (r^k - v)$. Now the xyz-property yields $v \to v + 1$. Hence G_v is connected.

Finally, we construct the required path through the positive integers inductively. Let $S_1 = (1)$. For $j \in \mathbb{N}$, let S_j be a finite list of distinct positive integers such that the sum of any two consecutive elements in the list is a kth power. We extend S_j to a longer such list S_{j+1} containing the smallest positive integer p not in S_j as follows. Let q be the last element of S_j, and let r be the largest element of S_j. Choose positive integers n and m such that $m^k - p > n^k - q > r$. Let $u = n^k - q$ and $v = m^k - p$. Choose a path P in G_u from u to v. Obtain S_{j+1} from S_j by appending P and then p. Since $u > r$, all the integers appended to the list have not previously occurred in the list, the first element that was missing is now included, and any two consecutive elements in the list sum to a kth power. Since we iteratively extend the list in a way that includes the least integer missing from the previous list, each positive integer appears eventually.

Also solved by E. J. Ionaşcu, J. R. Roche, K. Schilling, and R. Stong. Part **(a)** also solved by O. P. Lossers (Netherlands) and the proposer.

Integrating a Rational Function

12189 [2020, 563]. *Proposed by Hidefumi Katsuura, San Jose State University, San Jose, CA.* Evaluate

$$\int_0^1 \frac{(k+1)x^k - \sum_{m=0}^k x^{mk}}{x^{k(k+1)} - 1}\, dx,$$

where k is a positive integer.

Solution by Giuseppe Fera and Giorgio Tescaro, Vicenza, Italy. The value of the integral is $\ln(k+1)/k$. To prove this, we start with the fact that for $x \neq 1$,

$$\sum_{m=0}^{k} x^{mk} = \frac{x^{k(k+1)} - 1}{x^k - 1}.$$

Substituting this formula in the integrand, using a limit to avoid the singularity at $x = 1$, and then making the change of variable $y = x^{k+1}$, we see that

$$\begin{aligned}
\int_0^1 \frac{(k+1)x^k - \sum_{m=0}^k x^{mk}}{x^{k(k+1)} - 1}\, dx &= \lim_{a\to 1^-} \left(\int_0^a \frac{(k+1)x^k\, dx}{x^{k(k+1)} - 1} - \int_0^a \frac{dx}{x^k - 1} \right) \\
&= \lim_{a\to 1^-} \left(\int_0^{a^{k+1}} \frac{dy}{y^k - 1} - \int_0^a \frac{dx}{x^k - 1} \right) \\
&= \lim_{a\to 1^-} \int_{a^{k+1}}^a \frac{dx}{1 - x^k}.
\end{aligned}$$

To evaluate this limit, consider any a with $0 < a < 1$. When $a^{k+1} \le x \le a$, set

$$g(x) = \sum_{m=0}^{k-1} x^m = \frac{1 - x^k}{1 - x}.$$

Note that g is increasing on $[a^{k+1}, a]$, so

$$(1-x)g(a^{k+1}) \le (1-x)g(x) \le (1-x)g(a).$$

Inverting and substituting for $g(x)$, we get

$$\frac{1}{g(a)} \cdot \frac{1}{1-x} \le \frac{1}{1-x^k} \le \frac{1}{g(a^{k+1})} \cdot \frac{1}{1-x},$$

and integrating yields

$$\frac{1}{g(a)}\int_{a^{k+1}}^{a}\frac{dx}{1-x}\le\int_{a^{k+1}}^{a}\frac{dx}{1-x^k}\le\frac{1}{g(a^{k+1})}\int_{a^{k+1}}^{a}\frac{dx}{1-x}.$$

Since

$$\int_{a^{k+1}}^{a}\frac{dx}{1-x}=\ln\left(\frac{1-a^{k+1}}{1-a}\right)=\ln\left(\sum_{m=0}^{k}a^m\right),$$

we arrive at the bounds

$$\frac{1}{g(a)}\ln\left(\sum_{m=0}^{k}a^m\right)\le\int_{a^{k+1}}^{a}\frac{dx}{1-x^k}\le\frac{1}{g(a^{k+1})}\ln\left(\sum_{m=0}^{k}a^m\right).$$

Finally, we have $\lim_{a\to 1^-} g(a^{k+1})=\lim_{a\to 1^-} g(a)=\lim_{a\to 1^-}\sum_{m=0}^{k-1}a^m=k$, and

$$\lim_{a\to 1^-}\ln\left(\sum_{m=0}^{k}a^m\right)=\ln(k+1).$$

Therefore, by the squeeze theorem,

$$\int_0^1\frac{(k+1)x^k-\sum_{m=0}^{k}x^{mk}}{x^{k(k+1)}-1}\,dx=\lim_{a\to 1^-}\int_{a^{k+1}}^{a}\frac{dx}{1-x^k}=\frac{\ln(k+1)}{k}.$$

Also solved by U. Abel & V. Kushnirevych (Germany), T. Akhmetov (Russia), K. F. Andersen (Canada), N. Batir (Turkey), A. Berkane (Algeria), R. Boukharfane (Saudi Arabia), P. Bracken, B. Bradie, N. Caro (Brazil), R. Chapman (UK), H. Chen, R. Dempsey, A. Dixit (India) & S. Pathak (US), S. P. I. Evangelou (Greece), M. L. Glasser, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), H. Kwong, K.-W. Lau (China), G. Lavau (France), O. P. Lossers (Netherlands), L. Matejíčka (Slovakia), M. Omarjee (France), Á. Plaza (Spain), K. Sarma (India), V. Schindler (Germany), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), T. Wiandt, M. Wildon (UK), L. Zhou, and the proposer.

An Incenter is an Orthocenter

12190 [2020, 563]. *Proposed by Leonard Giugiuc, Drobeta-Turnu Severin, Romania, and Gabriela Negutescu, Telea, Romania.* Let ABC be a triangle, and let D, E, and F be points on BC, CA, and AB, respectively, such that AD, BE, and CF are concurrent at P. It is well known that if P is the orthocenter of ABC, then P is the incenter of DEF. Prove the converse.

Solution by Titu Zvonaru, Comăneşti, Romania. We show that if AD is the angle bisector of $\angle EDF$, then AD is perpendicular to BC. Combining this with similar statements about BE and CF, it then follows that if P is the incenter of DEF, then P is the orthocenter of ABC, as desired.

Let ℓ be the line through A parallel to BC, and let M and N be the points where DE and DF, respectively, intersect ℓ. Since $\triangle CDE$ is similar to $\triangle AME$ and $\triangle BDF$ is similar to $\triangle ANF$, we have

$$\frac{CE}{EA}=\frac{DC}{AM}\qquad\text{and}\qquad\frac{AF}{FB}=\frac{AN}{BD}.$$

By Ceva's theorem,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1.$$

Combining these three equations yields $AM = AN$. Consequently, in $\triangle MDN$, DA is both the angle bisector and median at D. It follows that $\triangle MDN$ is isosceles, with $MD = ND$, and hence AD is perpendicular to ℓ and thus to BC.

Editorial comment. The problem statement here corrects a typographical error that appeared in the original problem statement.

Also solved by R. Boukharfane (Saudi Arabia), R. B. Campos (Spain), H. Chen (China), C. Chiser (Romania), P. De (India), G. Fera (Italy), N. Hodges (UK), I. Patrascu & I. Cotoi (Romania), Y. Ionin, M. Kaplan & M. Goldenberg, K. T. L. Koo (China), O. Kouba (Syria), S. S. Kumar, Y. Lee (Korea), J. H. Lindsey II, M. Mihai & D. Ş. Marinescu (Romania), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), T. Wiandt, L. Zhou, and the proposers.

Fermat Strikes Twice

12192 [2020, 564]. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.* Find all triples (a, b, c) of positive integers such that (c, c^2) is a point on the graph of $y = x^2$ with minimum sum of distances to $(0, a)$ and $(0, b)$.

Solution by Nigel Hodges, Gloucestershire, UK. There are no such triples.

Let $f(x) = \sqrt{x^2 + (x^2 - a)^2} + \sqrt{x^2 + (x^2 - b)^2}$. We want to have f minimized at $x = c$, so we must have $f'(c) = 0$. The derivative of f is given by

$$f'(x) = \frac{x + 2(x^2 - a)x}{\sqrt{x^2 + (x^2 - a)^2}} + \frac{x + 2(x^2 - b)x}{\sqrt{x^2 + (x^2 - b)^2}}.$$

If $a = b$, then $f'(c) = 0$ implies $2c^2 = 2a - 1$, which cannot happen when a and c are integers. Hence we may assume $a \neq b$. The condition $f'(c) = 0$ with $c > 0$ becomes

$$\frac{2c^2 - 2a + 1}{\sqrt{c^2 + (c^2 - a)^2}} = -\frac{2c^2 - 2b + 1}{\sqrt{c^2 + (c^2 - b)^2}}.$$

Squaring both sides and simplifying yields

$$(b - a)\left(4c^4 + 2c^2 + a + b - 4ab\right) = 0.$$

Since $a \neq b$, this equation is equivalent to $(4c^2 + 1)^2 = (4a - 1)(4b - 1)$. Since $4a - 1 \equiv 3 \pmod 4$, the right side must have a prime factor p congruent to 3 modulo 4. This prime p must also divide the left side, so $(2c)^2 \equiv -1 \pmod p$. Now Fermat's little theorem and the fact that $(p - 1)/2$ is odd yield the contradiction

$$1 \equiv (2c)^{p-1} \equiv (-1)^{(p-1)/2} \equiv -1 \pmod p.$$

Editorial comment. Allen Stenger invoked Fermat in a different way, expressing the problem in terms of Fermat's principle of least time in optics, which corresponds to the angle of incidence equaling the angle of reflection.

Also solved by N. Caro (Brazil), R. Chapman (UK), H. Chen (China), K. Gatesman, E. J. Ionaşcu, O. Kouba (Syria), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), T. Wiandt, H. Widmer (Switzerland), L. Zhou, and the proposer.

CLASSICS

We solicit contributions of classics from readers, who should include the problem statement, solution, and references with their submission. The solution to the classic problem published in one issue will appear in the subsequent issue.

C2. *Ira Gessel [1972], contributed by the editors.* Prove that a positive integer n is a Fibonacci number if and only if $5n^2 + 4$ or $5n^2 - 4$ is a perfect square.

The Lion and the Man

C1. *Attributed to Richard Rado in the 1930s, contributed by the editors*. A lion and a man are in an enclosure. The maximum speed of the lion is equal to the maximum speed of the man. Can the lion catch the man?

Solution. We assume that the lion and the man start at different locations, and we show that the man can evade capture forever.

If the man starts on the boundary of the enclosure, then he first moves into the interior. As long as he does this by traveling less than half the distance to the lion, he won't be caught during this step. Once he is in the interior, we can let D be an open disk centered at the man's location that is entirely contained in the enclosure. We now give a strategy that the man can follow to evade capture while staying inside D and therefore inside the enclosure.

Let the unit of distance be chosen so that D has radius 2, and let the unit of time be chosen so that the maximum speed of both lion and man is 1. The strategy proceeds in stages. In stage 1, the man starts running directly away from the lion and runs at maximum speed in a straight line for 1 unit of time. Since the lion cannot run faster than the man, the man cannot be caught during stage 1. For $n \geq 2$, at stage n the man travels at maximum speed a distance $1/n$ in a direction that is perpendicular to the line L that passes through his location at the beginning of the stage and the center of D. There are two such directions to choose from, and the man chooses based on the location of the lion. If the lion is in one of the half planes determined by L, then the man runs into the other half plane. The man can run either way if the lion is on L. Every point that the man visits during stage n is closer to the man's position at the beginning of the stage than it is to the lion's position, so the man evades capture during stage n.

The time elapsed during the first n stages is $\sum_{k=1}^{n} 1/k$, which diverges as n approaches infinity. On the other hand, the distance between the man and the center of D after n stages, by repeated use of the Pythagorean theorem, is $\sqrt{\sum_{k=1}^{n} 1/k^2}$, which converges as n approaches infinity and in particular is bounded (generously) by 2. Thus the man evades capture forever while remaining inside D.

Editorial comment. We have treated the lion and man as points and assumed that to capture the man, the lion must reduce the distance between them to zero in finite time. The solution given shows that certain details of the problem don't matter, such as the shape of the enclosure or the initial positions of the man and lion (as long as they are distinct).

The problem has a colorful history. It was proposed by Richard Rado in the 1930s, with the enclosure being a disk, and solved as above by Abram Besicovitch in 1952. The problem was popularized by John Littlewood in his book *A Mathematician's Miscellany* (see B. Bollobás, ed. (1986), *Littlewood's Miscellany*, Cambridge: Cambridge Univ. Press, pp. 114–117). For further details and generalizations see Bollobás, B., Leader, I., and Walters, M. (2012), Lion and man—can both win?, *Israel J. Math.* 189: 267–286.

It is tempting to think that the man's best strategy is to stay as far from the lion as possible, and in the case of a circular enclosure this means that the man would run to the boundary and then run around the boundary (perhaps sometimes changing direction). However, if the man stays on the boundary, then the lion can catch the man by running outward from the center of the enclosure while staying on the radius from the center to the man. Thus, in order to avoid capture, the man must step into the interior of the enclosure. This gives him the freedom to move in any direction—a freedom that is exploited in Besicovitch's solution.

SOLUTIONS

Brianchon's Theorem on a Hidden Conic

12177 [2020, 372]. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam, and Cherng-tiao Perng, Norfolk, VA.* Let C be a nondegenerate conic, and let l be a line. Suppose that $A_1, \ldots, A_{2n}$ and $B_1, \ldots, B_{2n}$ are points on C such that $A_i A_{i+1}$ and $B_i B_{i+1}$ intersect at a point on l for $i = 1, \ldots, 2n - 1$.
(a) Show that $A_{2n} A_1$ and $B_{2n} B_1$ intersect at a point on l.
(b) Let $n = 3$ and take subscripts modulo 6. For $i = 1, \ldots, 6$, suppose that $A_i B_i$ and $A_{i+1} B_{i+1}$ intersect at a point D_i. Prove that the three lines $D_1 D_4$, $D_2 D_5$, and $D_3 D_6$ are concurrent.

Solution by Richard Stong, Center for Communications Research, San Diego, CA.
(a) The case $n = 1$ is trivial. For the case $n = 2$, we note that applying Pascal's theorem to the hexagon $A_1 A_2 A_3 B_1 B_2 B_3$ shows that the intersection points $A_1 A_2 \cap B_1 B_2$, $A_2 A_3 \cap B_2 B_3$, and $A_3 B_1 \cap B_3 A_1$ are collinear. Since the first two are on l, it follows that the third is as well. Applying Pascal's theorem again to the hexagon $A_1 B_3 B_4 B_1 A_3 A_4$ shows that $A_1 B_3 \cap B_1 A_3$, $B_3 B_4 \cap A_3 A_4$, and $B_4 B_1 \cap A_4 A_1$ are collinear. Again since the first two are on l, it follows that the third is as well, proving the case $n = 2$. The cases $n > 2$ follow immediately from the $n = 2$ case and induction. Using the $n = 2$ case we conclude that $A_1 A_4$ and $B_1 B_4$ meet on l, and therefore we can drop the indices 2 and 3 and use the induction hypothesis.

(b) By a projective transformation, we may assume l is the line at infinity. If C is disjoint from l, then C is an ellipse, and by a further affine transformation we may assume C is a circle. It suffices to prove the result in this case: If C is tangent to l, then C is a parabola. The result for parabolas follows from continuity by treating them as limits of ellipses. If C meets l in two points, then C is a hyperbola. This case follows from the circle case by an argument using analytic continuation. The result for the circle $x^2 + y^2 = 1$ means that a certain analytic function of the x-coordinates of the points $A_1, \ldots, A_6$, B_1 (which determine the remaining coordinates) vanishes for all real values of these coordinates between -1 and 1. By analytic continuation, the same function is 0 for purely imaginary values of these coordinates, which implies the result for the hyperbola $y^2 - x^2 = 1$; an affine transformation reduces any hyperbola to this case.

If C is a circle and l is the line at infinity, then the statement that A_iA_{i+1} and B_iB_{i+1} meet on l says that A_iA_{i+1} and B_iB_{i+1} are parallel and hence $\angle A_iA_{i+1}B_i = \angle A_{i+1}B_iB_{i+1}$. Thus the chords A_iB_i and $A_{i+1}B_{i+1}$ subtend the same arc of the circle and hence are congruent. It follows that there is a smaller concentric circle simultaneously tangent to all the chords A_iB_i at their midpoints. Thus the hexagon $D_1D_2D_3D_4D_5D_6$ has an inscribed circle. Brianchon's theorem then states that the principal diagonals of this hexagon are concurrent, which is the desired conclusion.

Also solved by L. Zhou and the proposer.

Every Function has Some Continuity

12178 [2020, 373]. *Proposed by Stephen Portnoy, University of Illinois, Urbana, IL.* Given any function $f\colon \mathbb{R} \to \mathbb{R}$, show that there is a real number x and a sequence $x_1, x_2, \dots$ of distinct real numbers such that $x_n \to x$ and $f(x_n) \to f(x)$ as $n \to \infty$.

Solution by Supravat Sarkar, Indian Statistical Institute, Bangalore, India. Let $A = \{(x, f(x)) : x \in \mathbb{R}\}$. The set A is an uncountable subset of $\mathbb{R}^2$, which implies that some point of A must be a limit point of A. To see this, suppose it is not true. Now every point in A has an open neighborhood in $\mathbb{R}^2$ that contains no other point in A. Thus the subspace topology of A is discrete. Any uncountable set with discrete topology is not second countable, but being a subspace of the second countable space $\mathbb{R}^2$, A must be second countable. This is a contradiction.

Let $(x, f(x))$ be an element of A that is a limit point of A. There exist distinct points $(x_n, f(x_n))$ in A converging to $(x, f(x))$ as $n \to \infty$. Hence the numbers $x_1, x_2, \dots$ are distinct and $x_n \to x$ and $f(x_n) \to f(x)$ as $n \to \infty$.

Editorial comment. Jacob Boswell and Charles Curtis gave an example showing that the analogous result for functions from $\mathbb{Q}$ to $\mathbb{Q}$ need not hold. Jean-Pierre Grivaux, Klaas Pieter Hart, Kenneth Schilling, and Richard Stong all proved the stronger statement that for all but countably many x, such a sequence can be found. Celia Schacht pointed out that a proof of this stronger statement can be found in W. H. Young (1907), A theorem in the theory of functions of a real variable, *Rendiconti del Circolo Matematico di Palermo* 24(1), 187–192. Éric Pité, Stephen Scheinberg, and George Stoica observed that the result in the problem follows from a theorem of H. Blumberg saying that for every function $f : \mathbb{R} \to \mathbb{R}$, there is a dense subset D of $\mathbb{R}$ such that the restriction of f to D is continuous; see H. Blumberg (1922), New properties of all real functions, *Trans. Amer. Math. Soc.* 24(2), 113–128.

Also solved by K. F. Andersen (Canada), J. Boswell & C. Curtis, R. Chapman (UK), H. Chen (China), T. Corso (Germany), G. A. Edgar, G. Fera & G. Tescaro (Italy), O. Geupel (Germany), J.-P. Grivaux (France), K. P. Hart (Netherlands), D. Hensley, E. A. Herman, E. J. Ionaşcu, B. Karaivanov (USA) & T. S. Vassilev (Canada), J. C. Kieffer, L. Matejíčka (Slovakia), A. Natian, J. Nieto (Venezuela), J. Olson, M. Omarjee (France), A. Pathak, L. J. Peterson, É. Pité, K. Sarma (India), C. Schacht, S. Scheinberg, K. Schilling, E. Schmeichel, A. Stadler (Switzerland), G. Stoica (Canada), R. Stong, R. Tauraso (Italy), Northwestern University Math Problem Solving Group, and the proposer.

Factorials are Rarely Good

12179 [2020, 373]. *Proposed by Nick MacKinnon, Winchester College, Winchester, UK.* A positive integer n is *good* if its prime factorization $2^{a_1}3^{a_2}\cdots p_m^{a_m}$ has the property that a_i/a_{i+1} is an integer whenever $1 \le i < m$. Find all n greater than 2 such that $n!$ is good.

Solution by Celia Schacht, North Carolina State University, Raleigh, NC. The values of n such that $n!$ is good are 3, 4, 5, 6, 7, 10, and 11.

We have $n! = \prod_{p\in P_n} p^{\alpha_p(n)}$, where P_n is the set of primes less than or equal to n and

$$\alpha_p(n) = \sum_{k=1}^{\lfloor \log_p n\rfloor} \left\lfloor \frac{n}{p^k}\right\rfloor.$$

We focus on the relative sizes of $\alpha_5(n)$ and $\alpha_7(n)$. Note that $n/5 \geq n/7+1$ implies $\lfloor n/5\rfloor > \lfloor n/7\rfloor$ and holds when $n \geq 17.5$. Explicit checking shows that $\lfloor n/5\rfloor > \lfloor n/7\rfloor$ also holds for $n \in \{15, 16, 17\}$. Since $\lfloor n/5^k\rfloor \geq \lfloor n/7^k\rfloor$ for all k, for $n \geq 15$ we conclude

$$\alpha_5(n) > \alpha_7(n). \tag{1}$$

We complete the argument by showing

$$\alpha_5(n) < 2\alpha_7(n) \tag{2}$$

for $n \geq 28$. When (1) and (2) both hold, n cannot be good, since $\alpha_5(n)/\alpha_7(n)$ is strictly between 1 and 2. Hence these inequalitites reduce the problem to checking explicitly which n less than 28 are good, and these turn out to be only 3, 4, 5, 6, 7, 10, and 11.

To prove (2), we need an upper bound on $\alpha_5(n)$ and a lower bound on $\alpha_7(n)$. We compute

$$\alpha_5(n) = \sum_{k=1}^{\lfloor \log_5 n\rfloor} \left\lfloor \frac{n}{5^k}\right\rfloor \leq \sum_{k=1}^{\lfloor \log_5 n\rfloor} \frac{n}{5^k} = \frac{n}{5}\cdot\frac{1-1/5^{\lfloor \log_5 n\rfloor}}{1-1/5} \leq \frac{n(1-1/n)}{4} = \frac{n-1}{4}$$

and

$$\begin{aligned}\alpha_7(n) &= \sum_{k=1}^{\lfloor \log_7 n\rfloor} \left\lfloor \frac{n}{7^k}\right\rfloor \geq \sum_{k=1}^{\lfloor \log_7 n\rfloor} \frac{n}{7^k} - \lfloor \log_7 n\rfloor \\ &= \frac{n}{7}\cdot\frac{1-1/7^{\lfloor \log_7 n\rfloor}}{1-1/7} - \lfloor \log_7 n\rfloor \geq \frac{n(1-7/n)}{6} - \lfloor \log_7 n\rfloor \\ &= \frac{n-7}{6} - \lfloor \log_7 n\rfloor.\end{aligned}$$

Hence to prove (2) it suffices to show $(n-1)/4 < (n-7)/3 - 2\lfloor \log_7 n\rfloor$, which simplifies to $24\lfloor \log_7 n\rfloor + 25 < n$ and holds when $n \geq 74$. It is also easily checked that (2) holds when $28 \leq n \leq 73$.

Also solved by S. Chandrasekhar (India), R. Chapman (UK), W. Chang, G. Fera (Italy), D. Fleischman, O. Geupel (Germany), N. Hodges (UK), Y. J. Ionin, W. Janous (Austria), M. Kaplan & M. Goldenberg, O. Kouba (Syria), S. S. Kumar, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany) J. H. Nieto (Venezuela), S. Omar (Morocco), É. Pité, C. R. Pranesachar (India), M. A. Prasad (Inda), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. Terr, F. A. Velandia & J. F. González (Columbia), L. Zhou, Eagle Problem Solvers, and the proposer.

A Combination of Betas

12180 [2020, 373]. *Proposed by Pablo Fernández Refolio, Madrid, Spain.* Prove

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^2}{2^{8n}(2n+1)} = \frac{2}{\pi} - \frac{\sqrt{2}\,C^2}{\pi^{3/2}} + \frac{\sqrt{2\pi}}{2C^2},$$

where $C = \int_0^\infty t^{-1/4}e^{-t}\,dt$.

Solution by Quan Minh Nguyen, William Academy, Toronto, ON, Canada. Let S denote the requested sum. Using Wallis's integral, we see that

$$S = \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^2}{2^{8n}(2n+1)} = \sum_{n=0}^{\infty} \left(\frac{\binom{4n}{2n}}{2^{4n}(2n+1)} \cdot \frac{2}{\pi} \int_0^{\pi/2} \sin^{4n} x \, dx \right)$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{(2n+1)} \left(\frac{\sin^2 x}{4} \right)^{2n} dx.$$

Recall the generating function for the Catalan numbers:

$$\sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} t^n = \frac{1 - \sqrt{1-4t}}{2t}, \qquad 0 < |t| \le \frac{1}{4}.$$

(The singularity at $t = 0$ is removable.) Replacing t with $-t$ in this equation and then averaging the two equations yields

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{2n+1} t^{2n} = \frac{\sqrt{1+4t} - \sqrt{1-4t}}{4t}, \qquad 0 < |t| \le \frac{1}{4}.$$

Setting $t = (\sin^2 x)/4$ in this equation, we obtain

$$S = \frac{2}{\pi} \int_0^{\pi/2} \frac{\sqrt{1+\sin^2 x} - \sqrt{1-\sin^2 x}}{\sin^2 x} \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \left(\sqrt{1+\sin^2 x} - \cos x \right) \csc^2 x \, dx.$$

To evaluate the integral, we begin by using integration by parts to get

$$S = -\frac{2}{\pi} \cot x \left(\sqrt{1+\sin^2 x} - \cos x \right) \Big|_0^{\pi/2} + \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\cos^2 x}{\sqrt{1+\sin^2 x}} + \cos x \right) dx$$

$$= \frac{2}{\pi} + \frac{2}{\pi} \int_0^{\pi/2} \frac{\cos^2 x}{\sqrt{1+\sin^2 x}} \, dx.$$

Substituting $u = \sin x$ and then $t = u^4$, and recognizing Beta functions, we obtain

$$S = \frac{2}{\pi} + \frac{2}{\pi} \int_0^1 \frac{\sqrt{1-u^2}}{\sqrt{1+u^2}} \, du = \frac{2}{\pi} + \frac{2}{\pi} \left(\int_0^1 \frac{1}{\sqrt{1-u^4}} \, du - \int_0^1 \frac{u^2}{\sqrt{1-u^4}} \, du \right)$$

$$= \frac{2}{\pi} + \frac{1}{2\pi} \left(\int_0^1 t^{-3/4}(1-t)^{-1/2} \, dt - \int_0^1 t^{-1/4}(1-t)^{-1/2} \, dt \right)$$

$$= \frac{2}{\pi} + \frac{1}{2\pi} \left(B\left(\frac{1}{4}, \frac{1}{2}\right) - B\left(\frac{3}{4}, \frac{1}{2}\right) \right).$$

Using Euler's reflection formula $\Gamma(3/4)\Gamma(1/4) = \pi\sqrt{2}$ and recognizing that $C = \Gamma(3/4)$, we compute

$$B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{\sqrt{2}\pi^{3/2}}{C^2}$$

and

$$B\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)} = \frac{2^{3/2}C^2}{\sqrt{\pi}}.$$

Hence

$$S = \frac{2}{\pi} - \frac{\sqrt{2}C^2}{\pi^{3/2}} + \frac{\sqrt{2\pi}}{2C^2}.$$

Also solved by A. Berkane (Algeria), P. Bracken, R. Chapman (UK), H. Chen, G. Fera (Italy), P. Fulop (Hungary), L. Glasser, O. Kouba (Syria), K.-W. Lau (China), A. D. Pirvuceanu (Romania), V. Schindler (Germany), F. Sinani (Kosovo), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T Wiandt, and the proposer.

A Sum of an Integral of a Fractional Part Yields Gamma

12181 [2020, 461]. *Proposed by Shivam Sharma, University of Delhi, New Delhi, India.* Prove

$$\sum_{k=2}^{\infty} \frac{1}{k} \int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} dx = \gamma,$$

where $\{x\}$ equals $x - \lfloor x \rfloor$, the fractional part of x, and γ is $\lim_{n\to\infty}\left(-\ln n + \sum_{i=1}^{n}(1/i)\right)$, the Euler–Mascheroni constant.

Solution by Gérard Lavau, Fontaine lès Dijon, France. For integers n and k with $n \geq 1$ and $k \geq 2$, we have $\lfloor 1/\sqrt[k]{x} \rfloor = n$ if and only if $1/(n+1)^k < x \leq 1/n^k$. For such x, we have $\{1/\sqrt[k]{x}\} = 1/\sqrt[k]{x} - n$, so

$$\begin{aligned}
\int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} dx &= \sum_{n=1}^{\infty} \int_{1/(n+1)^k}^{1/n^k} \left(\frac{1}{\sqrt[k]{x}} - n \right) dx \\
&= \sum_{n=1}^{\infty} \left[\frac{k}{k-1} \left(\frac{1}{n^{k-1}} - \frac{1}{(n+1)^{k-1}} \right) - n \left(\frac{1}{n^k} - \frac{1}{(n+1)^k} \right) \right] \\
&= \sum_{n=1}^{\infty} \left[\frac{1}{k-1} \left(\frac{1}{n^{k-1}} - \frac{1}{(n+1)^{k-1}} \right) - \frac{1}{(n+1)^k} \right] \\
&= \frac{1}{k-1} \sum_{n=1}^{\infty} \left(\frac{1}{n^{k-1}} - \frac{1}{(n+1)^{k-1}} \right) - \sum_{n=2}^{\infty} \frac{1}{n^k} = \frac{1}{k-1} - (\zeta(k) - 1),
\end{aligned}$$

where the first sum in the last line is a telescoping series and ζ is the Riemann zeta function. Therefore

$$\sum_{k=2}^{\infty} \frac{1}{k} \int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} dx = \sum_{k=2}^{\infty} \left(\frac{1}{k(k-1)} - \frac{\zeta(k)-1}{k} \right).$$

The desired result now follows from the formulas

$$\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1 \qquad \text{and} \qquad \sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k} = 1 - \gamma.$$

The first of these formulas can be derived by using partial fractions to rewrite the sum as a telescoping series. The second was proved by Euler (see page 111 in J. Havil (2003), *Gamma: Exploring Euler's Constant*, Princeton: Princeton University Press).

Also solved by Z. Ahmed (India), K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), N. Bhandari (Nepal), G. E. Bilodeau, R. Boukharfane (Saudi Arabia), J. Boswell & C. Curtis, P. Bracken, B. Bradie, B. S. Burdick, F. Cardona (Columbia), J. N. Caro Montoya (Brazil), W. Chang, R. Chapman (UK), H. Chen, C. Chiser (Romania), B. E. Davis, M. Dinca & D. S. Marinescu (Romania), A. Dixit (Canada) &

S. Pathak (USA), A. Eydelzon, G. Fera (Italy), M. L. Glasser, N. Grivaux (France), J. A. Grzesik, E. A. Herman, N. Hodges (UK), W. Janous (Austria), S. Kaczkowski, M. Kaplan, K. T. L. Koo (China), O. Kouba (Syria), S. S. Kumar, P. Lalonde (Canada), K.-W. Lau (China), R. Molinari, S. E. Muñoz (Venezuela), K. Nelson, Q. M. Nguyen (Canada), M. Omarjee (France), S.-H. Park (Korea), Á. Plaza (Spain), C. R. Pranesachar (India), M. A. Prasad (India), K. Sarma (India), E. Schmeichel, B. Shala (Slovenia), F. Sinani (Kosovo), S. Singhania (India), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Tauraso (Italy), H. Vinuesa (Spain), T. Wiandt, H. Widmer (Switzerland), M. Wildon (UK), Y. Xiang (China), L. Zhou, and the proposer.

Bounding Circumradii of Corner Triangles

12182 [2020, 461]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let R and r be the circumradius and inradius, respectively, of triangle ABC. Let D, E, and F be chosen on sides BC, CA, and AB so that AD, BE, and CF bisect the angles of ABC. Let R_A, R_B, and R_C denote the circumradii of triangles AEF, BFD, and CDE, respectively. Prove $R_A + R_B + R_C \le 3R^2/(4r)$.

Solution by Michel Bataille, Rouen, France. Let a, b, and c be the sides of $\triangle ABC$ opposite angles A, B, and C, respectively. The law of sines gives $a = 2R\sin A$ and $EF = 2R_A \sin A$, and hence $R_A = R \cdot EF/a$. Similar results hold for R_B and R_C, so the requested inequality is equivalent to

$$\frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} \le \frac{3R}{4r}.$$

Since BE bisects $\angle ABC$, $AE/c = EC/a = (EC + AE)/(a + c) = b/(a + c)$, so $AE = bc/(a + c)$. Similarly, $AF = bc/(a + b)$, and using the law of cosines, we obtain

$$\begin{aligned} EF^2 = AE^2 + AF^2 - 2AE \cdot AF \cdot \cos A &= \frac{b^2c^2}{(a+c)^2} + \frac{b^2c^2}{(a+b)^2} - \frac{bc(b^2+c^2-a^2)}{(a+b)(a+c)} \\ &= \frac{bc}{(a+b)^2(a+c)^2} \cdot \left(a^2(a+b)(a+c) - a(a+b+c)(b-c)^2\right) \\ &\le \frac{bc}{(a+b)^2(a+c)^2} \cdot a^2(a+b)(a+c) = \frac{a^2bc}{(a+b)(a+c)}. \end{aligned}$$

By the AM–GM inequality,

$$EF \le \frac{a\sqrt{bc}}{\sqrt{(a+b)(a+c)}} \le \frac{a\sqrt{bc}}{\sqrt{2\sqrt{ab} \cdot 2\sqrt{ac}}} = \frac{\sqrt{a}\sqrt[4]{b}\sqrt[4]{c}}{2} \le \frac{2a+b+c}{8}.$$

Similarly, $FD \le (2b + c + a)/8$ and $DE \le (2c + a + b)/8$. Therefore

$$\begin{aligned} \frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} &\le \frac{3}{4} + \frac{1}{8}\left(\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c}\right) \\ &= \frac{3}{8} + \frac{(a+b+c)(ab+bc+ca)}{8abc}. \end{aligned}$$

With $s = (a + b + c)/2$, we have $ab + bc + ca = s^2 + r^2 + 4rR$ and $abc = 4srR$. Applying Gerretsen's inequality $s^2 \le 4R^2 + 3r^2 + 4rR$ and Euler's inequality $R \ge 2r$, we obtain

$$\begin{aligned} \frac{EF}{a} + \frac{FD}{b} + \frac{DE}{c} &\le \frac{3}{8} + \frac{2s(s^2+r^2+4rR)}{32srR} = \frac{s^2+r^2+10rR}{16rR} \\ &\le \frac{2R^2+2r^2+7rR}{8rR} = \frac{6R^2-(R-2r)(4R+r)}{8rR} \le \frac{6R^2}{8rR} = \frac{3R}{4r}, \end{aligned}$$

which completes the proof.

Also solved by M. Dinča & M. Ursărescu (Romania), G. Fera (Italy), N. Hodges (UK), W. Janous (Austria), C. R. Pranesachar (India), V. Schindler (Germany), S. Singhania (India), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, and the proposer.

A Gaussian Binomial Identity

12183 [2020, 461]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denote the Gaussian binomial coefficient

$$\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.$$

For integers m, n, and r with $m \geq 1$ and $n \geq r \geq 0$, prove

$$\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{q^{rm}}{1-q^m} \begin{bmatrix} m+n \\ m \end{bmatrix}_q^{-1}.$$

Solution I by Albert Stadler, Herrliberg, Switzerland. We use induction on n. For $n = r = 0$, both sides of the proposed identity equal $1/(1-q^m)$.

We extend the definition of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ in the standard way by setting it to 0 when $k \notin [0, n]$. We use the well-known and easily-proved analogues of the Pascal identities for the Gaussian binomial coefficients:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q. \qquad (*)$$

For the induction step, we obtain the truth of the statement for $n+1$ from its truth for n. First suppose $0 \leq r \leq n$. Using the first equation in $(*)$ and then reindexing the second sum,

$$\begin{aligned}
\sum_{k=0}^{n+1} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q &= \sum_{k=0}^{n+1} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \left(\begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right) \\
&= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q + q^n \sum_{k=0}^{n} \frac{(-1)^{k+1} q^{\binom{k+2}{2}-r(k+1)-k}}{1-q^{k+1+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q \\
&= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{n+1-r} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-q^{k+1+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q \\
&= \frac{q^{rm}}{1-q^m} \begin{bmatrix} m+n \\ m \end{bmatrix}_q^{-1} - q^{n+1-r} \cdot \frac{q^{r(m+1)}}{1-q^{m+1}} \begin{bmatrix} m+n+1 \\ m+1 \end{bmatrix}_q^{-1} \\
&= \begin{bmatrix} m+n+1 \\ m \end{bmatrix}_q^{-1} \left(\frac{q^{rm}}{1-q^m} \cdot \frac{1-q^{m+n+1}}{1-q^{n+1}} - q^{n+1} \cdot \frac{q^{rm}}{1-q^{m+1}} \cdot \frac{1-q^{m+1}}{1-q^{n+1}} \right) \\
&= \begin{bmatrix} m+n+1 \\ m \end{bmatrix}_q^{-1} \frac{q^{rm}}{1-q^m},
\end{aligned}$$

as required.

It remains to prove the statement for $r = n+1$. The computation is similar:

$$\sum_{k=0}^{n+1} \frac{(-1)^k q^{\binom{k+1}{2}-(n+1)k}}{1-q^{k+m}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \sum_{k=0}^{n+1} \frac{(-1)^k q^{\binom{k+1}{2}-(n+1)k}}{1-q^{k+m}} \left(q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right)$$

$$= \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-nk}}{1-q^{k+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q - q^{-n} \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-nk}}{1-q^{k+1+m}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

$$= \frac{q^{nm}}{1-q^m} \begin{bmatrix} m+n \\ m \end{bmatrix}_q^{-1} - q^{-n} \cdot \frac{q^{n(m+1)}}{1-q^{m+1}} \begin{bmatrix} m+n+1 \\ m+1 \end{bmatrix}_q^{-1}$$

$$= \begin{bmatrix} m+n+1 \\ m \end{bmatrix}_q^{-1} \left(\frac{q^{nm}}{1-q^m} \cdot \frac{1-q^{m+n+1}}{1-q^{n+1}} - \frac{q^{nm}}{1-q^{m+1}} \cdot \frac{1-q^{m+1}}{1-q^{n+1}} \right)$$

$$= \begin{bmatrix} m+n+1 \\ m \end{bmatrix}_q^{-1} \frac{q^{(n+1)m}}{1-q^m},$$

completing the induction step.

Solution II by Pierre Lalonde, Plessisville, QC, Canada. More generally, we see that

$$\sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-xq^k} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{x^r}{1-x} \prod_{k=1}^{n} \frac{1-q^k}{1-xq^k},$$

for $n \geq r \geq 0$. The proposed problem is the case $x = q^m$.

The right side of the identity is a rational function in x with numerator of degree r and denominator of degree $n+1$, which is greater than r. The zeros of the denominator are $1/q^k$ for $0 \leq k \leq n$, which are formally distinct. Therefore, the partial fraction decomposition of the right side is

$$\frac{x^r}{1-x} \prod_{k=1}^{n} \frac{1-q^k}{1-xq^k} = \sum_{k=0}^{n} \frac{A_k}{1-xq^k},$$

where

$$A_k = \lim_{x \to 1/q^k} (1-xq^k) \cdot \frac{x^r}{1-x} \prod_{i=1}^{n} \frac{1-q^i}{1-xq^i} = q^{-rk} \frac{\prod_{i=1}^{n}(1-q^i)}{\prod_{i=1}^{k}(1-q^{-i}) \prod_{i=1}^{n-k}(1-q^i)}$$

$$= (-1)^k q^{(\sum_{i=1}^{k} i)-rk} \cdot \frac{\prod_{i=n+1-k}^{n}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)} = (-1)^k q^{\binom{k+1}{2}-rk} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Hence

$$\frac{x^r}{1-x} \prod_{k=1}^{n} \frac{1-q^k}{1-xq^k} = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k+1}{2}-rk}}{1-xq^k} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

as claimed.

Editorial comment. Several solvers used the method of the first solution. It can be adapted to prove the generalization in the second solution. Hacer Bozdag mentioned a still more general result, with two additional parameters and implying the claim, from E. Kılıç and H. Prodinger (2016), Evaluation of sums involving Gaussian q-binomial coefficients with rational weight functions, *Int. J. Number Theory* 12, 495–504.

Also solved by H. Bozdag (Turkey), R. Chapman (UK), N. Hodges (UK), W. P. Johnson, H. Kwong, M. A. Prasad (India), R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

The Distance Between Norms

12186 [2020, 462]. *Proposed by Anatoly Eydelzon, University of Texas at Dallas, Richardson, TX.* For $v = \langle x_1, \ldots, x_n \rangle$ in $\mathbb{R}^n$, let $\|v\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ and $\|v\|_\infty =$

$\max_{1\le i\le n}|x_i|$; these are the usual p-norm and ∞-norm on $\mathbb{R}^n$. For what v does the series

$$\sum_{p=1}^{\infty}\Big(\|v\|_p-\|v\|_\infty\Big)$$

converge?

Solution by Óscar Ciaurri, Universidad de La Rioja, Logroño, Spain. When v is the zero vector, the terms of the series are identically zero, and hence the series converges. Excluding this trivial case, we show that the given series S converges if and only if there is a unique $j\in\{1,\ldots,n\}$ such that $|x_j|=\|v\|_\infty$.

Suppose there is a unique such j. By symmetry, we may assume $j=1$. We have

$$\|v\|_p-\|v\|_\infty=\|v\|_\infty\left(\left(1+\sum_{i=2}^{n}\frac{|x_i|^p}{|x_1|^p}\right)^{1/p}-1\right)$$

and, by Bernoulli's inequality $(1+z)^r\le 1+rz$ for $0\le r\le 1$ and $z>-1$, we have

$$\|v\|_p-\|v\|_\infty\le\|v\|_\infty\sum_{i=2}^{n}\frac{|x_i|^p}{p|x_1|^p}.$$

Summing over p, we obtain

$$S\le\|v\|_\infty\sum_{i=2}^{n}\sum_{p=1}^{\infty}\frac{|x_i|^p}{p|x_1|^p}.$$

The inner series all converge since $|x_i|/|x_1|<1$, and hence S converges.

Now suppose that there are at least two values $j,k\in\{1,\ldots,n\}$ such that $\|v\|_\infty=|x_j|=|x_k|$. In this case, $\|v\|_p\ge(|x_j|^p+|x_k|^p)^{1/p}=2^{1/p}\|v\|_\infty$, so $\|v\|_p-\|v\|_\infty\ge\|v\|_\infty(2^{1/p}-1)$. Since

$$\lim_{p\to\infty}\frac{2^{1/p}-1}{1/p}=\lim_{t\to 0}\frac{2^t-1}{t}=\log 2>0,$$

the series $\sum_{p=1}^{\infty}(2^{1/p}-1)$ diverges by comparison to the harmonic series, and hence S diverges.

Also solved by K. F. Andersen (Canada), N. Caro (Brazil), R. Chapman (UK), H. Chen (China), C. Curtis & A. Appuhamy & J. Boswell, J. Freeman (Netherlands), J.-P. Grivaux (France), L. Han, E. A. Herman, N. Hodges (UK), E. J. Ionaşcu, K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II, U. Milutinović (Slovenia), M. Omarjee (France), Á. Plaza & K. Sasdarangani (Spain), M. A. Prasad (India), K. Sarma (India), K. Schilling, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, M. Wildon (UK), C.-Y. Wu, and the proposer.

CLASSICS

Here each month we feature one classic problem, whose solution will appear in the subsequent issue. Classics are problems of unusual elegance that are not new but deserve to be better known. We solicit contributions of Classic problems from readers, who should include the problem statement, solution, and references with their submission. We will not be soliciting or publishing reader solutions to Classic problems that appear here.

C1. *Attributed to Richard Rado in the 1930s, contributed by the editors.* A lion and a man are in an enclosure. The maximum speed of the lion is equal to the maximum speed of the man. Can the lion catch the man?

SOLUTIONS

Optimizing an Inequality

12169 [2020, 274]. *Proposed by Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let n be an integer with $n \geq 2$. Find the least positive real number α such that

$$(n-1)\cdot\sqrt{1+\alpha\sum\nolimits_{1\leq i<j\leq n}(x_i-x_j)^2}+\prod_{i=1}^{n}x_i \geq \sum_{i=1}^{n}x_i$$

for all nonnegative real numbers $x_1,\ldots,x_n$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. If $x_1 = x_2 = \cdots = x_{n-1} = R$ and $x_n = 0$, then the inequality states

$$(n-1)\cdot\sqrt{1+(n-1)\alpha R^2} \geq (n-1)R,$$

or

$$\alpha \geq \frac{1}{n-1}-\frac{1}{(n-1)R^2}.$$

Letting $R\to\infty$ we find $\alpha \geq 1/(n-1)$. We claim that the given inequality holds for $\alpha = 1/(n-1)$, and therefore this is the smallest possible value of α. Thus, we must show

$$(n-1)\cdot\sqrt{1+\frac{1}{n-1}\sum\nolimits_{1\leq i<j\leq n}(x_i-x_j)^2}+\prod_{i=1}^{n}x_i \geq \sum_{i=1}^{n}x_i. \qquad (*)$$

Note that since both sides of this inequality are continuous in each x_k, it suffices to prove the inequality for $x_k > 0$.

Let $S=\sum_{i=1}^n x_i$ and $P=\prod_{i=1}^n x_i$. Since $\sum_{1\leq i<j\leq n}(x_i-x_j)^2 = n\sum_{i=1}^n x_i^2 - S^2$, the inequality can be written

$$(n-1)\cdot\sqrt{1+\frac{1}{n-1}\left(n\sum_{i=1}^{n}x_i^2-S^2\right)}+P \geq S.$$

Thus, for fixed S and P it suffices to prove ($*$) when the x_i are chosen to minimize $\sum_{i=1}^n x_i^2$. Let $g(x_1,\ldots,x_n)=\sum_{i=1}^n x_i$ and $h(x_1,\ldots,x_n)=\sum_{i=1}^n \log x_i$. Since our two constraints $g=S$ and $h=\log P$ define a smooth compact manifold, this minimum must exist, and it must occur either at a point that satisfies the Lagrange multiplier equations

$$2x_i = \lambda + \frac{\mu}{x_i}$$

for some λ and μ or at a point where ∇g and ∇h are linearly dependent. The Lagrange multiplier equations are quadratic in x_i, so they can be satisfied only at points where the

x_i take at most two distinct values. Also, ∇g and ∇h are linearly dependent only at points where the x_i are all equal. Therefore, it will suffice to prove $(*)$ at all points where the x_i take either one or two distinct positive values.

If the x_i all have the same value y, then inequality $(*)$ becomes $n-1+y^n \geq ny$, which is precisely Bernoulli's inequality. Now assume that k of the x_i equal y and $n-k$ of them equal z, where $0 < k < n$ and $z < y$. In that case, inequality $(*)$ becomes

$$(n-1)\cdot\sqrt{1+\frac{k(n-k)}{n-1}(y-z)^2}+y^k z^{n-k} \geq ky+(n-k)z.$$

Bernoulli's inequality gives $y^k \geq 1+k(y-1)$ and $z^{n-k} \geq 1+(n-k)(z-1)$. Thus, if $y > z \geq 1$ then

$$\begin{aligned} y^k z^{n-k} &\geq (1+k(y-1))(1+(n-k)(z-1)) \\ &\geq 1+k(y-1)+(n-k)(z-1) = ky+(n-k)z-(n-1), \end{aligned}$$

and the desired inequality follows.

Next, suppose $z < y \leq 1$. If $1+k(y-1)$ and $1+(n-k)(z-1)$ are both nonnegative, then we can reason as in the previous paragraph. If either one of them is negative, say $1+(n-k)(z-1) < 0$, then $(*)$ follows from

$$ky+(n-k)z \leq ky+(n-k-1) \leq k+(n-k-1) = n-1.$$

Thus, for the rest of the solution we may assume $z < 1 < y$. Suppose

$$z \leq (n-k-1)/(n-k).$$

Inequality $(*)$ would follow from

$$(n-1)\cdot\sqrt{1+\frac{k(n-k)}{n-1}(y-z)^2} \geq ky+(n-k)z.$$

Since the left side is a decreasing function of z and the right side is an increasing function of z, it suffices to prove this in the case $z = (n-k-1)/(n-k)$; that is, it suffices to prove

$$(n-1)\cdot\sqrt{1+\frac{k(n-k)}{n-1}\left(y-\frac{n-k-1}{n-k}\right)^2} \geq ky+(n-k-1).$$

After squaring and canceling a factor of $k/(n-k)$ this reduces to

$$(n-k-1)(n-k)n(y-1)^2+(n-1) \geq 0,$$

which is clearly true.

Finally, suppose $z > (n-k-1)/(n-k)$. By Bernoulli's inequality, it suffices to show

$$(n-1)\cdot\sqrt{1+\frac{k(n-k)}{n-1}(y-z)^2}+(1+k(y-1))(1+(n-k)(z-1)) \geq ky+(n-k)z.$$

After some simplification, this reduces to

$$\left((n-1)-k(n-k)(1-z)^2\right)(y-1)^2+(n-1)(1-z)^2 \geq 0,$$

which is true since $(1-z)^2 < 1/(n-k)^2$ and $n-1 > k/(n-k)$, so the coefficient of $(y-1)^2$ is positive.

Also solved by A. Stadler (Switzerland) and the proposer.

The Base-5 Expansion of a Reciprocal

12170 [2020, 274]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Let p be a prime number congruent to 1 modulo 15. Show that the minimal period of the base 5 expansion of $1/p$ cannot be equal to $(p-1)/15$.

Solution by Joel Schlosberg, Bayside, NY. Let $n = (p-1)/30$. Since p is odd, $(p-1)/15$ is even, so $n \in \mathbb{N}$. Since $p \equiv 1^2 \pmod 5$, the quadratic reciprocity theorem implies that 5 is a quadratic residue modulo p. Therefore, $5 \equiv z^2 \pmod p$ for some $z \in \mathbb{Z}$. Since $p \nmid 5$, also $p \nmid z$. Therefore, by Fermat's little theorem,

$$5^{15n} \equiv 5^{(p-1)/2} \equiv z^{p-1} \equiv 1 \pmod p.$$

Suppose that $2n$ is the minimal period of the base 5 expansion of $1/p$. This means that $2n$ is the least positive integer m such that $p \mid (5^m - 1)$. Since also $p \mid (5^{15n} - 1)$, the multiplicative order of 5 modulo p must divide $\gcd(2n, 15n)$, which equals n. Now $p \mid (5^n - 1)$, a contradiction.

Also solved by R. Chapman (UK), A. Dixit (Canada) & S. Pathak (USA), S. M. Gagola, Jr., K. T. L. Koo (China), O. P. Lossers (Netherlands), A. Nakhash, M. A. Prasad, A. Stadler (Switzerland), A. Stenger, R. Stong, D. Terr, E. White & R. White, and the proposer.

A Tetrahedron and the Midpoints of its Edges

12172 [2020, 275]. *Proposed by Hidefumi Katsuura, San Jose State University, San Jose, CA.* Let A, B, C, and D be four points in three-dimensional space, and let U, V, W, X, Y, and Z be the midpoints of AB, AC, AD, BC, BD, and CD, respectively.
(a) Prove

$$4\left(UZ^2 + VY^2 + WX^2\right) = AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2.$$

(b) Prove

$$\begin{aligned} 4\Big((AB \cdot CD \cdot UZ)^2 &+ (AC \cdot BD \cdot VY)^2 + (AD \cdot BC \cdot WX)^2\Big) \\ &\geq (AB \cdot BC \cdot CA)^2 + (BC \cdot CD \cdot DB)^2 + (CD \cdot DA \cdot AC)^2 + (DA \cdot AB \cdot BD)^2, \end{aligned}$$

and determine when equality holds.

Solution by Li Zhou, Polk State College, Winter Haven, FL.
(a) By Apollonius's theorem,

$$4WB^2 = 2AB^2 + 2BD^2 - AD^2,$$
$$4WC^2 = 2AC^2 + 2CD^2 - AD^2,$$

and

$$4WX^2 = 2WB^2 + 2WC^2 - BC^2 = AB^2 + AC^2 + BD^2 + CD^2 - AD^2 - BC^2.$$

Adding the last equation to the analogous expressions for $4UZ^2$ and $4VY^2$ establishes the identity.
(b) Using the expressions above for $4WX^2$, $4VY^2$, $4UZ^2$, a computation shows that

$$\begin{aligned} 4\cdot\big((AB \cdot CD \cdot UZ)^2 &+ (AC \cdot BD \cdot VY)^2 + (AD \cdot BC \cdot WX)^2\big) \\ &- (AB\cdot BC\cdot CA)^2 - (BC\cdot CD\cdot DB)^2 - (CD\cdot DA\cdot AC)^2 - (DA\cdot AB\cdot BD)^2 \qquad (*) \end{aligned}$$

is equal to

$$\frac{1}{2}\det\begin{bmatrix} 2AD^2 & AD^2+BD^2-AB^2 & AD^2+CD^2-AC^2 \\ AD^2+BD^2-AB^2 & 2BD^2 & BD^2+CD^2-BC^2 \\ AD^2+CD^2-AC^2 & BD^2+CD^2-BC^2 & 2CD^2 \end{bmatrix}.$$

The determinant here is the Cayley–Menger determinant for the tetrahedron $ABCD$ and its value is $288\Delta^2$, where Δ is the volume of $ABCD$. Hence $(*)$ is equal to $144\Delta^2$, which is clearly nonnegative. This yields the desired inequality, and equality holds if and only if $\Delta = 0$, in other words A, B, C, and D are coplanar.

Editorial comment. The Cayley–Menger determinant generalizes Heron's formula for the area of a triangle to simplices of higher dimension.

Also solved by M. Bataille (France), R. Chapman (UK), G. Fera & G. Tescaro (Italy), D. Fleischman, E. A. Herman, W. Janous (Austria), M. Kaplan & M. Goldenberg, B. Karaivanov (USA) & T. S. Vassilev (Canada), K. T. L. Koo (China), A. Stadler (Switzerland), R. Stong, T. Wiandt, and the proposer.

A Matrix Equation

12173 [2020, 275]. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.* Suppose that X and Y are n-by-n complex matrices such that $2Y^2 = XY - YX$ and the rank of $X - Y$ is 1. Prove $Y^3 = YXY$.

Solution by Roger A. Horn, Tampa, FL. Let z and w be nonzero complex n-vectors such that $X - Y = zw^*$. It suffices to show that if

$$2Y^2 = zw^*Y - Yzw^*, \tag{1}$$

then $Yzw^*Y = 0$. Jacobson's lemma (see page 126 of R. Horn and C. Johnson (2018), *Matrix Analysis,* 2nd ed., New York: Cambridge University Press) states that if $BC - CB$ commutes with C, then $BC - CB$ is nilpotent. Consequently, Y^2 (and hence Y) is nilpotent. The rank of Y^2 is at most 2, since it is the sum of two matrices whose ranks are at most 1. Therefore, the Jordan canonical form of Y is a direct sum of nilpotent Jordan blocks that are not larger than 4-by-4. There are three cases.

Case (a): $Y^2 = 0$ *(no block larger than 2-by-2).* If $Y^2 = 0$, then $Y^2z = 0$ and

$$0 = 2Y^3 = Y\,2Y^2 = Yzw^*Y - Y^2zw^* = Yzw^*Y. \tag{2}$$

Case (b): $Y^2 \neq 0$ *and* $Y^3 = 0$ *(the largest block is 3-by-3).* We compute

$$0 = 2Y^4 = Y^2\,2Y^2 = Y^2zw^*Y - Y^3zw^* = (Y^2z)(w^*Y).$$

Either $w^*Y = 0$ and we are done, or $w^*Y \neq 0$ and $Y^2z = 0$. In the latter case, (2) also holds, and it ensures that $Yzw^*Y = 0$.

Case (c): $Y^3 \neq 0$ *and* $Y^4 = 0$ *(the largest block is 4-by-4).* Let v be a complex n-vector such that $Y^3v \neq 0$. Suppose $Yz \neq 0$. We compute

$$0 = 2Y^5v = 2Y^2\,Y^3v = zw^*Y^4v - Yzw^*Y^3v = -(w^*Y^3v)Yz,$$

so $w^*Y^3v = 0$. We also have

$$0 = 2Y^4v = 2Y^2\,Y^2v = zw^*Y^3v - Yzw^*Y^2v = -(w^*Y^2v)Yz,$$

so $w^*Y^2v = 0$ as well. Now compute

$$2Y^3v = 2Y^2\,Yv = zw^*Y^2v - Yzw^*Yv = -(w^*Yv)Yz, \tag{3}$$

which ensures $w^*Yv \neq 0$ since $Y^3 v \neq 0$. Multiply (3) by Y to obtain

$$0 = 2Y^4 v = -(w^*Yv)Y^2 z,$$

so $Y^2 z = 0$. Finally, use (1) to compute

$$2Y^3 v = Y\,2Y^2 v = Yzw^*Yv - Y^2 zw^*v = (w^*Yv)Yz,$$

which contradicts (3). We conclude $Yz = 0$ and hence $Yzw^*Y = 0$.

Editorial comment. Kyle Gatesman observed that the result holds when the hypothesis $2Y^2 = XY - YX$ is replaced by the more general $kY^2 = XY - YX$ for some nonzero $k \in \mathbb{C}$. Several solvers noted that the conclusion $Y^3 = YXY$ can be strengthened to $Y^3 = 0 = YXY$.

Also solved by M. Bataille (France), C. Chiser (Romania), K. Gatesman, N. Grivaux (France), L. Han, E. A. Herman, N. Hodges (UK), K. T. L. Koo (China), C. P. A. Kumar (India), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), K. Sarma (India), A. Stadler (Switzerland), R. Stong, J. Stuart, R. Tauraso (Italy), E. I. Verriest, and the proposer.

Powers of 4 and 5 with the Same Leading Digits

12174 [2020, 372]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.*
(a) Let n be a positive integer, and suppose that the three leading digits of the decimal expansion of 4^n are the same as the three leading digits of 5^n. Find all possibilities for these three leading digits.
(b) Prove that for any positive integer k there exists a positive integer n such that the k leading digits of the decimal expansion of 4^n are the same as the k leading digits of 5^n.

Solution by Oliver Geupel, Brühl, Germany. We prove the stronger claim that for any positive integer k there are exactly two k-digit numbers a that occur as the k leading digits of the decimal expansions of 4^n and 5^n for some positive integer n. In particular, the condition for such a number a is the existence of a positive integer n and nonnegative integers p and q such that $10^p a \le 5^n < 10^p(a+1)$ and $10^q a \le 4^n < 10^q(a+1)$. Notice that if $10^p a = 5^n$ then $a \le 10^q a \le 4^n < 5^n = 10^p a$, so $p > 0$. This implies that $10^p a$ is even and 5^n is odd, which is a contradiction. Therefore we can strengthen the first inequality to $10^p a < 5^n < 10^p(a+1)$. The product of the second inequality with the square of the first yields

$$10^{2p+q} a^3 < 5^{2n} 2^{2n} < 10^{2p+q}(a+1)^3.$$

Thus a power 10^m lies between a^3 and $(a+1)^3$. Since a has k digits, we have

$$10^{3k-3} \le a^3 < 10^m < (a+1)^3 \le 10^{3k}.$$

Thus $m \in \{3k-2, 3k-1\}$, leaving only two candidates for a: $\lfloor 10^{k-2/3} \rfloor$ and $\lfloor 10^{k-1/3} \rfloor$. In the case $k = 3$, these numbers are 215 and 464.

Now suppose that $a = \lfloor 10^\beta \rfloor$, where $\beta \in \{k - 2/3, k - 1/3\}$. We prove that a occurs as the k leading digits of the decimal expansions of 4^n and 5^n for some positive integer n. This confirms that 215 and 464 are solutions to part **(a)** and proves the claim in part **(b)**.

The inequality $10^p a < 5^n < 10^p(a+1)$ can be rewritten as

$$p + \log_{10} a < n \log_{10} 5 < p + \log_{10}(a+1).$$

Since $\log_{10} a < \beta < \log_{10}(a+1)$, to satisfy the inequality we need to have $p + \beta$ close to $n \log_{10} 5$. Thus we begin by finding p and n for which these numbers are close.

Let $\varepsilon = \min\{\beta - \log_{10} a, \log_{10}(a+1) - \beta\}$, which is positive. Kronecker's approximation theorem asserts that the positive integer multiples of an irrational number modulo 1 are dense in (0, 1) (see, for example, Chapter XXIII of G. H. Hardy and E. M. Wright (1975), *An Introduction to the Theory of Numbers*, 4th ed., Oxford: Clarendon Press). Therefore, there exist infinitely many pairs of positive integers n and p such that $|n \log_{10} 5 - p - \beta| < \varepsilon/2$. Consider such pairs (n, p).

Taking logarithms in $4 = 100/25$ yields $\log_{10} 4 = 2(1 - \log_{10} 5)$, so

$$|n \log_{10} 4 - (2n - 2p - 3\beta) - \beta| = 2|p + \beta - n \log_{10} 5| < \varepsilon,$$

which can be rewritten as $|n \log_{10} 4 - q - \beta| < \varepsilon$, where q is the integer $2n - 2p - 3\beta$. Among the pairs (n, p) satisfying the restriction involving ε, choose a pair with n large enough so that q is positive. We obtain

$$q + \log_{10} a \le q + \beta - \varepsilon < n \log_{10} 4 < q + \beta + \varepsilon \le q + \log_{10}(a+1)$$

and, analogously,

$$p + \log_{10} a < n \log_{10} 5 < p + \log_{10}(a+1).$$

Thus $10^q a < 4^n < 10^q(a+1)$ and $10^p a < 5^n < 10^p(a+1)$. Consequently, the k-digit number a occurs as the k leading digits of the decimal expansions of 4^n and 5^n.

Also solved by R. Chapman (UK), G. Fera (Italy), N. Hodges (UK), O. P. Lossers (Netherlands), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer. Part (**a**) also solved by D. Terr.

An Incenter-Centroid Inequality

12175 [2020, 372]. *Proposed by Giuseppe Fera, Vicenza, Italy.* Let I be the incenter and G be the centroid of a triangle ABC. Prove

$$2 < \frac{IA^2}{GA^2} + \frac{IB^2}{GB^2} + \frac{IC^2}{GC^2} \le 3.$$

Solution by Arkady Alt, San Jose, CA. Let a, b, and c be the lengths of the sides opposite A, B, and C, let m_a, m_b, and m_c be the corresponding median lengths, and let l_A, l_B, and l_C be the corresponding angle bisector lengths. Let r be the inradius and s the semiperimeter.

By the Pythagorean theorem, $IA^2 = r^2 + (s-a)^2$. From Heron's formula and the inradius/semiperimeter formula for the area of a triangle, we have

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

Using $2s = a + b + c$, we obtain

$$IA^2 = \frac{(s-a)[(s-b)(s-c) + (s-a)s]}{s} = \frac{bc(s-a)}{s}.$$

It is well known that $GA = (2/3)m_a$. By Apollonius's theorem,

$$m_a^2 = (2b^2 + 2c^2 - a^2)/4.$$

Therefore $GA^2 = (2b^2 + 2c^2 - a^2)/9$, so

$$\frac{IA^2}{GA^2} = \frac{9bc(s-a)}{s(2b^2 + 2c^2 - a^2)}.$$

To establish the upper bound, we observe that

$$\begin{aligned} 2b^2 + 2c^2 - a^2 &= (b+c)^2 + (b-c)^2 - a^2 \ge (b+c)^2 - a^2 \\ &= (b+c+a)(b+c-a) = 4s(s-a), \end{aligned}$$

and therefore

$$\frac{IA^2}{GA^2} \le \frac{9bc(s-a)}{4s^2(s-a)} = \frac{9bc}{4s^2}.$$

Similarly, $IB^2/GB^2 \le 9ac/(4s^2)$ and $IC^2/GC^2 \le 9ab/(4s^2)$, so

$$\frac{IA^2}{GA^2} + \frac{IB^2}{GB^2} + \frac{IC^2}{GC^2} \le \frac{9(ab+bc+ca)}{4s^2}.$$

By the Cauchy–Schwarz inequality, $a^2 + b^2 + c^2 \ge ab + bc + ca$, so

$$4s^2 = (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca) \ge 3(ab+bc+ca).$$

Therefore

$$\frac{IA^2}{GA^2} + \frac{IB^2}{GB^2} + \frac{IC^2}{GC^2} \le \frac{9(ab+bc+ca)}{3(ab+bc+ca)} = 3.$$

For the lower bound, we start with

$$2b^2 + 2c^2 - a^2 = (b+c)^2 - (a^2 - (b-c)^2) < (b+c)^2,$$

which holds because $a^2 > (b-c)^2$, which follows from the triangle inequality. Therefore

$$\frac{IA^2}{GA^2} > \frac{9bc(s-a)}{s(b+c)^2} = \frac{9l_A^2}{4s^2},$$

where in the last step we have used the known formula $l_A^2 = 4bcs(s-a)/(b+c)^2$. Similarly,

$$\frac{IB^2}{GB^2} > \frac{9l_B^2}{4s^2}$$

and

$$\frac{IC^2}{GC^2} > \frac{9l_C^2}{4s^2},$$

so

$$\frac{IA^2}{GA^2} + \frac{IB^2}{GB^2} + \frac{IC^2}{GC^2} > \frac{9}{4s^2}(l_A^2 + l_B^2 + l_C^2).$$

The required lower bound now follows from the inequality

$$l_A^2 + l_B^2 + l_C^2 > (8/9)s^2$$

(see page 218, inequality 11.7 in D. S. Mitrinović, J. E. Pečarić, V. Volenec (1989), *Recent Advances in Geometric Inequalities*, Dordrecht: Springer).

Editorial comment. Li Zhou cited experimental evidence from Geometer's Sketchpad for the following conjectures: The order of IA/GA, IB/GB, and IC/GC corresponds inversely to the order of a, b, and c, and hence also to the order of angles A, B, and C. Moreover, the sum of the two largest of IA^2/GA^2, IB^2/GB^2, and IC^2/GC^2 is already at least 2.

Walter Janous strengthened the inequality to

$$2 + \frac{r}{8R} \le \frac{IA^2}{GA^2} + \frac{IB^2}{GB^2} + \frac{IC^2}{GC^2} \le \frac{41}{16} + \frac{7r}{8R},$$

where R is the circumradius of $\triangle ABC$. That this upper bound is stronger follows from $2r \le R$.

Also solved by H. Bailey, S. Gayen (India), W. Janous (Austria), M. Kaplan & M. Goldenberg, P. Khalili, K.-W. Lau (China), J. H. Lindsey II, C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, T. Zvonaru (Romania), and the proposer.

A Diophantine Equation

12176 [2020, 372]. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Solve

$$xy^3 + y^2 - x^5 - 1 = 0$$

in positive integers.

Solution by Mandyam A. Prasad, Mumbai, India. We show that the only solution in positive integers is $(x, y) = (1, 1)$. When $x = 1$, the equation becomes $y^3 + y^2 - 2 = 0$, whose only solution is $y = 1$. When $y = 1$, the equation becomes $x - x^5 = 0$, whose only positive solution is $x = 1$.

Hence we may assume $x \geq 2$ and $y \geq 2$. The polynomial $x^4 - x - 1$ is positive at $x = 2$ and has positive derivative for $x \geq 2$, so $x^4 - x - 1 > 0$ for $x \geq 2$. Therefore

$$(x - 1)(x^4 - x - 1) > 0.$$

Expanding yields $x^4 + x^2 < x^5 + 1$. If $y \leq x$, then

$$xy^3 + y^2 \leq x^4 + x^2 < x^5 + 1,$$

which contradicts the original equation. Therefore, we may assume $x < y$.

If $x^2 \leq y$, then

$$xy^3 + y^2 = x^5 + 1 \leq xy^2 + 1,$$

which yields $xy^2(y - 1) \leq 1 - y^2$, contradicting $y > 1$. Hence $y < x^2$.

Rewritten as $(y^2 - 1)(xy + 1) = x(x^4 - y)$, the original equation implies $x(x^4 - y) \equiv 0 \pmod{xy + 1}$. Multiplying by $-x(y + y^3)$ yields

$$-x^6y - x^6y^3 + x^2y^2 + x^2y^4 \equiv -x(y + y^3)x(x^4 - y) \equiv 0 \pmod{x(xy + 1)}.$$

The extra factor of x in the modulus is allowed because we multiplied by a multiple of x. Using $x^2y \equiv -x \pmod{x(xy + 1)}$ and the original equation, we obtain

$$x^5 + x^3 - xy - xy^3 = x^3 - xy - 1 + y^2 \equiv 0 \pmod{x(xy + 1)}.$$

Since

$$x^3 - xy - 1 + y^2 > x^2 - xy + y^2 - 1 > 0$$

when $x, y \geq 2$, we must have

$$x^3 - xy - 1 + y^2 \geq x(xy + 1),$$

because the left side is a multiple of the right side. This inequality can be rewritten as

$$(x^2 - y)(x - y) - x - 1 \geq 0.$$

Since $x < y < x^2$, the left side is negative, which is a contradication. This forbids all solutions with $x, y > 1$.

Also solved by the proposer.

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SOLUTIONS

Strengthening the Cauchy–Schwarz Inequality

12163 [2020, 179]. *Proposed by Thomas Speckhofer, Attnang-Puchheim, Austria.* Let $\mathbb{R}^n$ have the usual dot product and norm. When $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\Sigma v = x_1 + \cdots + x_n$. Prove

$$\|v\|^2 \|w\|^2 \geq (v \cdot w)^2 + \frac{1}{n}(\|v\| \, |\Sigma w| - \|w\| \, |\Sigma v|)^2$$

for all $v, w \in \mathbb{R}^n$.

Solution by the Davis Problem Solving Group, Davis, CA. If either $v = 0$ or $w = 0$ then both sides of the requested inequality are zero, so we may assume $v \neq 0$ and $w \neq 0$.

First assume $\Sigma v \neq 0$ and $\Sigma w \neq 0$. By homogeneity, we may assume $\Sigma v = \Sigma w = 1$. We have $(v \cdot w)^2 = \|v\|^2 \|w\|^2 \cos^2 \theta$, where θ is the angle between the vectors v and w. Thus we must prove $\|v\|^2 \|w\|^2 \geq \|v\|^2 \|w\|^2 \cos^2 \theta + (1/n)(\|v\| - \|w\|)^2$, or

$$\|v\|^2 \|w\|^2 \sin^2 \theta \geq \frac{1}{n}(\|v\| - \|w\|)^2. \tag{1}$$

Let S denote the area of the triangle whose vertices are the origin, v, and w. If h is the altitude of the triangle from the origin, then $h \geq 1/\sqrt{n}$, since $1/\sqrt{n}$ is the minimum distance from the origin to a point in the hyperplane $x_1 + \cdots + x_n = 1$. Thus

$$\|v\| \|w\| \sin \theta = 2S = h\|v - w\| \geq \frac{1}{\sqrt{n}} \|v - w\|,$$

and squaring yields

$$\|v\|^2 \|w\|^2 \sin^2 \theta \geq \frac{1}{n} \|v - w\|^2 \geq \frac{1}{n}(\|v\| - \|w\|)^2,$$

where the final inequality is a consequence of the triangle inequality. This establishes (1). Equality holds if and only v and w are linearly dependent.

If $\Sigma v = 0$ and $\Sigma w = 0$, then the inequality reduces to the Cauchy–Schwarz inequality, and once again equality holds if and only if v and w are linearly dependent. Finally, assume that one of Σv or Σw is zero and the other is nonzero. It suffices to consider the case where $\Sigma w = 0$ and $\Sigma v \neq 0$, and again we may assume $\quad v = 1$. As before, if θ is the angle between v and w then the inequality to be proved reduces to $\|v\|^2 \|w\|^2 \sin^2 \theta \geq (1/n)\|w\|^2$, and since we have assumed $w \neq 0$, this is equivalent to

$$\|v\| \sin \theta \geq \frac{1}{\sqrt{n}}. \tag{2}$$

The left side of (2) is the distance from v, which is in the hyperplane $x_1 + \cdots + x_n = 1$, to a point in the hyperplane $x_1 + \cdots + x_n = 0$. This distance must be at least $1/\sqrt{n}$, the distance between the two parallel hyperplanes, showing that (2) is true. In this case, equality is attained if and only if $\lambda v = \mu w + (1/n, \ldots, 1/n)$ for some real λ and μ; that is, if and only if $(1, \ldots, 1)$ is in the span of v and w.

Also solved by R. A. Agnew, K. F. Andersen (Canada), J. N. Caro Montoya (Brazil), R. Chapman (UK), L. Giugiuc (Romania), L. Han, E. A. Herman, W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), R. Mansuy (France), M. Omarjee (France), E. Schmeichel, A. Stadler (Switzerland), G. Stoica (Canada), R. Stong, Florida Atlantic University Problem Solving Group, and the proposer.

A Pell-Type Diophantine Equation

12164 [2020, 179]. *Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia.* Characterize the positive integers d such that $(d^2 + d)x^2 - y^2 = d^2 - 1$ has a solution in positive integers x and y.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. There are solutions exactly when $d + 1$ is a square. Write $d + 1 = gm^2$, where g is squarefree. If $g = 1$, then $d = m^2 - 1$, and $(x, y) = (1, m)$ is a solution. We show that there is no solution (x, y) when $g \geq 2$.

Fix a squarefree g with $g \geq 2$. Let d be minimal such that the specified equation has a solution (x, y) when d has the form $gm^2 - 1$. Note that $d > 1$, since when $d = 1$ the equation is $2x^2 - y^2 = 0$, which famously has no positive integer solutions. With d minimized, we reduce to checking finitely many cases by first showing $x < 2\sqrt{d}$ for the solution with smallest positive x and then showing $d < 14$.

It is convenient to work in the ring $\mathbb{Z}[\sqrt{D}]$, where $D = d(d + 1)$, which is the set of real numbers of the form $a + b\sqrt{D}$, where $a, b \in \mathbb{Z}$ and elements multiply as real numbers. The *norm* of an element $a + b\sqrt{D}$ is defined to be

$$(a + b\sqrt{D})(a - b\sqrt{D}),$$

which equals $a^2 - b^2 D$. With this definition, it is easy to confirm that the norm of a product is the product of the norms of the factors.

A solution (u, v) to an equation of the form $u^2 - kv^2 = c$ corresponds to an element $u + v\sqrt{k}$ in $\mathbb{Z}[\sqrt{k}]$ with norm c. In particular, the Pell equation $u^2 - Dv^2 = 1$ has the solution $(u, v) = (2d + 1, 2)$, which corresponds to the number $2d + 1 + 2\sqrt{D}$ of norm 1. Let α be this number.

Now choose $\beta = y + x\sqrt{D}$ with $x, y > 0$ so that β is the smallest real number in $\mathbb{Z}[\sqrt{D}]$ having norm $1 - d^2$. Thus (x, y) is a solution to $y^2 - Dx^2 = 1 - d^2$ with minimal positive x and y.

Because the norm of α is 1, we have $\alpha^{-1} = 2d + 1 - 2\sqrt{D}$, and hence α^{-1} is in $\mathbb{Z}[\sqrt{D}]$ and has norm 1. For suitable integers x' and y', we have

$$\alpha^{-1}\beta = (2d + 1 - 2\sqrt{D})(y + x\sqrt{D}) = y' + x'\sqrt{D}.$$

By the multiplicativity of the norm, $\alpha^{-1}\beta$ has norm $1 - d^2$. Also $\alpha^{-1}\beta < \beta$, since $\alpha^{-1} < 1$. By the minimality of the positive coefficients in β, at least one of x' and y' is nonpositive. Furthermore, since $\alpha^{-1}\beta$ is a positive real number, x' or y' is positive. We compute

$$\alpha^{-1}\beta(-y' + x'\sqrt{D}) = (y' + x'\sqrt{D})(-y' + x'\sqrt{D}) = d^2 - 1,$$

where the final equality holds because the middle expression is the negative of the norm of $\alpha^{-1}\beta$. Thus

$$\frac{(d^2-1)\alpha}{\beta} = \frac{d^2-1}{\alpha^{-1}\beta} = -y' + x'\sqrt{D}.$$

Since d^2-1 and $\alpha^{-1}\beta$ are positive, so is $-y'+x'\sqrt{D}$. With the restrictions above on x' and y', we conclude $y' \le 0 < x'$. Since $(y')^2-(x')^2D = 1-d^2$, setting $y'=0$ would give $(x')^2 = (d^2-1)/(d(d+1)) = (d-1)/d < 1$; hence $y' < 0$.

Since $-y'+x'\sqrt{D}$ has norm $1-d^2$ with $-y'$ and x' both positive, the minimality of β implies that $(d^2-1)\alpha/\beta$ is at least β, so

$$x\sqrt{d(d+1)} < \beta \le \sqrt{(d^2-1)\alpha} = \sqrt{d^2-1}(\sqrt{d}+\sqrt{d+1}).$$

Therefore,

$$x < \sqrt{d-1} + \sqrt{(d^2-1)/d} < 2\sqrt{d}.$$

Next we bound d. Write the original equation as

$$y^2 = (d+1)((x^2-1)d+1) = gm^2((x^2-1)d+1).$$

It follows that $(x^2-1)d+1 = gn^2$ for some positive integer n. Since $g \ge 2$, we have $x \ne 1$ and

$$n^2-(x^2-1)m^2 = \frac{(x^2-1)d+1-(x^2-1)(d+1)}{g} = \frac{2-x^2}{g}.$$

In the ring $\mathbb{Z}[\sqrt{x^2-1}]$, consider γ and δ given by

$$\gamma = x+\sqrt{x^2-1} \quad\text{and}\quad \delta = n+m\sqrt{x^2-1},$$

with norms 1 and $(2-x^2)/g$, respectively. Let n_1 and m_1 be positive integers such that $n_1+m_1\sqrt{x^2-1}$ has norm $(2-x^2)/g$ in this ring. Setting $(x,y)=(x,gm_1n_1)$ yields a solution to the original equation with $d+1=gm_1^2$. The minimality of d for this g implies that δ is minimal among all elements of $\mathbb{Z}[\sqrt{x^2-1}]$ having positive coefficients and norm $(2-x^2)/g$.

The same argument given earlier for $(d^2-1)\alpha/\beta$ shows that $(x^2-2)\gamma/(g\delta)$ has norm $(2-x^2)/g$ and can be written as $n'+m'\sqrt{x^2-1}$ with n' and m' being positive integers. The minimality of δ now implies

$$gm^2(x^2-1) < g\delta^2 \le (x^2-2)\gamma < 2x(x^2-1),$$

and hence

$$d+1 = gm^2 < 2x < 4\sqrt{d}.$$

Treating this as an inequality in $\sqrt{d}$ and applying the quadratic formula yields

$$d < (2+\sqrt{3})^2 < 14.$$

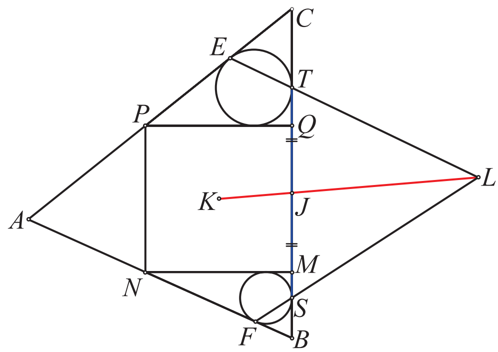
Since these minimal solutions require $d < 14$ and $x < 2\sqrt{d}$, there remain only finitely many cases to consider. The casework is streamlined by reducing the equation modulo $d-1$, requiring $2x^2 \equiv y^2 \pmod{d-1}$. If $d-1$ has as a factor any prime congruent to ± 3 modulo 8 (such as 3, 5, or 11), then x must also be a multiple of this factor, since 2 is not a square modulo any such number. Since $x < 2\sqrt{d}$, these possibilities are easily eliminated. For example, when $d-1=9$, we need only consider 3 and 6 for x in the original equation, and neither $110 \cdot 9 - y^2 = 99$ nor $110 \cdot 36 - y^2 = 99$ has an integer solution. If $d - 1$ has 4 as a factor, then x must be even because 2 is not a square mod 4, and these possibilities can similarly be checked quickly.

For $d < 14$, in each case $d - 1$ is a multiple of some element of $\{3, 4, 5, 11\}$ except for the remaining cases where d is 2, 3, or 8. The last two of these have $g = 1$, so only $d = 2$ needs to be analyzed. In this case, the equation reads $6x^2 - y^2 = 3$, hence $y = 3z$ for some integer z, and $2x^2 - 3z^2 = 1$. Taking the equation modulo 3 shows that this also fails.

Also solved by the proposer.

An Unexpected Bisection

12165 [2020, 180]. *Proposed by Tran Quang Hung and Nguyen Minh Ha, Hanoi, Vietnam.* Let $MNPQ$ be a square with center K inscribed in triangle ABC with N and P lying on sides AB and AC, respectively, while M and Q lie on side BC. Let the incircle of $\triangle BMN$ touch side BM at S and side BN at F, and let the incircle of $\triangle CQP$ touch side CQ at T and side CP at E. Let L be the point of intersection of lines FS and ET. Prove that KL bisects the segment ST.



Solution I by Haoran Chen, Suzhou, China. Let G and H be the feet of the altitudes to BC from L and K, respectively. Let J be the intersection of KL and ST, and let I be the midpoint of ST. Our goal is to show that I and J are the same point.

Let s be the side length of the square $MNPQ$. Let $\alpha = \angle CTE = \angle STL$ and $\beta = \angle BSF = \angle TSL$. We establish formulas for $\cot\alpha$ and $\cot\beta$. To derive these formulas, let D be the foot of the perpendicular from E to CT, so that $\cot\alpha = DT/DE$. Let $x = QT$, $y = CT = CE$, and $z = PE$. This gives $x + z = PQ = s$. Since $\triangle CDE \sim \triangle CQP$, we have $DE/CE = QP/CP$, so

$$DE = CE \cdot QP/CP = y(x+z)/(y+z).$$

Similarly, $CD = y(x+y)/(y+z)$, so

$$DT = y - CD = y(z-x)/(y+z).$$

We conclude

$$\cot\alpha = \frac{DT}{DE} = \frac{y(z-x)/(y+z)}{y(x+z)/(y+z)} = 1 - \frac{2x}{x+z} = 1 - \frac{2x}{s}.$$

Similarly, if we let $u = MS$, then $\cot\beta = 1 - 2u/s$.

If $x = u$, then $\cot\alpha = \cot\beta$, so $\alpha = \beta$, and the desired conclusion follows by symmetry. Now assume without loss of generality that $x > u$, so $\cot\alpha = 1 - 2x/s < 1 - 2u/s = \cot\beta$. Letting $t = GL$, we have $GT = t\cot\alpha < t\cot\beta = GS$, so $GT < ST/2$. Also, $HT = x + s/2 > u + s/2 = HS$, so $HT > ST/2$. Thus G lies between H and T, and I lies between G and H. Clearly J is also between G and H, so to show that $I = J$ it suffices to prove $IG/IH = JG/JH$.

By similar triangles, we have

$$\frac{JG}{JH} = \frac{LG}{KH} = \frac{t}{s/2} = \frac{2t}{s}.$$

Also,

$$IG = \frac{ST}{2} - GT = \frac{GS + GT}{2} - GT = \frac{GS - GT}{2} = \frac{t(\cot\beta - \cot\alpha)}{2}$$

and

$$IH = IS - HS = \frac{x + s + u}{2} - \left(\frac{s}{2} + u\right) = \frac{x - u}{2}.$$

Therefore

$$\frac{IG}{IH} = \frac{t(\cot\beta - \cot\alpha)/2}{(x-u)/2} = \frac{t[(1 - 2u/s) - (1 - 2x/s)]}{x - u} = \frac{2t}{s},$$

which completes the proof.

Solution II by L. Richie King, Davidson, NC. Let the bisector of $MNPQ$ parallel to MQ and NP intersect line ET at U and line FS at V. We show that K is the midpoint of UV. The result follows from this, since LK is the median of $\triangle LUV$ from L, and so it bisects every section parallel to UV, including ST.

Let O be the center of the incircle of $\triangle PQC$. Note that QO bisects $\angle PQC$. Let P', E', and T' be the reflections of P, E, and T in QO. The line PQ is tangent to the incircle at T', and the lines $P'E'$ and $P'T$ are also tangent to the incircle.

We use some known results about polars. The *polar* of a point Z with respect to the incircle of $\triangle PQC$ is the line perpendicular to ZO that passes through the image of Z under inversion in the incircle. A fundamental fact about polars is that if the polar of Z passes through a point Y then the polar of Y passes through Z.

Since E is fixed under inversion in the incircle, the polar of E is PC, the line tangent to the incircle at E. Similarly, the polar of T' is PQ. Since the polars of both E and T' pass through P, the polar of P must pass through both E and T', so it must be the line ET'. Similarly, the polar of P' is $E'T$. Let X be the point of intersection of ET' and $E'T$. Then X lies on the polars of both P and P', so the polar of X is the line PP', which is perpendicular to QO.

The point X is one of the vertices of the *diagonal triangle* of the concyclic quadrilateral $ETT'E'$. The other two vertices are the point Y where the lines ET and $E'T'$ intersect, which lies on QO, and the point Z at infinity on the lines EE' and TT'. We now use one more known fact about polars: the polar of each vertex of the diagonal triangle of a concyclic quadrilateral is the line through the other two vertices (see H. S. M. Coxeter, (1998), *Non-Euclidean Geometry*, 6th ed., Washington, DC: Mathematical Association of America, p. 57). In particular, PP', which is the polar of X, passes through Y, and therefore Y is the intersection point of PP' and QO. We conclude that PQY is an isosceles right triangle, with right angle at Y. Therefore Y lies on the bisector of $MNPQ$ parallel to MQ and NP, so $U = Y$ and UK has length equal to the side length of the square. Similar reasoning shows that VK has the same length, which establishes our claim that K is the midpoint of UV.

Editorial comment. Marty Getz and Dixon Jones generalized the problem to a rectangle inscribed in a triangle, as did the Davis Problem Solving Group. Giuseppe Fera and Giorgio Tescaro generalized to an inscribed parallelogram.

Also solved by W. Burleson & C. Helms & L. Ide & A. Liendo & M. Thomas, W. Chang, P. De (India), G. Fera & G. Tescaro (Italy), M. Getz & D. Jones, O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), N. Hodges (UK), W. Hu (China), E.-Y. Jang (Korea), W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, C. R. Pranesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, T. Zvonaru (Romania), Davis Problem Solving Group, and the proposers.

Asymptotics of a Recursive Sequence

12166 [2020, 180]. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden.* Let $a_0 = 0$, and define a_k recursively by $a_k = e^{a_{k-1}-1}$ for $k \geq 1$.
(a) Prove $k/(k+2) < a_k < k/(k+1)$ for all $k \geq 1$.
(b) Is there a number c such that $a_k < (k+c)/(k+c+2)$ for all k?

Solution by Jean-Pierre Grivaux, Paris, France. We prove part **(a)** by induction on k. The base case $k = 1$ follows from $2 < e < 3$. For the induction step, the inductive hypothesis implies that

$$e^{-2/(k+2)} < a_{k+1} < e^{-1/(k+1)}.$$

Thus it suffices to show that

$$e^{-2/(k+2)} > \frac{k+1}{k+3} \quad \text{and} \quad e^{-1/(k+1)} < \frac{k+1}{k+2}.$$

The first of these is a rearrangement of the inequality

$$e^{2x} = 1 + 2x + 2x^2 + \cdots + \frac{2^n x^n}{n!} + \cdots < 1 + 2x + 2x^2 + 2x^3 + \cdots = \frac{1+x}{1-x}$$

for $0 < x < 1$ applied at $x = 1/(k+2)$, and the second is a rearrangement of the inequality $e^x > 1 + x$ for $x \neq 0$ applied at $x = 1/(k+1)$.

The answer to part **(b)** is no. To establish this, we first study the asymptotics of a_k more carefully. Let $v_k = a_k - 1 = e^{v_{k-1}} - 1$. From part **(a)** we conclude that v_k tends to 0 as $k \to \infty$. Thus we compute

$$\frac{1/v_{k+1} - 1/v_k}{(k+1) - k} = \frac{1}{v_{k+1}} - \frac{1}{v_k} = \frac{1 + v_k - e^{v_k}}{v_k(e^{v_k} - 1)} \sim \frac{-v_k^2/2}{v_k^2} = -\frac{1}{2}.$$

Hence by the Stolz–Cesàro theorem we have $\lim_{k\to\infty}(1/v_k)/k = -1/2$, or equivalently $v_k \sim -2/k$.

Now we compute

$$\frac{1}{v_{k+1}} - \frac{1}{v_k} + \frac{1}{2} = \frac{e^{v_k}(v_k - 2) + v_k + 2}{2v_k(e^{v_k} - 1)} \sim \frac{v_k^3/6}{2v_k^2} = \frac{v_k}{12} \sim -\frac{1}{6k}.$$

Therefore

$$\frac{\left(\frac{1}{v_{k+1}} + \frac{k+1}{2}\right) - \left(\frac{1}{v_k} + \frac{k}{2}\right)}{H_k - H_{k-1}} = k\left(\frac{1}{v_{k+1}} - \frac{1}{v_k} + \frac{1}{2}\right) \sim -\frac{1}{6},$$

where H_k is the kth harmonic number. Applying the Stolz–Cesàro theorem again, we obtain

$$\frac{1}{v_k} + \frac{k}{2} \sim -\frac{H_{k-1}}{6} \sim -\frac{\ln k}{6}.$$

Thus

$$a_k - 1 + \frac{2}{k} = v_k + \frac{2}{k} = v_k \cdot \frac{2}{k} \cdot \left(\frac{1}{v_k} + \frac{k}{2}\right) \sim \left(-\frac{2}{k}\right)\left(\frac{2}{k}\right)\left(-\frac{\ln k}{6}\right) = \frac{2\ln k}{3k^2},$$

and therefore

$$\lim_{k\to\infty} k^2\left(a_k - 1 + \frac{2}{k}\right) = \infty.$$

However, if a bound of the type given in part **(b)** held, we would have

$$k^2\left(a_k - 1 + \frac{2}{k}\right) < \frac{2(c+2)k}{k+c+2},$$

which is bounded above. Thus no such bound can hold.

Also solved by K. F. Andersen (Canada), R. Chapman (UK), L. Han (USA) & X. Tang (China), N. Hodges (UK), M. Kaplan, O. Kouba (Syria), G. Lavau (France), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Stadler (Switzerland), and A. Stenger. Part **(a)** only solved by P. Bracken, D. Fleischman, O. Geupel (Germany), W. Janous (Austria), A. Natian, and the proposer.

Bounds on a Function of the Angles and Sides of a Triangle

12168 [2020, 274]. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, North Macedonia.* Let a, b, and c be the side lengths of a triangle ABC with circumradius R and inradius r. Prove

$$\frac{2}{R} \le \frac{\sec(A/2)}{a} + \frac{\sec(B/2)}{b} + \frac{\sec(C/2)}{c} \le \frac{1}{r}.$$

Solution by S. S. Kumar, Portola High School, Irvine, California. Let s and K denote the semiperimeter and area of ABC, respectively. We first prove the second inequality. Note that by the half-angle formula and the law of cosines,

$$\sec(A/2) = \sqrt{\frac{2}{1+\cos A}} = \sqrt{\frac{4bc}{(b+c)^2 - a^2}} = \sqrt{\frac{bc}{s(s-a)}}.$$

By the AM-GM inequality, we have $2\sqrt{bc} \le b + c$ and $2\sqrt{(s-b)(s-c)} \le a$. Applying Heron's formula and the relation $K = rs$, it follows that

$$\frac{\sec(A/2)}{a} = \frac{1}{a}\sqrt{\frac{bc(s-b)(s-c)}{s(s-a)(s-b)(s-c)}} \le \frac{b+c}{4K} = \frac{b+c}{4rs}.$$

Combining this with similar formulas for the other angles, we have

$$\frac{\sec(A/2)}{a} + \frac{\sec(B/2)}{b} + \frac{\sec(C/2)}{c} \le \frac{b+c}{4rs} + \frac{c+a}{4rs} + \frac{a+b}{4rs} = \frac{4s}{4rs} = \frac{1}{r}.$$

To prove the first inequality, we note that by the law of sines, $a = 2R\sin A$, and similarly for the other sides, so the inequality is equivalent to

$$\frac{\sec(A/2)}{\sin A} + \frac{\sec(B/2)}{\sin B} + \frac{\sec(C/2)}{\sin C} \ge 4.$$

Define $f(x) = \sec(x/2)/\sin x$. It is tedious but straightforward to compute that on $(0, \pi)$,

$$f''(x) = \frac{1}{4}\sec(x/2)\csc(x)\left(4\csc^2(x) + (2\cot(x) - \tan(x/2))^2 + \sec^2(x/2)\right) > 0.$$

Hence, by Jensen's inequality, we obtain

$$\frac{\sec(A/2)}{\sin A} + \frac{\sec(B/2)}{\sin B} + \frac{\sec(C/2)}{\sin C} \ge 3f\left(\frac{A+B+C}{3}\right) = 4,$$

as desired.

Editorial comment. As noted by Omran Kouba, one can also deduce the first inequality by applying Jensen's inequality to the function $g(x) = -\log(\cos^2(x)\sin(x))$ on the interval

$(0, \pi/2)$, which is more easily computed to be convex than is $f(x)$. In fact this yields the stronger inequality

$$\frac{2}{R} \leq 3\sqrt[3]{\frac{\sec(A/2)}{a} \cdot \frac{\sec(B/2)}{b} \cdot \frac{\sec(C/2)}{c}},$$

which along with the AM-GM inequality implies the first inequality.

Also solved by A. Alt, M. Bataille (France), H. Chen, C. Chiser (Romania), G. Fera (Italy), S. Gayen (India), O. Geupel (Germany), N. Hodges (UK), M. Kaplan & M. Goldenberg, P. Khalili, K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), V. Schindler (Germany), A. Stadler (Switzerland), N. Stanciu & M. Drăgan (Romania), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Wimmer, M. R. Yegan (Iran), T. Zvonaru (Romania), and the proposer.

Estimating the Logarithmic Derivative of a Chebyshev Polynomial

12171 [2020, 275]. *Proposed by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Giessen, Germany.* Let T_n be the nth Chebyshev polynomial, defined by $T_n(\cos\theta) = \cos(n\theta)$. Prove

$$\frac{T_n'(1/z)}{T_n(1/z)} = \frac{nz}{\sqrt{1-z^2}} + O\left(z^{2n+1}\right)$$

as $z \to 0$.

Solution by Kenneth F. Andersen, Edmonton, Canada. We prove the equivalent statement, with $x = 1/z$,

$$\frac{T_n'(x)}{T_n(x)} = \frac{n}{\sqrt{x^2-1}} + O\left(\frac{1}{x^{2n+1}}\right) \quad \text{as } x \to \infty.$$

We begin with the fact that for $x \geq 1$, $T_n(x) = (A(x)^n + A(x)^{-n})/2$, where $A(x) = x + \sqrt{x^2-1}$. This can be proved by induction, using the well-known recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Alternatively, if we extend $A(x)$ to $x < 1$ by an appropriate choice of a branch of the square root function in the complex numbers, then with $x = \cos\theta$ for $0 \leq \theta \leq \pi$ we have $A(x) = \cos\theta + i\sin\theta = e^{i\theta}$, and therefore

$$T_n(x) = T_n(\cos\theta) = \cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{A(x)^n + A(x)^{-n}}{2}.$$

This equation can then be extended to $x \geq 1$ by analytic continuation.

Since $A'(x) = 1 + x/\sqrt{x^2-1} = A(x)/\sqrt{x^2-1}$ for $x > 1$, we have

$$T_n'(x) = \frac{nA(x)^{n-1} - nA(x)^{-n-1}}{2} \cdot A'(x) = \frac{n(A(x)^n - A(x)^{-n})}{2\sqrt{x^2-1}}.$$

Therefore

$$\left|\frac{T_n'(x)}{T_n(x)} - \frac{n}{\sqrt{x^2-1}}\right| = \frac{n}{\sqrt{x^2-1}}\left|\frac{A(x)^n - A(x)^{-n}}{A(x)^n + A(x)^{-n}} - 1\right| = \frac{2n}{(A(x)^{2n}+1)\sqrt{x^2-1}}.$$

The desired conclusion now follows because $A(x) \sim 2x$ and $\sqrt{x^2-1} \sim x$ as $x \to \infty$.

Editorial comment. The problem statement above corrects a typographical error from the original printing.

Also solved by A. Berkane (Algeria), R. Chapman (UK), H. Chen, O. Geupel (Germany), J.-P. Grivaux (France), L. Han (USA) & X. Tang (China), N. Hodges (UK), K. T. L. Koo (China), O. Kouba (Syria), M. Omarjee (France), A. Stadler (Switzerland), R. Tauraso (Italy), D. Terr, E. I. Verriest, T. Wiandt, and the proposer.

SOLUTIONS

An (Almost) Impossible Integral

12158 [2020, 86]. *Proposed by Hervé Grandmontagne, Paris, France.* Prove

$$\int_0^1 \frac{(\ln x)^2 \arctan x}{1+x}\,dx = \frac{21}{64}\pi\zeta(3) - \frac{1}{24}\pi^2 G - \frac{1}{32}\pi^3 \ln 2,$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$ and G is Catalan's constant $\sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$.

Solution by the proposer. Let R be the function defined by

$$R(x) = \int_0^x \frac{\ln^2 t}{1+t}dt = \int_0^1 \frac{x\ln^2(tx)}{1+tx}dt.$$

Integrating the given integral J by parts yields

$$\begin{aligned} J &= \big[R(x)\arctan x\big]_0^1 - \int_0^1 \frac{R(x)}{1+x^2}dx \\ &= \frac{\pi}{4}\int_0^1 \frac{\ln^2 x}{1+x}dx - \int_0^1\int_0^1 \frac{x\ln^2(tx)}{(1+x^2)(1+tx)}dt\,dx. \end{aligned}$$

Observe that

$$\frac{x}{(1+tx)(1+x^2)} + \frac{t}{(1+tx)(1+t^2)} = \frac{x}{(1+t^2)(1+x^2)} + \frac{t}{(1+t^2)(1+x^2)}.$$

Multiplying by $\ln^2(tx)$, integrating both sides, and exploiting symmetry under interchange of x and t gives

$$\int_0^1\int_0^1 \frac{x\ln^2(tx)}{(1+x^2)(1+tx)}dt\,dx = \int_0^1\int_0^1 \frac{x\ln^2(tx)}{(1+t^2)(1+x^2)}dt\,dx.$$

Thus after rewriting $\ln^2(tx)$ as $(\ln t + \ln x)^2$ we find

$$\begin{aligned}
J &= \frac{\pi}{4}\int_0^1 \frac{\ln^2 x}{1+x}dx - \int_0^1\int_0^1 \frac{x\ln^2(tx)}{(1+t^2)(1+x^2)}dt\,dx \\
&= \frac{\pi}{4}\int_0^1 \frac{\ln^2 x}{1+x}dx - \int_0^1 \frac{\ln^2 t}{1+t^2}dt\int_0^1 \frac{x}{1+x^2}dx - 2\int_0^1 \frac{\ln t}{1+t^2}dt\int_0^1 \frac{x\ln x}{1+x^2}dx \\
&\qquad - \int_0^1 \frac{1}{1+t^2}dt\int_0^1 \frac{x\ln^2 x}{1+x^2}dx.
\end{aligned}$$

The component integrals of this last expression are all fairly standard. The nonelementary ones are

$$\int_0^1 \frac{\ln t}{1+t^2}dt = -G, \qquad \int_0^1 \frac{\ln^2 t}{1+t^2}dt = \frac{\pi^3}{16},$$

$$\int_0^1 \frac{x\ln x}{1+x^2}dx = \frac{1}{4}\int_0^1 \frac{\ln y}{1+y}dy = \frac{1}{4}\mathrm{Li}_2(-1) = -\frac{\pi^2}{48},$$

and

$$\int_0^1 \frac{x\ln^2 x}{1+x^2}dx = \frac{1}{8}\int_0^1 \frac{\ln^2 y}{1+y}dy = -\frac{1}{4}\mathrm{Li}_3(-1) = \frac{3}{16}\zeta(3),$$

where we have substituted $y = x^2$ in the last two integrals. Plugging these all in, we get

$$\begin{aligned}
J &= \frac{\pi}{4}\cdot\frac{3}{2}\zeta(3) - \frac{\pi^3}{16}\cdot\frac{\ln 2}{2} - 2(-G)\cdot\frac{-\pi^2}{48} - \frac{\pi}{4}\cdot\frac{3}{16}\zeta(3) \\
&= \frac{21}{64}\pi\zeta(3) - \frac{1}{24}\pi^2 G - \frac{1}{32}\pi^3\ln 2.
\end{aligned}$$

Editorial comment. Several solvers noted that this integral appears in Section 1.24, pp. 14–15 of C. I. Vălean (2019), *(Almost) Impossible Integrals, Sums, and Series*, Cham: Springer, both explicitly and as the special case $n = 1$ of the more general integral

$$\begin{aligned}
\int_0^1 \frac{(\ln x)^{2n}\arctan x}{1+x}\,dx &= \frac{\pi}{4}(1-2^{-2n})\zeta(2n+1)(2n)! + \frac{1}{2}\beta(2n+2)(2n)! \\
&- \frac{\pi}{16}\lim_{s\to 0}\left(\frac{d^{2n}}{ds^{2n}}\left(\csc\frac{\pi s}{2}\left(\psi\left(\frac{3}{4}-\frac{s}{4}\right)-\psi\left(\frac{1}{4}-\frac{s}{4}\right)\right)\right.\right. \\
&\qquad \left.\left. + \sec\frac{\pi s}{2}\left(\psi\left(1-\frac{s}{4}\right)-\psi\left(\frac{1}{2}-\frac{s}{4}\right)\right) - 2\pi\csc(\pi s)\right)\right),
\end{aligned}$$

where ζ is the Riemann zeta function, ψ is the digamma function, and β is the Dirichlet beta function.

Also solved by A. Berkane (Algeria), P. Bracken, H. Chen, G. Fera (Italy), B. Huang, K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), M. A. Prasad (India), S. Sharma (India), F. Sinani (Kosovo), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, R. Tauraso (Italy), C. I. Vălean (Romania), J. Van Casteren & L. Kempeneers (Belgium), T. Wiandt, T. Wilde (UK), and Y. Zhou & M. L. Glasser.

The Neyman–Pearson Lemma in Disguise

12159 [2020, 86]. *Proposed by Rudolf Avenhaus, Universität der Bundeswehr München, Neubiberg, Germany, and Thomas Krieger, Forschungszentrum Jülich, Jülich, Germany.* Let Φ denote the distribution function of a standard normal random variable, and let U

denote its inverse function. Let n be a positive integer, and suppose $0 < \alpha < 1$ and $\mu \geq 0$. Prove

$$\Phi\left(U(\alpha) - \sqrt{n}\mu\right) \leq \left(\Phi\left(U(\sqrt[n]{\alpha}) - \mu\right)\right)^n.$$

Solution by the proposers. The inequality in the problem is an equality if $\mu = 0$. Thus we may assume $\mu > 0$.

Consider the following hypothesis testing problem: Let $X_1, \ldots, X_n$ be independent and identically normally distributed random variables with variance 1, where under the null hypothesis H_0 their expected values are all zero, and under the alternative hypothesis H_1 they are μ. In other words,

$$X_i \sim \begin{cases} \mathcal{N}(0, 1) & \text{under } H_0, \\ \mathcal{N}(\mu, 1) & \text{under } H_1. \end{cases}$$

We consider two decision procedures for testing these hypotheses: a simple intuitive test and the Neyman–Pearson test. In the simple test, we reject the null hypothesis if $\max_{i=1,\ldots,n} X_i$ is larger than a constant k, in other words, if the sample $(x_1, \ldots, x_n)$ belongs to the critical region C_s defined by

$$C_s = \left\{(x_1, \ldots, x_n) : \max_{i=1,\ldots,n} x_i > k\right\}.$$

We choose the threshold k so that the probability of a type I error is $1 - \alpha$; that is, $P_{H_0}(C_s) = 1 - \alpha$. This means

$$\alpha = P_{H_0}(\overline{C_s}) = P_{H_0}\left(\max_{i=1,\ldots,n} X_i \leq k\right) = \prod_{i=1}^{n} P_{H_0}(X_i \leq k) = (\Phi(k))^n,$$

and solving for k yields $k = U(\sqrt[n]{\alpha})$. If we let β_s denote the probability of a type II error for the simple test, then

$$\beta_s = P_{H_1}\left(\max_{i=1,\ldots,n} X_i \leq k\right) = (\Phi(k - \mu))^n = (\Phi(U(\sqrt[n]{\alpha}) - \mu))^n. \tag{1}$$

The Neyman–Pearson test uses the critical region C_{NP} defined by

$$C_{NP} = \left\{(x_1, \ldots, x_n) : \frac{\phi_{H_1}(x_1, \ldots, x_n)}{\phi_{H_0}(x_1, \ldots, x_n)} > k'\right\},$$

for some positive constant k', where the joint density functions ϕ_{H_0} under H_0 and ϕ_{H_1} under H_1 are given by

$$\phi_{H_0}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} \quad \text{and} \quad \phi_{H_1}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\mu)^2/2}.$$

Using these joint density functions, the critical region can be rewritten as

$$C_{NP} = \left\{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i > k''\right\}$$

for some constant k''. Once again we choose k', and therefore k'', so that the probability of a type I error is $1 - \alpha$. Because $\sum_{i=1}^{n} X_i$ is normally distributed, with distribution given by

$$\sum_{i=1}^{n} X_i \sim \begin{cases} \mathcal{N}(0, n) & \text{under } H_0, \\ \mathcal{N}(n\mu, n) & \text{under } H_1, \end{cases}$$

we obtain

$$\alpha = P_{H_0}(\overline{C_{NP}}) = P_{H_0}\left(\sum_{i=1}^{n} X_i \le k''\right) = \Phi\left(\frac{k''}{\sqrt{n}}\right),$$

and therefore $k'' = \sqrt{n}U(\alpha)$. The probability β_{NP} of a type II error is then given by the formula

$$\beta_{NP} = P_{H_1}(\overline{C_{NP}}) = P_{H_1}\left(\sum_{i=1}^{n} X_i \le k''\right) = \Phi\left(\frac{k'' - n\mu}{\sqrt{n}}\right) = \Phi(U(\alpha) - \sqrt{n}\mu). \quad (2)$$

According to the Neyman–Pearson lemma, $\beta_{NP} \le \beta_s$, and by (1) and (2), this is equivalent to the required inequality.

Editorial comment. The proposers' solution shows that the inequality can be proved without performing any calculations on the formulas on the two sides of the inequality. Richard Stong showed that the inequality can also be proved by direct calculations with these formulas. Letting $y = \sqrt{n}\mu$, the requested inequality reads

$$\Phi(U(\alpha) - y) \le (\Phi(U(\alpha^{1/n}) - y/\sqrt{n}))^n.$$

Since this inequality is an equality when $n = 1$, it suffices to show that the right side is a nondecreasing function of n for all real $n \ge 1$. Taking a logarithmic derivative and letting $x = U(\alpha^{1/n}) - y/\sqrt{n}$, we find that this is equivalent to

$$\frac{\Phi(x)\log\Phi(x)}{\phi(x)} - \frac{x}{2} \ge \alpha^{1/n}\log(\alpha^{1/n})U'(\alpha^{1/n}) - \frac{U(\alpha^{1/n})}{2}, \quad (3)$$

where ϕ is the density function for the standard normal distribution, that is,

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Next we note that $x \le U(\alpha^{1/n})$, with equality if $\mu = 0$ and $y = 0$, and in this case (3) is an equality. Thus it suffices to show that the left side is a nonincreasing function of x, or equivalently, taking a derivative, that

$$\frac{1}{2} + \log\Phi(x) + \frac{x\Phi(x)\log\Phi(x)}{\phi(x)} \le 0.$$

At this point, all of the parameters n, α, and μ have been eliminated, and the problem has been reduced to an inequality involving the standard normal distribution and density functions. Some further elaborate calculations verify this inequality.

No solutions were received other than the proposers' solution and the solution of R. Stong.

Fibonacci and Lucas: A Golden Braid

12160 [2020, 179]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Univerità di Roma "Tor Vergata," Rome, Italy.* Let F_n be the nth Fibonacci number, and let L_n be the nth Lucas number. (These numbers are defined recursively by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ when $n \ge 1$, and by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ when $n \ge 1$.) Prove

$$\sum_{k=0}^{n}\binom{2n+1}{n-k}F_{2k+1} = 5^n \qquad \text{and} \qquad \sum_{k=0}^{n}\binom{2n+1}{n-k}L_{2k+1} = \sum_{k=0}^{n}\binom{2k}{k}5^{n-k}$$

for all $n \in \mathbb{N}$.

Solution by Robin Chapman, University of Exeter, Exeter, UK. Let ϕ be the golden ratio $(\sqrt{5}+1)/2$. The familiar formulas for the Fibonacci numbers (Binet's formula) and the Lucas numbers are

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \qquad \text{and} \qquad L_n = \phi^n + (-\phi)^{-n}.$$

Combining the two formulas, we get $\phi^n = (L_n + \sqrt{5}\, F_n)/2$.

Let

$$S_n = \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{L_{2k+1} + \sqrt{5}\, F_{2k+1}}{2} = \sum_{k=0}^{n} \binom{2n+1}{n-k} \phi^{2k+1} = \sum_{k=0}^{n} \binom{2n+1}{k} \phi^{2n-2k+1}.$$

Pascal's formula yields $\binom{2n+1}{k} = \binom{2n-1}{k} + 2\binom{2n-1}{k-1} + \binom{2n-1}{k-2}$, a formula that holds even for $k \in \{0, 1\}$ if we take $\binom{m}{j} = 0$ when j is negative. We use this to compute

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} \binom{2n-1}{k} \phi^{2n-2k+1} + 2\sum_{k=1}^{n} \binom{2n-1}{k-1} \phi^{2n-2k+1} + \sum_{k=2}^{n} \binom{2n-1}{k-2} \phi^{2n-2k+1} \\
&= \sum_{k=0}^{n} \binom{2n-1}{k} \phi^{2n-2k+1} + 2\sum_{k=0}^{n-1} \binom{2n-1}{k} \phi^{2n-2k-1} + \sum_{k=0}^{n-2} \binom{2n-1}{k} \phi^{2n-2k-3} \\
&= \sum_{k=0}^{n-1} \binom{2n-1}{k} \phi^{2n-2k-1}(\phi^2 + 2 + \phi^{-2}) + \binom{2n-1}{n} \phi - \binom{2n-1}{n-1} \phi^{-1} \\
&= 5S_{n-1} + \frac{1}{2}\binom{2n}{n}.
\end{aligned}$$

In the last step, we used $\binom{2n-1}{n} = \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}$, along with $\phi + \phi^{-1} = \sqrt{5}$ and $\phi - \phi^{-1} = 1$. With the initial condition $S_0 = \phi$, the recurrence gives

$$S_n = \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{L_{2k+1} + \sqrt{5}\, F_{2k+1}}{2} = \frac{1}{2}\left(5^n \sqrt{5} + \sum_{k=0}^{n} \binom{2k}{k} 5^{n-k}\right).$$

As $\sqrt{5}$ is irrational, this separates into the two required identities.

Also solved by U. Abel & G. Arends (Germany), A. Berkane (Algeria), B. Bradie, B. Burdick, W. Chang, H. Chen (China), G. Fera (Italy), P. Fulop (Hungary), J. Grivaux (France), N. Hodges (UK), Y. Ionin, K. T. L. Koo (China), O. Kouba (Syria), P. Lalonde (Canada), G. Lavau (France), O. P. Lossers (Netherlands), C. Pranesachar (India), L. Shapiro, A. Stadler (Switzerland), R. Stong, B. Sury (India), D. Terr, J. Van hamme (Belgium), M. Vowe (Switzerland) M. Wildon (UK), and the proposer.

Integer Pairs on an Ellipse

12161 [2020, 179]. *Proposed by José Hernández Santiago, Guerrero, Mexico.* Let $N(C)$ be the number of pairs $(u, v) \in \mathbb{Z} \times \mathbb{Z}$ satisfying $u^2 + uv + v^2 = C$. Prove that 6 divides $N(C)$ for every positive integer C.

Solution by Allen Stenger, Boulder, CO. The number of pairs (u, v) satisfying the given equation is the same as the number of pairs satisfying $u^2 - uv + v^2 = C$ due to the mapping of (u, v) to $(u, -v)$. We work with the second equation.

We work in the ring $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$. The elements of this ring have the form $u + v\omega$, where u and v are integers, and the norm of this element is $u^2 - uv + v^2$. Thus our number $N(C)$ is equal to the number of elements of $\mathbb{Z}[\omega]$ whose norm is C. The ring has six units, namely ± 1, $\pm\omega$, and $\pm\omega^2$, and so each nonzero ring element has six associates (including itself). All associates have the same norm, so the total number of elements with a given norm is a multiple of 6.

The number $N(C)$ is finite, since $4C = (2u - v)^2 + 3v^2$, which implies that v is bounded and then also u is bounded. Since u and v are integers, the number of solutions (u, v) is finite.

Editorial comment. A related result is mentioned in H. L. Keng (1982), *Introduction to Number Theory*, Berlin: Springer. Exercise 2 on p. 308 states, "The number of solutions to $x^2 + xy + y^2 = k$ is $6E(k)$, where $E(k)$ is the number of divisors of k of the form $3h + 1$ minus the number of divisors of the form $3h + 2$." An anonymous solver noted that the result is given with three solutions as Problem 195 in M. I. Krusemeyer, G. T. Gilbert, and L. C. Larson (2012), *A Mathematical Orchard: Problems and Solutions*, Washington, DC: MAA, 338–340.

Solvers used various techniques, such as (a) showing that if (u, v) is a solution to $u^2 + uv + v^2 = C$, then so is $(v, -(u + v))$, and that iterating this observation yields six distinct solutions, (b) bringing in group actions, linear algebra, and/or the ring of integers $Z(\omega)$, where $\omega = \exp(2\pi i/3)$, and (c) using automorphisms of binary quadratic forms. Most solvers tacitly assumed that $N(C)$ is finite.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), A. J. Bevelacqua, J. N. Caro Montoya (Brazil), N. Caro (Brazil), W. Chang, R. Chapman (UK), C. Curtis & J. Boswell, R. Dempsey, A. Dixit (Canada) & S. Pathak (USA), G. Fera (Italy), N. Garson (Canada), K. Gatesman, O. Geupel (Germany), J.-P. Grivaux (France), J. W. Hagood, Y. J. Ionin, W. Janous (Austria), K. T. L. Koo (China), O. Kouba (Syria), C. P. A. Kumar (India), P. Lalonde (Canada), G. Lavau (France), O. P. Lossers (Netherlands), C. Moe, A. Natian, A. Pathak, L. J. Peterson, C. R. Pranesachar (India), J. Schlosberg, E. Schmeichel, J. H. Smith, A. Stadler (Switzerland), D. Stone & J. Hawkins, R. Stong, R. Tauraso (Italy), D. Terr, M. Vowe (Switzerland), the Missouri State University Problem Solving Group, and the proposer.

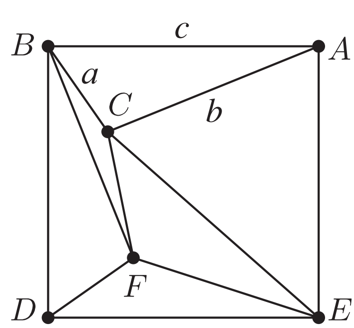
A Triangle Inequality from the Triangle Inequality

12162 [2020, 179]. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Consider a triangle with sides of lengths a, b, and c and with area S. Prove

$$\sqrt{a^2 + b^2 - 4S} + \sqrt{a^2 + c^2 - 4S} \geq \sqrt{b^2 + c^2 - 4S},$$

and determine when equality holds.

Solution by Yagub Aliyev, Baku, Azerbaijan. In the figure, $ABDE$ is a square and $\triangle BDF \cong \triangle ABC$. Applying the law of cosines in $\triangle ACE$, we get



$$CE = \sqrt{b^2 + c^2 - 2bc\cos\angle CAE}$$
$$= \sqrt{b^2 + c^2 - 2bc\sin A} = \sqrt{b^2 + c^2 - 4S}.$$

Similar calculations show that $FE = \sqrt{a^2 + c^2 - 4S}$ and $CF = \sqrt{a^2 + b^2 - 4S}$. By the triangle inequality, $CF + FE \geq EC$, and equality holds if and only if F lies on the segment CE.

Editorial comment. Most solvers used analytical approaches and provided one of various equivalent conditions for equality:

- With notation as in the diagram above, C lies on the upper-left quarter of the circle with diameter DE;
- a is the shortest side and $5(a^4 + b^4 + c^4) = 6(a^2b^2 + b^2c^2 + c^2a^2)$;
- a is the shortest side and $a^4 + b^4 + c^4 = 24S^2$;
- a is the shortest side and $a^2 + b^2 + c^2 = 8S$;
- A is the smallest angle and $\cot A + \cot B + \cot C = 2$;
- A is the smallest angle and $\cot \omega = 2$, where ω is the Brocard angle;
- $\sqrt{\cot A} = \sqrt{\cot B} + \sqrt{\cot C}$; or
- for some real $k > 0$,

$$\cot A = \frac{(k+1)^2}{k^2+k+1}, \quad \cot B = \frac{1}{k^2+k+1}, \quad \cot C = \frac{k^2}{k^2+k+1}.$$

Also solved by M. Bataille (France), R. Chapman (UK), C. Curtis, G. Fera & G. Tescaro (Italy), K. Gatesman, N. Hodges (UK), W. Janous (Austria), B. Karaivanov (USA) & T. S. Vassilev (Canada), P. Khalili, K. T. L. Koo (China), O. Kouba (Syria), K.-W. Lau (China), D. J. Moore, K. S. Palacios (Peru), C. R. Pranesachar (India), J. Schlosberg, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), F. Visescu (Romania), T. Wiandt, L. Zhou, T. Zvonaru (Romania), Davis Problem Solving Group, and the proposer.

Arithmetic Progressions and Fibonacci Numbers

12167 [2020, 274]. *Proposed by Nick MacKinnon, Winchester College, Winchester, UK.* Let S be the set of positive integers expressible as the sum of two nonzero Fibonacci numbers. Show that there are infinitely many six-term arithmetic progressions of numbers in S but only finitely many such seven-term arithmetic progressions.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. Since $2F_n = F_{n+1} + F_{n-2}$, we may view each element of S as a sum of two distinct Fibonacci numbers. Also note that any sum $F_n + F_k$ with $k < n$ lies in the interval $(F_n, F_{n+1}]$. Hence the elements of S in this interval are precisely the sums of F_n with smaller Fibonacci numbers. In particular, the expression of any given $s \in S$ as a sum of two distinct Fibonacci numbers is unique, and the larger is the largest F_n with $F_n < s$ (except for $s = 2$).

To find 6-term arithmetic progressions, start with F_n (for some $n \geq 3$, so that $F_n = F_{n-1} + F_{n-2} \in S$) and let the common difference in the progression be F_{n+3}. The resulting 6-term arithmetic progression with its terms shown to be in S is

$$\begin{aligned}
F_n &= F_{n-1} + F_{n-2}, \\
F_n + F_{n+3} &= F_n + F_{n+3}, \\
F_n + 2F_{n+3} &= F_{n+4} + F_{n+2}, \\
F_n + 3F_{n+3} &= F_{n+5} + F_{n+2}, \\
F_n + 4F_{n+3} &= F_{n+5} + F_{n+4}, \\
F_n + 5F_{n+3} &= F_{n+6} + F_{n+3}.
\end{aligned}$$

Such a progression cannot be extended to seven terms, since (a) the preceding term $F_n - F_{n+3}$ is negative, and (b) the next term $F_{n+6} + 2F_{n+3}$, being smaller than F_{n+7}, can only be in S if $2F_{n+3}$ is a Fibonacci number. Since $F_{n+4} < 2F_{n+3} < F_{n+5}$, it is not a Fibonacci number.

To complete the solution, we prove a stronger statement, namely that except for small values, these progressions are the only 6-term progressions in S. (The exceptions are subsets of the 10-term progression $2, 3, \ldots, 11$ and the two 7-term progressions $2, 6, 10, \ldots, 26$ and $3, 5, 7, \ldots, 15$; this requires checking small cases.)

For any 6-term progression $\{a_0 + kd\}_{k=0}^5$, we have

$$\frac{a_0 + 5d}{a_0 + 3d} < \frac{5}{3} \le \frac{F_{n+1}}{F_n}$$

when $n \ge 4$. Thus at least two of the last three terms in this progression lie in the same interval of the form $(F_n, F_{n+1}]$. Since we may ignore cases with $n \le 8$, we may assume we have a 5-term arithmetic progression $\{a_j\}_{j=1}^5$ whose last two terms lie in the interval $(F_{n+5}, F_{n+6}]$ for some $n \ge 4$. (We have chosen the indices here to match the example above.) We now consider two cases.

Case 1: *The top three terms lie in the interval* $(F_{n+5}, F_{n+6}]$. These terms (a_3, a_4, a_5) must be $(F_{n+5} + F_j, F_{n+5} + F_k, F_{n+5} + F_l)$, where $j < k < l \le n + 4$. Since the terms are in progression, $F_j + F_l = 2F_k = F_{k-2} + F_{k+1}$. Because representations as the sum of two Fibonacci numbers are unique, $l = k + 1$ and $j = k - 2$. Hence $k \le n + 3$, the common difference is F_{k-1}, and the preceding term a_2 must satisfy

$$a_2 = F_{n+5} + F_{k-2} - F_{k-1} = F_{n+5} - F_{k-3} = F_{n+4} + F_m$$

for some m. This forces $F_{k-3} + F_m = F_{n+3} = F_{n+2} + F_{n+1}$, which cannot hold since $k - 3 \le n$ and expressions as sums of distinct Fibonacci numbers are unique.

Case 2: *Only the top two terms of the 5-term progression lie in* $(F_{n+5}, F_{n+6}]$. Those terms a_4 and a_5 must be $F_{n+5} + F_k$ and $F_{n+5} + F_l$, where $k < l \le n + 4$. The previous term a_3 is $F_{n+5} + 2F_k - F_l$; it must satisfy

$$a_3 = \frac{a_1 + a_5}{2} > \frac{F_{n+5} + F_l}{2}.$$

Eliminating F_l (by summing $1/3$ of the equality and $2/3$ of the inequality for a_3) yields $a_3 > \frac{2}{3}F_{n+5} + \frac{2}{3}F_k$. By several applications of the Fibonacci recurrence, $\frac{2}{3}F_{n+5} = F_{n+4} + \frac{1}{3}F_{n+1}$, so

$$a_3 > F_{n+4} + \frac{1}{3}F_{n+1} + \frac{2}{3}F_k.$$

Since a_3 exceeds F_{n+4}, we conclude $a_3 = F_{n+4} + F_j$ for some $j \le n + 3$. Furthermore, since $\frac{1}{3}F_{n+1} + \frac{2}{3}F_k > \max(F_{k-1}, 2)$, we have $j \ge \max(k, 4)$. From $a_3 = F_{n+5} + 2F_k - F_l = F_{n+4} + F_j$, we conclude

$$F_{n+3} + 2F_k = F_l + F_j,$$

and hence at least one of j and l is at least as large as $n + 3$.

Since $F_{n+1}/F_n < 2 < F_{n+2}/F_n$ whenever $n \ge 3$, one Fibonacci number is twice another only for the initial values $1, 1, 2$. If $j = n + 3$, then $F_l = 2F_k$, so $F_k = 1$, and the last three terms of the progression are F_{n+5}, $F_{n+5} + 1$, and $F_{n+5} + 2$, but $F_{n+5} - 1 \notin S$. If $l = n + 3$, then $F_j = 2F_k$, but we already have $F_j > 2$.

Thus $l = n + 4$, which yields $F_j + F_{n+2} = 2F_k = F_{k+1} + F_{k-2}$. If $j = n + 2 = k$, then we obtain the family described earlier, extending to 6-term progressions. Otherwise, F_j and F_{n+2} are distinct, and hence one of j and $n + 2$ must equal $k - 2$. It is not j because $j \ge k$, and it is not $n + 2$ because $k < n + 4$. Hence we cannot produce such a 6-term arithmetic progression outside the family described earlier.

Also solved by J. Christopher, N. Hodges (UK), Y. J. Ionin, J. H. Nieto (Venezuela), A. Pathak (India), A. Stadler (Switzerland), R. Tauraso (Italy), T. Wilde (UK), and the proposer.